Section 14.8

1.) Find extrema for $f(x,y) = xy$
s.t. $x^2 + 2y^2 = 1$:
Let $F(x,y,\lambda) = xy - \lambda(x^2 + 2y^2 - 1) \longrightarrow$

$$\left.\begin{array}{l} F_x = y - 2\lambda x = 0 \\ F_y = x - 4\lambda y = 0 \end{array}\right\} \quad \left.\begin{array}{l} \lambda = y/2x \\ \lambda = x/4y \end{array}\right\} \quad \frac{y}{2x} = \frac{x}{4y} \longrightarrow \boxed{2y^2 = x^2}$$

$F_\lambda = -x^2 - 2y^2 + 1 = 0$ (sub.)

$\longrightarrow -x^2 - x^2 + 1 = 0 \longrightarrow 2x^2 = 1 \longrightarrow$
$x^2 = 1/2 \Rightarrow x = \pm 1/\sqrt{2}$; if $x = \frac{1}{\sqrt{2}}$, then
$y = \pm 1/2$ so $\left(\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$ and $\left(\frac{1}{\sqrt{2}}, \frac{-1}{2}\right)$ are
critical points ; if $x = -1/\sqrt{2}$ then
$y = \pm 1/2$ so $\left(\frac{-1}{\sqrt{2}}, \frac{1}{2}\right)$ and $\left(\frac{-1}{\sqrt{2}}, -1/2\right)$ are
critical points :

crit. pts.	values of $f(x,y)$	
$\left(\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$	$1/2\sqrt{2}$	MAX
$\left(\frac{1}{\sqrt{2}}, \frac{-1}{2}\right)$	$-1/2\sqrt{2}$	MIN
$\left(\frac{-1}{\sqrt{2}}, \frac{1}{2}\right)$	$-1/2\sqrt{2}$	MIN
$\left(\frac{-1}{\sqrt{2}}, \frac{-1}{2}\right)$	$1/2\sqrt{2}$	MAX

3.) maximize $f(x,y) = 49 - x^2 - y^2$ s.t.
$x + 3y = 10$:
Let $F(x,y,\lambda) = (49 - x^2 - y^2) - \lambda(x + 3y - 10)$

$$\longrightarrow \left.\begin{array}{l} F_x = -2x - \lambda = 0 \\ F_y = -2y - 3\lambda = 0 \end{array}\right\} \quad \left.\begin{array}{l} \lambda = -2x \\ \lambda = -\frac{2}{3}y \end{array}\right\} \quad -2x = -\frac{2}{3}y \longrightarrow \boxed{y = 3x}$$

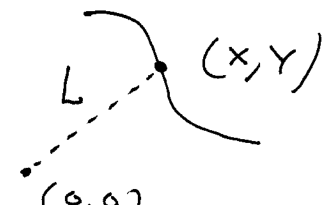
$F_\lambda = -x - 3y + 10 = 0$ (sub.)

$\longrightarrow -x - 3(3x) + 10 = 0 \longrightarrow$

$-10x + 10 = 0 \rightarrow x = 1$ and $y = 3$ so

$\boxed{(1,3)}$ is critical point with maximum value $f(1,3) = 39$.

8.) Distance



$L = \sqrt{(x-0)^2 + (y-0)^2} \rightarrow$

$L^2 = x^2 + y^2$;

Minimize and maximize

$f(x,y) = x^2 + y^2$ s.t. $x^2 + xy + y^2 = 1$:

Let $F(x,y,\lambda) = (x^2+y^2) - \lambda(x^2+xy+y^2-1) \rightarrow$

$$\left.\begin{array}{l} F_x = 2x - \lambda(2x+y) = 0 \rightarrow \lambda = \dfrac{2x}{2x+y} \\[2ex] F_y = 2y - \lambda(x+2y) = 0 \rightarrow \lambda = \dfrac{2y}{x+2y} \end{array}\right\} \rightarrow$$

$$\frac{2x}{2x+y} = \frac{2y}{x+2y} \rightarrow 2x^2 + \cancel{4xy} = \cancel{4xy} + 2y^2 \rightarrow$$

$\boxed{y^2 = x^2} \rightarrow \underline{y = x}$ or $\underline{y = -x}$;

$F_\lambda = -x^2 - xy - y^2 + 1 = 0$ (sub.)

If $y = x$, then $-x^2 - x^2 - x^2 + 1 = 0 \rightarrow$

$3x^2 = 1 \rightarrow x = \pm 1/\sqrt{3} \rightarrow \underline{\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)}$ and

$\underline{\left(\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)}$ are critical points ;

If $y = -x$, then $\cancel{-x^2} + \cancel{x^2} - x^2 + 1 = 0 \rightarrow$

$x^2 = 1 \rightarrow x = \pm 1 \rightarrow \underline{(1,-1)}$ and $\underline{(-1,1)}$ are critical points ; then

crit. pts.	values of $f(x,y)$	
$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$	$\frac{1}{3}+\frac{1}{3}=\frac{2}{3}$	MIN
$\left(\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$	$\frac{1}{3}+\frac{1}{3}=\frac{2}{3}$	MIN
$(1,-1)$	$1+1=2$	MAX
$(-1,1)$	$1+1=2$	MAX

So min. distance is $\sqrt{\frac{2}{3}}$; max. distance is $\sqrt{2}$.

14.) maximize and minimize

$f(x,y) = 3x-y+6$ s.t. $x^2+y^2=4$:

Let $F(x,y,\lambda) = (3x-y+6)-\lambda(x^2+y^2-4) \rightarrow$

$$\left.\begin{aligned} F_x &= 3-\lambda(2x)=0 \rightarrow \lambda=\frac{3}{2x} \\ F_y &= -1-\lambda(2y)=0 \rightarrow \lambda=\frac{-1}{2y} \end{aligned}\right\} \frac{3}{\cancel{2}x}=\frac{-1}{\cancel{2}y} \rightarrow \boxed{y=\frac{-1}{3}x} \;;$$

$F_\lambda = -x^2-y^2+4=0$ (sub.)

$\rightarrow -x^2-\left(\frac{-1}{3}x\right)^2+4=0 \rightarrow 4=\frac{10}{9}x^2 \rightarrow$

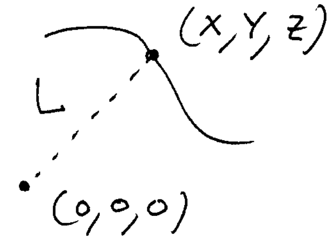
$x^2=\frac{18}{5} \rightarrow x=\pm\sqrt{\frac{18}{5}}=\pm 3\sqrt{\frac{2}{5}}$;

if $x=3\cdot\sqrt{\frac{2}{5}}$, then $y=-\sqrt{\frac{2}{5}}$ and $\left(3\sqrt{\frac{2}{5}}, -\sqrt{\frac{2}{5}}\right)$ is a critical point ;

if $x=-3\sqrt{\frac{2}{5}}$, then $y=\sqrt{\frac{2}{5}}$ and $\left(-3\sqrt{\frac{2}{5}}, \sqrt{\frac{2}{5}}\right)$ is a critical point ; then

crit. pts.	values of $f(x,y)$	
$\left(3\sqrt{\frac{2}{5}}, -\sqrt{\frac{2}{5}}\right)$	$9\sqrt{\frac{2}{5}}+\sqrt{\frac{2}{5}}+6=10\sqrt{\frac{2}{5}}+6$	MAX
$\left(-3\sqrt{\frac{2}{5}}, \sqrt{\frac{2}{5}}\right)$	$-9\sqrt{\frac{2}{5}}-\sqrt{\frac{2}{5}}+6=-10\sqrt{\frac{2}{5}}+6$	MIN

21.) Distance



$L = \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2}$

$\rightarrow \quad L^2 = x^2 + y^2 + z^2$;

minimize $f(x,y,z) = x^2 + y^2 + z^2$
s.t. $\quad z^2 = xy + 4$:

Let $F(x,y,z,\lambda) = (x^2 + y^2 + z^2) - \lambda(z^2 - xy - 4) \rightarrow$

$F_x = 2x - \lambda(-y) = 0 \rightarrow \lambda = \frac{-2x}{y}$ $\quad 1 = \frac{-2x}{y} \rightarrow \underline{y = -2x}$,

$F_y = 2y - \lambda(-x) = 0 \rightarrow \lambda = \frac{-2y}{x}$ $\quad 1 = \frac{-2y}{x} \rightarrow \underline{y = \frac{-1}{2}x}$

$F_z = 2z - \lambda(2z) = 0 \rightarrow \lambda = 1$ $\quad \rightarrow -2x = \frac{-1}{2}x$

$\rightarrow 4x = x \rightarrow 3x = 0 \rightarrow x = 0,\ y = 0$;

$F_\lambda = -z^2 + xy + 4 = 0 \rightarrow -z^2 + (0)(0) + 4 = 0$

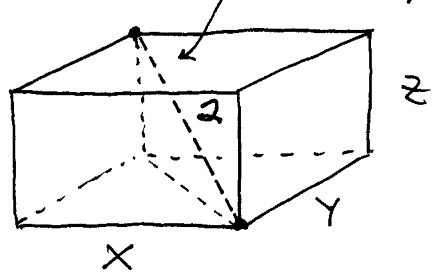
$\rightarrow z^2 = 4 \rightarrow z = \pm 2$ so $\boxed{(0,0,2)}$

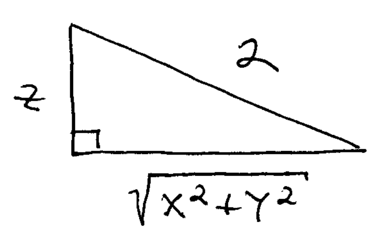
and $\boxed{(0,0,-2)}$ are critical points ;

<u>crit. pts.</u>	<u>values of f(x,y)</u>	
$(0,0,2)$	$0+0+4 = 4$	MIN
$(0,0,-2)$	$0+0+4 = 4$	MIN

So minimum distance is $\sqrt{4} = 2$.

27.) diagonal of box is 2





then $x^2 + y^2 + z^2 = 4$;

maximize volume $V = XYZ$

s.t. $X^2+Y^2+Z^2=4$: $(X>0, Y>0, Z>0)$

Let $F(x,Y,z,\lambda) = XYZ - \lambda(X^2+Y^2+Z^2-4) \rightarrow$

$$\left.\begin{aligned} F_X &= YZ - \lambda(2X) = 0 \rightarrow \lambda = YZ/2X \\ F_Y &= XZ - \lambda(2Y) = 0 \rightarrow \lambda = XZ/2Y \\ F_Z &= XY - \lambda(2Z) = 0 \rightarrow \lambda = \frac{XY}{2Z} \end{aligned}\right\} \text{(pair them)} \rightarrow$$

$$\left.\begin{aligned} \frac{YZ}{2X} &= \frac{XZ}{2Y} \longrightarrow \boxed{X^2 = Y^2} \\ \frac{XZ}{2Y} &= \frac{XY}{2Z} \longrightarrow \boxed{Z^2 = Y^2} \end{aligned}\right\} \text{(sub)}$$

$$F_\lambda = -X^2 - Y^2 - Z^2 + 4 = 0$$

$$\rightarrow -(Y^2) - Y^2 - (Y^2) + 4 = 0$$

$$\rightarrow 3Y^2 = 4 \rightarrow Y = \pm 2/\sqrt{3} = \frac{2}{\sqrt{3}} \text{ (ONLY +!)}$$

if $Y = \frac{2}{\sqrt{3}}$, then $X = \frac{2}{\sqrt{3}}$ and $Z = \frac{2}{\sqrt{3}}$

so critical point is $\boxed{\left(\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)}$ and maximum volume is

$$V = \left(\frac{2}{\sqrt{3}}\right)^3 = \frac{8}{3\sqrt{3}}$$

30.) Maximize and minimize $T = 400XYZ^2$

s.t. $X^2+Y^2+Z^2=1$:

Let $F(X,Y,Z,\lambda) = 400XYZ^2 - \lambda(X^2+Y^2+Z^2-1) \rightarrow$

$$F_X = 400YZ^2 - \lambda(2X) = 0 \rightarrow \lambda = \frac{400YZ^2}{2X}$$

$$\rightarrow \boxed{\lambda = \frac{200YZ^2}{X}};$$

$$F_Y = 400XZ^2 - \lambda(2Y) = 0 \rightarrow \lambda = \frac{400XZ^2}{2Y}$$

$$\rightarrow \boxed{\lambda = \frac{200XZ^2}{Y}};$$

$$F_Z = 800XYZ - \lambda(2Z) = 0 \rightarrow \lambda = \frac{800XYZ}{2Z}$$

$$\rightarrow \boxed{\lambda = \frac{400XYZ}{Z}} = 400XY;$$

(pair λ's) $\rightarrow$

$$\left.\begin{array}{l} \frac{200YZ^2}{X} = \frac{200XZ^2}{Y} \rightarrow Y^2Z^2 = X^2Z^2 \\ \frac{200XZ^2}{Y} = 400XY \rightarrow XZ^2 = 2XY^2 \end{array}\right\} \rightarrow$$

$$\left.\begin{array}{l} Y^2Z^2 - X^2Z^2 = 0 \rightarrow (Y^2 - X^2)Z^2 = 0 \\ XZ^2 - 2XY^2 = 0 \rightarrow X(Z^2 - Y^2) = 0 \end{array}\right\} \rightarrow$$

$$(*) \quad \left.\boxed{\begin{array}{l} Y^2 = X^2 \text{ or } Z = 0 \\ Z^2 = Y^2 \text{ or } X = 0 \end{array}}\right\} \quad \text{(sub)}$$

$$F_\lambda = \boxed{-X^2 - Y^2 - Z^2 + 1 = 0}$$; then

<u>case</u> 1: If $Y^2 = X^2$ and $Z^2 = Y^2$, then
$-(Y^2) - Y^2 - (Y^2) + 1 = 0 \rightarrow 3Y^2 = 1 \rightarrow$
$Y = \pm 1/\sqrt{3}$; if $Y = \frac{1}{\sqrt{3}}$, then $X = \pm\frac{1}{\sqrt{3}}$
and $Z = \pm 1/\sqrt{3}$ and critical points are

$$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right), \left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right);$$

if $Y = \frac{-1}{\sqrt{3}}$, then $X = \frac{\pm 1}{\sqrt{3}}$ and $Z = \frac{\pm 1}{\sqrt{3}}$
and critical points are
$\left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right), \left(\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$;

<u>case</u> 2: If $Y^2 = X^2$ and $X = 0$, then $Y = 0$ and
$0 - 0 - Z^2 + 1 = 0 \rightarrow Z^2 = 1 \rightarrow Z = \pm 1$
so critical points are $(0,0,1), (0,0,-1)$;

<u>case</u> 3: If $Z = 0$ and $Z^2 = Y^2$, then $Y = 0$
and $-X^2 - 0 - 0 + 1 = 0 \rightarrow X^2 = 1 \rightarrow X = \pm 1$
so critical points are $(1,0,0), (-1,0,0)$;

<u>Case</u> 4: If $Z = 0$ and $X = 0$, then
$0 - Y^2 - 0 + 1 = 0 \rightarrow Y^2 = 1 \rightarrow Y = \pm 1$
so critical points are $(0,1,0), (0,-1,0)$;
then

critical pts.	values of $f(x,y)$	
$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$	$\frac{400}{9}\ ^\circ C$	MAX
$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$	$\frac{400}{9}\ ^\circ C$	MAX
$\left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$	$-\frac{400}{9}\ ^\circ C$	MIN
$\left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$	$-\frac{400}{9}\ ^\circ C$	MIN
$\left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$	$-\frac{400}{9}\ ^\circ C$	MIN
$\left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$	$-\frac{400}{9}\ ^\circ C$	MIN
$\left(\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$	$\frac{400}{9}\ ^\circ C$	MAX
$\left(\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$	$\frac{400}{9}\ ^\circ C$	MAX
$(0,0,1)$	$0\ ^\circ C$	
$(0,0,-1)$	$0\ ^\circ C$	
$(1,0,0)$	$0\ ^\circ C$	
$(-1,0,0)$	$0\ ^\circ C$	
$(0,1,0)$	$0\ ^\circ C$	
$(0,-1,0)$	$0\ ^\circ C$	

37.) maximize $f(x,y,z) = x^2 + 2y - z^2$
s.t. $2x - y = 0$ and $y + z = 0$:

Let $F(x,y,z,\lambda,\mu) = (x^2 + 2y - z^2)$
$- \lambda(2x - y) - \mu(y + z) \rightarrow$

$$\left.\begin{array}{l} F_x = 2x - \lambda(2) - \mu(0) = 2x - 2\lambda = 0 \rightarrow \lambda = x \\ F_y = 2 - \lambda(-1) - \mu(1) = 2 + \lambda - \mu = 0 \quad \text{(sub.)} \\ F_z = -2z - \lambda(0) - \mu(1) = -2z - \mu = 0 \rightarrow \mu = -2z \end{array}\right\}$$

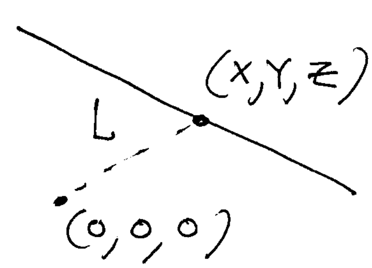
$\rightarrow 2 + x - (-2z) = 0 \rightarrow \boxed{x + 2z = -2}$;

$F_\lambda = -2x + y = 0 \rightarrow \boxed{2x - y = 0}$;
$F_\mu = -y - z = 0 \rightarrow \boxed{y + z = 0}$; then solve

$$\left.\begin{array}{l} x + 2z = -2 \\ 2x - y = 0 \\ y + z = 0 \end{array}\right\} \quad \left.\begin{array}{l} x + 2z = -2 \\ 2x + z = 0 \end{array}\right\} \rightarrow$$

$-3z = 4 \rightarrow z = -4/3$, $y = 4/3$, $x = 2/3$ so
critical point is $\boxed{(2/3, 4/3, -4/3)}$ and

maximum value is
$f\left(\frac{2}{3}, \frac{4}{3}, \frac{-4}{3}\right) = \frac{4}{9} + \frac{24}{9} - \frac{16}{9} = \frac{12}{9} = \frac{4}{3}$.



39.) Distance
$L = \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2}$

$\rightarrow L^2 = x^2 + y^2 + z^2$;

minimize $f(x,y,z) = x^2+y^2+z^2$
s.t. $y+2z=12$ and $x+y=6$:

Let $F(x,y,z,\lambda,\mu) = (x^2+y^2+z^2)$
$-\lambda(y+2z-12)-\mu(x+y-6) \rightarrow$

$F_x = 2x-\lambda(0)-\mu(1) = 2x-\mu=0 \rightarrow \mu=2x$
$F_y = 2y-\lambda(1)-\mu(1) = 2y-\lambda-\mu=0$ (sub.)
$F_z = 2z-\lambda(2)-\mu(0) = 2z-2\lambda=0 \rightarrow \lambda=z$
$\rightarrow 2y-z-2x=0 \rightarrow \boxed{2x-2y+z=0}$;

$F_\lambda = -y-2z+12=0 \rightarrow \boxed{y+2z=12}$;
$F_\mu = -x-y+6=0 \rightarrow \boxed{x+y=6}$; solve

$$\left.\begin{aligned} x+y &= 6 \\ y+2z &= 12 \\ 2x-2y+z &= 0 \end{aligned}\right\} \quad \left.\begin{aligned} x-2z &= -6 \\ 2x+5z &= 24 \end{aligned}\right\} \quad 9z=36 \rightarrow$$

$z=4$, $x=2$, $y=4$ so critical point
is $\boxed{(2,4,4)}$ and minimum
function value is $f(2,4,4)=4+16+16=36$
so minimum distance is $\sqrt{36} = 6$.