Section 10.3

1.) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ (Let $f(x) = \frac{1}{x^2}$, which is $+$, $\downarrow$, cont. for $x \geq 1$), then

$$\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{A\to\infty} \int_1^A \frac{1}{x^2}\,dx$$

$$= \lim_{A\to\infty} \left.\frac{-1}{x}\right|_1^A = \lim_{A\to\infty}\left(\frac{-1}{A} - \frac{-1}{1}\right) = 0 + 1 = 1 < \infty,$$

so series converges.

3.) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+4}$ (Let $f(x) = \frac{1}{x^2+4}$, which is $+$, $\downarrow$, cont. for $x \geq 1$), then

$$\int_1^{\infty} \frac{1}{x^2+4}\,dx = \lim_{A\to\infty} \int_1^A \frac{1}{x^2+2^2}$$

$$= \lim_{A\to\infty} \left.\frac{1}{2}\arctan\left(\frac{x}{2}\right)\right|_1^A$$

$$= \lim_{A\to\infty}\left(\frac{1}{2}\arctan\left(\frac{A}{2}\right) - \frac{1}{2}\arctan\frac{1}{2}\right)$$

$$= \frac{1}{2}\overset{\pi/2}{\arctan(\infty)} - \frac{1}{2}\arctan\frac{1}{2}$$

$$= \frac{\pi}{4} - \frac{1}{2}\arctan\left(\frac{1}{2}\right) < \infty,$$ so series converges

4.) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+4}$ (Let $f(x) = \frac{1}{x+4}$, which is $+$, $\downarrow$, cont. for $x \geq 1$), then

$$\int_1^{\infty} \frac{1}{x+4}\,dx = \lim_{A\to\infty} \int_1^A \frac{1}{x+4}\,dx$$

$$= \lim_{A\to\infty} \ln|x+4|\Big|_1^A = \lim_{A\to\infty}\left(\ln(A+4) - \ln 5\right)$$

$= \ln(\infty) - \ln 5 = \infty$, so series diverges

6.) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ (Let $f(x) = \dfrac{1}{x(\ln x)^2}$, which is $+, \downarrow$, cont. for $x \geq 2$), then

$$\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx = \lim_{A\to\infty} \int_2^{A} \frac{1}{x(\ln x)^2}\,dx$$

$$= \lim_{A\to\infty} \left.\frac{-1}{\ln x}\right|_2^A = \lim_{A\to\infty} \left(\frac{-1}{\ln A} - \frac{-1}{\ln 2}\right)$$

$$= \cancel{\frac{-1}{\infty}}^{\,0} + \frac{1}{\ln 2} = \frac{1}{\ln 2} < \infty,$$ so series converges

7.) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+4}$ (Let $f(x) = \dfrac{x}{x^2+4}$, which is $+, \downarrow$, cont. for $x \geq 2$ (graphing calc.)), then

$$\int_2^{\infty} \frac{x}{x^2+4}\,dx = \lim_{A\to\infty} \int_2^{A} \frac{x}{x^2+4}\,dx$$

$$= \lim_{A\to\infty} \left.\tfrac{1}{2}\ln(x^2+4)\right|_2^A$$

$$= \lim_{A\to\infty} \left(\tfrac{1}{2}\ln(A^2+4) - \tfrac{1}{2}\ln(8)\right)$$

$$= \tfrac{1}{2}\ln(\infty) - \tfrac{1}{2}\ln 8 = \infty,$$ so series diverges

8.) $\displaystyle\sum_{n=2}^{\infty} \frac{\ln(n^2)}{n} = \sum_{n=2}^{\infty} \frac{2\ln n}{n} = 2\sum_{n=2}^{\infty} \frac{\ln n}{n}$

(Let $f(x) = \dfrac{\ln x}{x}$, which is $+$, cont., and $\downarrow$ for $x \geq 3$ since

$$f'(x) = \frac{x \cdot \frac{1}{x} - \ln x}{x^2} = \frac{1 - \ln x}{x^2} = 0 \to x = e$$

NO $\quad + \quad 0 \quad - \quad f'$

$x=0 \qquad x = e \approx 2.718$), then

$$\int_3^{\infty} \frac{\ln x}{x}\,dx = \lim_{A\to\infty} \int_3^{A} \frac{\ln x}{x}\,dx$$

$$= \lim_{A\to\infty} \frac{1}{2}(\ln x)^2 \Big|_3^A = \lim_{A\to\infty} \left(\frac{1}{2}(\ln A)^2 - \frac{1}{2}(\ln 3)^2\right)$$

$= \infty$, so $2\displaystyle\sum_{n=3}^{\infty} \frac{\ln n}{n}$ diverges and

so does $2\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n}$.

12.) $\displaystyle\sum_{n=1}^{\infty} e^{-n} = \sum_{n=1}^{\infty} \frac{1}{e^n} = \sum_{n=1}^{\infty} \left(\frac{1}{e}\right)^n$

converges by the geometric series test since $r = \frac{1}{e}$ and $-1 < r < 1$.

13.) ON NEXT PAGE

14.) $\sum_{n=1}^{\infty} \frac{5}{n+1}$; $0 < \frac{5}{n+n} \leq \frac{5}{n+1}$

and $\sum_{n=1}^{\infty} \frac{5}{n+n} = \sum_{n=1}^{\infty} \frac{5}{2n} = \frac{5}{2} \sum_{n=1}^{\infty} \frac{1}{n}$,

which diverges by p-series test since $p = 1 \leq 1$, so

$\sum_{n=1}^{\infty} \frac{5}{n+1}$ diverges by

comparison test

13.) $\sum_{n=1}^{\infty} \frac{n}{n+1}$; $\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}}$

$= \frac{1}{1+0} = 1 \neq 0$, so series diverges by nth-term test.

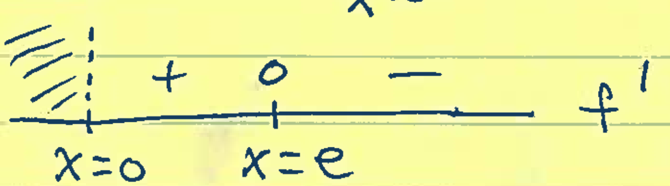
15.) $\sum_{n=1}^{\infty} \frac{3}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{3}{n^{1/2}}$; series diverges by p-series test since $p = \frac{1}{2} \leq 1$.

16.) $\sum_{n=1}^{\infty} \frac{-2}{n\sqrt{n}} = -2 \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$; series converges by p-series test since $p = 3/2 > 1$.

17.) $\sum_{n=1}^{\infty} \frac{-1}{8^n} = -\sum_{n=1}^{\infty} \left(\frac{1}{8}\right)^n$; series converges by geometric series test since $r = \frac{1}{8}$ and $-1 < r < 1$.

19.) $\sum_{n=2}^{\infty} \frac{\ln n}{n}$; let $f(x) = \frac{\ln x}{x} \xrightarrow{D}$

$f'(x) = \frac{x \cdot \frac{1}{x} - \ln x \cdot 1}{x^2} = \frac{1 - \ln x}{x}$, so



f is +, ↓ for $x \geq 3$,

and continuous for $x \geq 3$; then

$$\int_3^\infty \frac{\ln x}{x}\,dx = \lim_{A\to\infty} \int_3^A \frac{\ln x}{x}\,dx$$

$$= \lim_{A\to\infty} \frac{1}{2}(\ln x)^2 \Big|_3^A = \lim_{A\to\infty} \left(\frac{1}{2}(\ln A)^2 - \frac{1}{2}(\ln 3)^2\right)$$

$= \infty$, so series diverges by the integral test.

22.) $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{4^n+3}$; $\displaystyle\lim_{n\to\infty} \frac{5^n}{4^n+1} = \lim_{n\to\infty} \frac{\left(\frac{5}{4}\right)^n}{1+\frac{1}{4^n}}$

$= \dfrac{\infty}{1+0} = \infty \neq 0$, so series diverges by nth-term test.

24.) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n-1}$; Let $f(x) = \dfrac{1}{2x-1}$, then f is $+$, $\downarrow$, and continuous for $x \geq 1$; thus,

$$\int_1^\infty \frac{1}{2x-1}\,dx = \lim_{A\to\infty} \int_1^A \frac{1}{2x-1}\,dx$$

$$= \lim_{A\to\infty} \frac{1}{2}\ln|2x-1| \Big|_1^A$$

$$= \lim_{A\to\infty} \left(\frac{1}{2}\ln|2A-1| - \frac{1}{2}\cancelto{0}{\ln 1}\right) = \infty,$$

so series diverges by integral test.

28.) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}(\sqrt{n}+1)}$; Let $f(x) = \frac{1}{\sqrt{x}(\sqrt{x}+1)}$,

then f is $+$, $\downarrow$, and continuous for $x \geq 1$; thus,

$$\int_1^{\infty} \frac{1}{\sqrt{x}(\sqrt{x}+1)}\,dx = \lim_{A\to\infty} \int_1^A \frac{1}{\sqrt{x}(\sqrt{x}+1)}\,dx$$

$$= \lim_{A\to\infty} 2\ln|\sqrt{x}+1| \Big|_1^A$$

$$= \lim_{A\to\infty} \left(2\ln|\sqrt{A}+1| - 2\ln 2\right) = \infty,$$

so series <u>diverges</u> by <u>integral test</u>

30.) $\sum_{n=1}^{\infty} \frac{1}{(\ln 3)^n} = \sum_{n=1}^{\infty} \left(\frac{1}{\ln 3}\right)^n$ <u>converges</u>

by <u>geometric</u> series test since $r = \frac{1}{\ln 3}$ and $-1 < r < 1$.

32.) $\sum_{n=1}^{\infty} \frac{1}{n(1+(\ln n)^2)}$; Let $f(x) = \frac{1}{x(1+(\ln x)^2)}$,

then f is $+$, $\downarrow$, and continuous for $x \geq 1$; thus, $\int_1^{\infty} \frac{1}{x(1+(\ln x)^2)}\,dx = \lim_{A\to\infty} \int_1^A \frac{1}{x(1+(\ln x)^2)}\,dx$

$$= \lim_{A\to\infty} \arctan(\ln x) \Big|_1^A$$

$$= \lim_{A\to\infty} \left(\arctan(\ln A) - \arctan(\overset{0}{\cancel{\ln 1}})\right)$$

$$= \arctan(\infty) - \arctan(0) = \frac{\pi}{2} - 0 = \frac{\pi}{2},$$

so series converges by integral test.

33.) $\sum_{n=1}^{\infty} n \cdot \sin\left(\frac{1}{n}\right)$; $\lim_{n\to\infty} n \sin\left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}}$

$= \lim_{w\to 0} \frac{\sin w}{w} = 1 \neq 0$, so series

diverges by nth-term test

35.) $\sum_{n=1}^{\infty} \frac{e^n}{1+e^{2n}}$; Let $f(x) = \frac{e^x}{1+e^{2x}} \xrightarrow{D}$

$$f'(x) = \frac{(1+e^{2x}) \cdot e^x - e^x \cdot 2e^{2x}}{(1+e^{2x})^2}$$

$= \frac{1-e^{3x}}{(1+e^{2x})^2}$; $+ \quad 0 \quad -$ f', $x=0$

then f is $+$, $\downarrow$, and continuous for $x \geq 1$;

thus, $\int_1^{\infty} \frac{e^x}{1+e^{2x}}\,dx = \lim_{A\to\infty} \int_1^A \frac{e^x}{1+(e^x)^2}\,dx$

$= \lim_{A\to\infty} \arctan(e^x) \Big|_1^A$

$= \lim_{A\to\infty} \left(\arctan(e^A) - \arctan(e)\right)$

$= \arctan(\infty) - \arctan(e)$

$= \frac{\pi}{2} - \arctan(e)$, so series

converges.

38.) $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$; Let $f(x) = \frac{x}{x^2+1} \xrightarrow{D}$

$$f'(x) = \frac{(x^2+1)\cdot(1) - x\cdot(2x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} \; ;$$

then f is $+$, $\downarrow$, and continuous for $x \geq 1$; thus,

$$\begin{array}{c} + \quad \overset{0}{|} \quad - \\ \hline x=1 \end{array} \; f'$$

$$\int_1^{\infty} \frac{x}{x^2+1}\,dx = \lim_{A\to\infty} \int_1^A \frac{x}{x^2+1}\,dx$$

$$= \lim_{A\to\infty} \frac{1}{2}\ln(x^2+1)\Big|_1^A$$

$$= \lim_{A\to\infty} \left(\frac{1}{2}\ln(A^2+1) - \frac{1}{2}\ln 2\right) = \infty \; , \text{ so}$$

series diverges by integral test.

41.) $\sum_{n=1}^{\infty}\left(\frac{a}{n+2} - \frac{1}{n+4}\right)$, let $f(x) = \frac{a}{x+2} - \frac{1}{x+4} > 0$ for all $x \geq 1 \rightarrow \frac{a}{x+2} > \frac{1}{x+4} \rightarrow$

$a > \frac{x+2}{x+4} \rightarrow \boxed{a \geq 1}$ (since $\lim_{x\to\infty} \frac{x+2}{x+4} = 1$) ;

for convergence :

$$\int_1^{\infty} f(x)\,dx = \lim_{A\to\infty} \int_1^A f(x)\,dx < \infty .$$

$$\lim_{A\to\infty} \int_1^A \left(\frac{a}{x+2} - \frac{1}{x+4}\right) dx$$

$$= \lim_{A\to\infty} \left(a\ln(x+2) - \ln(x+4)\right)\Big|_1^A$$

$$= \lim_{A\to\infty} \left(\ln(x+2)^a - \ln(x+4)\right)\Big|_1^A$$

$$= \lim_{A \to \infty} \ln \frac{(x+2)^a}{x+4} \Big|_1^A$$

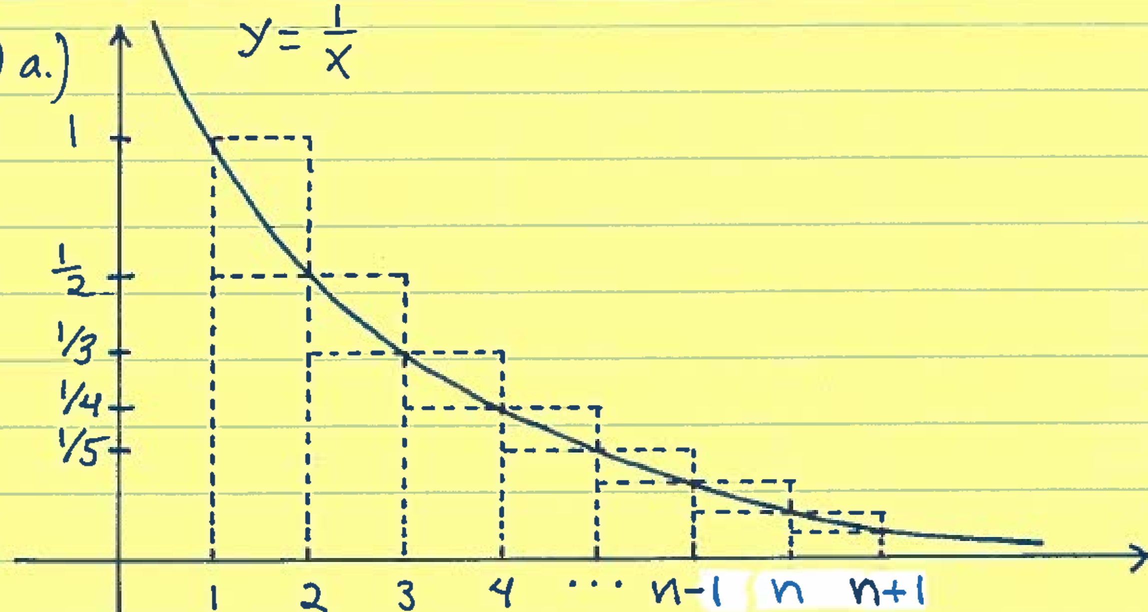
$$= \lim_{A \to \infty} \left[\ln \frac{(A+2)^a}{A+4} - \ln \frac{3^a}{5} \right]$$

$$\overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{A \to \infty} \ln \frac{a(A+2)^{a-1}}{1} - \ln \frac{3^a}{5} ;$$

if $a > 1$, then $\lim_{A \to \infty} \ln a(A+2)^{a-1} = \infty$,

so $\boxed{a \leq 1}$; thus $\boxed{a = 1}$.

43.) a.)



Consider the integral $\int_1^{n+1} \frac{1}{x}\, dx$

and rectangles above the graph of $y = \frac{1}{x}$ on the interval $[1, n+1]$. Comparing areas results in

$$\int_1^{n+1} \frac{1}{x}\,dx = \ln x \Big|_1^{n+1} = \ln(n+1) - \cancel{\ln 1}^{\,0}$$

$$\le \left(\frac{1}{1}\right)(1) + \left(\frac{1}{2}\right)(1) + \left(\frac{1}{3}\right)(1) + \cdots + \left(\frac{1}{n}\right)(1) \text{, i.e.,}$$

(I) $\boxed{\ln(n+1) \le 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}}$;

Consider the integral $\int_1^n \frac{1}{x}\,dx$ and rectangles below the graph of $y = \frac{1}{x}$ on the interval $[1, n]$. Comparing areas results in

$$\left(\frac{1}{2}\right)(1) + \left(\frac{1}{3}\right)(1) + \left(\frac{1}{4}\right)(1) + \cdots + \left(\frac{1}{n}\right)(1) \le \int_1^n \frac{1}{x}\,dx$$

$$= \ln x \Big|_1^n = \ln n - \cancel{\ln 1}^{\,0} \text{, i.e.,}$$

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} \le \ln n \longrightarrow$$

(II) $\boxed{1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \le 1 + \ln n}$.

b.) $(13{,}000{,}000{,}000 \text{ yrs.})$

$$\times \left(\frac{365 \text{ days}}{\text{yr.}}\right) \times \left(\frac{24 \text{ hrs.}}{\text{day}}\right)$$

$$\times \left(\frac{60 \text{ min.}}{\text{hr.}}\right) \times \left(\frac{60 \text{ sec.}}{\text{min.}}\right) \approx 4.09968 \times 10^{17}$$

$$= A \text{ ;}$$

then by (I) and (II)

$$\ln(A+1) \le 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{A} \le 1+\ln A \rightarrow$$

$$40.5548 \le 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{A} \le 41.5548$$

49.) $\sum_{n=1}^{\infty} \frac{1}{n^3}$; by (*)(*) where $f(x)=\frac{1}{x^3}$

$$\underbrace{f(n+1)+f(n+2)+f(n+3)+\cdots}_{\text{Remainder}} < \int_n^{\infty} \frac{1}{x^3}\,dx \ ;$$

require that $\int_n^{\infty} \frac{1}{x^3}\,dx < 0.01 \rightarrow$

$$\lim_{A\to\infty} \int_n^A x^{-3}\,dx = \lim_{A\to\infty} -\frac{1}{2}x^{-2}\Big|_n^A$$

$$= \lim_{A\to\infty}\left(\cancel{\frac{-1}{2A^2}}^{\,0} - \frac{-1}{2n^2}\right) = \frac{1}{2n^2} < \frac{1}{100} \rightarrow$$

$n^2 > 50 \rightarrow n > \sqrt{50} \approx 7.07$ so

choose $\boxed{n=8}$; then within 0.01

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \approx \sum_{n=1}^{8} \frac{1}{n^3} = \frac{1}{1^3}+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}$$

$$+\frac{1}{5^3}+\frac{1}{6^3}+\frac{1}{7^3}+\frac{1}{8^3} \approx 1.195$$

52.) $\sum_{n=4}^{\infty} \frac{1}{n(\ln n)^3}$; by (*)(*) where $f(x)=\frac{1}{x(\ln x)^3}$

$$\underbrace{f(n+1)+f(n+2)+f(n+3)+\cdots}_{\text{Remainder}} < \int_n^{\infty} \frac{1}{x(\ln x)^3}\,dx \ ;$$

require that $\int_n^{\infty} \frac{1}{x(\ln x)^3}\,dx < 0.01 \rightarrow$

$$\lim_{A\to\infty} \int_n^A \frac{1}{x(\ln x)^3}\,dx = \lim_{A\to\infty} \left. \frac{-1}{2}(\ln x)^{-2} \right|_n^A$$

$$= \lim_{A\to\infty} \left(\frac{-1}{2(\ln A)^2} \xrightarrow{} 0 - \frac{-1}{2(\ln n)^2} \right) = \frac{1}{2(\ln n)^2} < \frac{1}{100} \rightarrow$$

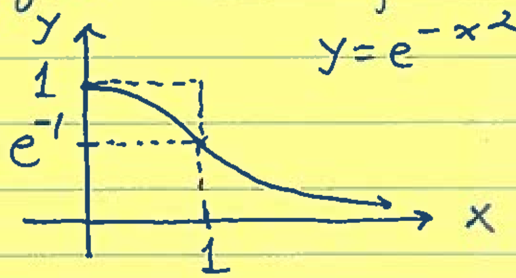
$(\ln n)^2 > 50 \rightarrow \ln n > \sqrt{50} \approx 7.07$, so

let $\ln n = 7.08 \rightarrow n = e^{7.08} \approx 1188$; then

within 0.01

$$\sum_{n=4}^{\infty} \frac{1}{n(\ln n)^3} \approx \sum_{n=4}^{1188} \frac{1}{n(\ln n)^3}$$

58.) $\sum_{n=0}^{\infty} e^{-n^2}$; Let $f(x) = e^{-x^2}$, then f is $+$, $\downarrow$, and continuous for $x \geq 0$; then



$$\int_0^{\infty} e^{-x^2} = \int_0^1 e^{-x^2} dx + \int_1^{\infty} e^{-x^2} dx$$

$$\leq (1)(1) + \int_1^{\infty} e^{-x} dx$$

$$= 1 + \lim_{A \to \infty} \int_1^A e^{-x} dx$$

$$= 1 + \lim_{A \to \infty} -e^{-x} \Big|_1^A$$

$$= 1 + \lim_{A \to \infty} \left(\frac{-1}{e^A} - \frac{-1}{e} \right)$$

$$= 1 + \left(0 + \frac{1}{e}\right) = 1 + \frac{1}{e}$$; so

$\int_0^{\infty} e^{-x^2} dx$ converges and so

$\sum_{n=0}^{\infty} e^{-n^2}$ converges.