Section 10.4

1.) $\sum_{n=1}^{\infty} \frac{1}{n^2+30}$; $0 < \frac{1}{n^2+30} \le \frac{1}{n^2+0} = \frac{1}{n^2}$ and

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by p-series test $(p=2>1)$

so $\sum_{n=1}^{\infty} \frac{1}{n^2+30}$ converges by comparison test

2.) $\sum_{n=1}^{\infty} \frac{n-1}{n^4+2}$; $0 < \frac{n-1}{n^4+2} \le \frac{n-1+1}{n^4+0} = \frac{n}{n^4} = \frac{1}{n^3}$

and $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges by p-series test $(p=3>1)$

so $\sum_{n=1}^{\infty} \frac{n-1}{n^4+2}$ converges by comparison test

4.) $\sum_{n=2}^{\infty} \frac{n+2}{n^2-n}$; $\frac{n+2}{n^2-n} \ge \frac{n+0}{n^2-n+n} = \frac{n}{n^2} = \frac{1}{n} > 0$

and $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges by p-series test $(p=1\le1)$ so

$\sum_{n=2}^{\infty} \frac{n+2}{n^2-n}$ diverges by comparison test

5.) $\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^{3/2}}$; $0 < \frac{\cos^2 n}{n^{3/2}} \le \frac{1}{n^{3/2}}$ and

$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges by p-series test $(p=3/2>1)$ so

$\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^{3/2}}$ converges by comparison test

6.) $\sum_{n=1}^{\infty} \frac{1}{n3^n}$; $0 < \frac{1}{n3^n} \le \frac{1}{(1)3^n} = \frac{1}{3^n} = \left(\frac{1}{3}\right)^n$

and $\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$ converges by geometric series test $\left(r = \frac{1}{3},\ -1 < r < 1\right)$

so $\sum_{n=1}^{\infty} \frac{1}{n3^n}$ converges by comparison test

7.) $\sum_{n=1}^{\infty} \sqrt{\frac{n+4}{n^4+4}}$; $0 < \sqrt{\frac{n+4}{n^4+4}} \le \sqrt{\frac{n+4n}{n^4+0}}$

$= \sqrt{\frac{5n}{n^4}} = \sqrt{5} \cdot \sqrt{\frac{1}{n^3}} = \sqrt{5} \cdot \frac{1}{n^{3/2}}$ and

$\sum_{n=1}^{\infty} \sqrt{5} \cdot \frac{1}{n^{3/2}} = \sqrt{5} \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges by p-series test $(p = 3/2 > 1)$ so

$\sum_{n=1}^{\infty} \sqrt{\frac{n+4}{n^4+4}}$ converges by comparison test

8.) $\sum_{n=1}^{\infty} \frac{\sqrt{n}+1}{\sqrt{n^2+3}}$; $\frac{\sqrt{n}+1}{\sqrt{n^2+3}} \ge \frac{\sqrt{n}+0}{\sqrt{n^2+3n^2}}$

$= \frac{\sqrt{n}}{\sqrt{4n^2}} = \frac{n^{1/2}}{2n} = \frac{1}{2} \cdot \frac{1}{n^{1/2}} > 0$ and

$\sum_{n=1}^{\infty} \frac{1}{2} \cdot \frac{1}{n^{1/2}} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ diverges by p-series test $(p = 1/2 \le 1)$ so

$\sum_{n=1}^{\infty} \frac{\sqrt{n}+1}{\sqrt{n^2+3}}$ diverges by comparison test

9.) $\displaystyle\sum_{n=1}^{\infty} \frac{n-2}{n^3-n^2+3}$; limit compare to $\frac{1}{n^2}$:

$$\lim_{n\to\infty} \frac{\frac{n-2}{n^3-n^2+3}}{1/n^2} = \lim_{n\to\infty} \frac{n-2}{n^3-n^2+3} \cdot \frac{n^2}{1}$$

$$= \lim_{n\to\infty} \frac{n^3-2n^2}{n^3-n^2+3} \cdot \frac{1/n^3}{1/n^3}$$

$$= \lim_{n\to\infty} \frac{1-2/n}{1-1/n+3/n^3} = \frac{1-0}{1-0+0} = 1 ;$$

and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by p-series test ($p=2>1$) so series

$\displaystyle\sum_{n=1}^{\infty} \frac{n-2}{n^3-n^2+3}$ converges by limit comparison test

11.) $\displaystyle\sum_{n=2}^{\infty} \frac{n(n+1)}{(n^2+1)(n-1)} = \sum_{n=2}^{\infty} \frac{n^2+n}{n^3-n^2+n-1}$;

limit compare to $\frac{1}{n}$:

$$\lim_{n\to\infty} \frac{\frac{n^2+n}{n^3-n^2+n-1}}{1/n} = \lim_{n\to\infty} \frac{n^2+n}{n^3-n^2+n-1} \cdot \frac{n}{1}$$

$$= \lim_{n\to\infty} \frac{n^3+n^2}{n^3-n^2+n-1} \cdot \frac{\frac{1}{n^3}}{1/n^3} = \lim_{n\to\infty} \frac{1+1/n}{1-\frac{1}{n}+\frac{1}{n^2}-\frac{1}{n^3}}$$

$$= \frac{1+0}{1-0+0-0} = 1 ;$$ and $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n}$ diverges

by p-series test ($p = 1 \le 1$) so series

$$\sum_{n=2}^{\infty} \frac{n(n+1)}{(n^2+1)(n-1)}$$ diverges by limit comparison test

12.) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{3+4^n}$; limit compare to

$$\frac{2^n}{4^n} = \left(\frac{2}{4}\right)^n = \left(\frac{1}{2}\right)^n : \lim_{n\to\infty} \frac{\frac{2^n}{3+4^n}}{\left(\frac{1}{2}\right)^n}$$

$$= \lim_{n\to\infty} \frac{2^n}{3+4^n} \cdot \frac{2^n}{1^n} = \lim_{n\to\infty} \frac{(2\cdot 2)^n}{3+4^n}$$

$$= \lim_{n\to\infty} \frac{4^n}{3+4^n} \cdot \frac{\frac{1}{4^n}}{\frac{1}{4^n}} = \lim_{n\to\infty} \frac{1}{\frac{3}{4^n}+1}$$

$= \dfrac{1}{0+1} = 1$; and $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ converges

by geometric series test

($r = \frac{1}{2}$, $-1 < r < 1$) so series

$$\sum_{n=1}^{\infty} \frac{2^n}{3+4^n}$$ converges by limit comparison test

13.) $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{\sqrt{n}\, 4^n}$; limit compare to $\dfrac{1}{\sqrt{n}}$:

$$\lim_{n\to\infty} \frac{\frac{5^n}{\sqrt{n}\,4^n}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{5^n}{\cancel{\sqrt{n}}\,4^n}\cdot\frac{\cancel{\sqrt{n}}}{1}$$

$$= \lim_{n\to\infty}\left(\frac{5}{4}\right)^n = \infty \text{ ; and } \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$$

diverges by p-series $\left(p=\frac{1}{2}\leq 1\right)$,

so $\displaystyle\sum_{n=1}^{\infty}\frac{5^n}{\sqrt{n}\,4^n}$ <u>diverges</u> by

<u>limit comparison test</u>

14.) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{2n+3}{5n+4}\right)^n$; limit compare to

$\left(\frac{2}{5}\right)^n$: $\displaystyle\lim_{n\to\infty}\frac{\left(\frac{2n+3}{5n+4}\right)^n}{\left(\frac{2}{5}\right)^n} = \lim_{n\to\infty}\left(\frac{2n+3}{5n+4}\right)^n\cdot\left(\frac{5}{2}\right)^n$

$$= \lim_{n\to\infty}\left(\frac{2n+3}{5n+4}\cdot\frac{5}{2}\right)^n = \lim_{n\to\infty}\left(\frac{10n+15}{10n+8}\right)^n$$

$$= \lim_{n\to\infty}\left(\frac{10n+8+7}{10n+8}\right)^n = \lim_{n\to\infty}\left(\frac{10n+8}{10n+8}+\frac{7}{10n+8}\right)^n$$

$$= \lim_{n\to\infty}\left[\left(1+\frac{1}{\frac{10n+8}{7}}\right)^{\frac{10n+8}{7}}\right]^{\frac{7}{10n+8}\cdot n\cdot\frac{\frac{1}{n}}{\frac{1}{n}}}$$

$$= \lim_{n\to\infty}\left[\left(1+\frac{1}{\frac{10n+8}{7}}\right)^{\frac{10n+8}{7}}\right]^{\frac{7}{10+\frac{8}{n}}}$$

$= [e]^{\frac{7}{10+0}} = e^{7/10}$; and $\sum_{n=1}^{\infty}\left(\frac{2}{5}\right)^n$ converges by geometric series test ($r = 2/5$, $-1 < r < 1$) so series $\sum_{n=1}^{\infty}\left(\frac{2n+3}{5n+4}\right)^n$ converges by limit comparison test

15.) $\sum_{n=2}^{\infty}\frac{1}{\ln n}$; limit compare to $\frac{1}{n}$:

$$\lim_{n\to\infty}\frac{\frac{1}{\ln n}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{1}{\ln n}\cdot\frac{n}{1} = \lim_{n\to\infty}\frac{n}{\ln n}$$

$$\overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty}\frac{1}{1/n} = \lim_{n\to\infty} n = \infty; \text{ and}$$

$\sum_{n=2}^{\infty}\frac{1}{n}$ diverges by p-series test ($p = 1 \le 1$) so series

$\sum_{n=2}^{\infty} \frac{1}{\ln n}$ diverges by limit comparison test

16.) $\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n^2}\right)$; limit compare to $\frac{1}{n^2}$: $\lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n^2}\right)}{\frac{1}{n^2}} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{n\to\infty} \frac{\frac{1}{1+1/n^2} \cdot \frac{-2}{\cancel{n^3}}}{\frac{-2}{\cancel{n^3}}}$

$= \frac{1}{1+0} = 1$; and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

by p-series test ($p=2>1$) so series $\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n^2}\right)$ converges by limit comparison test

17.) $\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n}+n^{1/3}}$; $0 < \frac{1}{0+n^{1/3}} < \frac{1}{2\sqrt{n}+n^{1/3}}$

and $\sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$ diverges by p-series test ($p=1/3 \le 1$), so $\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n}+n^{1/3}}$

<u>diverges</u> by <u>comparison test</u>.

18.) $\frac{3}{n+\sqrt{n}} \geq \frac{3}{n+n} = \frac{3}{2n} = \frac{3}{2}\cdot\frac{1}{n} > 0$ and

$\sum_{n=1}^{\infty} \frac{3}{2}\cdot\frac{1}{n} = \frac{3}{2}\sum_{n=1}^{\infty}\frac{1}{n}$ diverges by p-series test since $p = 1 \leq 1$; so $\sum_{n=1}^{\infty}\frac{3}{n+\sqrt{n}}$ diverges by comparison test

19.) $0 \leq \frac{\sin^2 n}{2^n} \leq \frac{1}{2^n} = \left(\frac{1}{2}\right)^n$ and

$\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n$ converges by geometric series test; so $\sum_{n=1}^{\infty}\frac{\sin^2 n}{2^n}$ converges by comparison test

20.) $\sum_{n=1}^{\infty}\frac{1+\cos n}{n^2}$; $0 < \frac{1+\cos n}{n^2} \leq \frac{1+1}{n^2} = \frac{2}{n^2}$

and $\sum_{n=1}^{\infty}\frac{2}{n^2}$ converges by p-series test $(p = 2 > 1)$, so $\sum_{n=1}^{\infty}\frac{1+\cos n}{n^2}$ converges by comparison test.

21.) $\sum_{n=1}^{\infty}\frac{2n}{3n-1}$; $\lim_{n\to\infty}\frac{2n}{3n-1}$

$\overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty}\frac{2}{3} = \frac{2}{3} \neq 0$, so

$$\sum_{n=1}^{\infty} \frac{2n}{3n-1}$$ diverges by

nth-term test.

22.) $$\sum_{n=1}^{\infty} \frac{n+1}{n^2 \cdot \sqrt{n}} = \sum_{n=1}^{\infty} \left(\frac{n}{n^2\sqrt{n}} + \frac{1}{n^2\sqrt{n}} \right)$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} + \sum_{n=1}^{\infty} \frac{1}{n^{5/2}}$$; each of these series converges by p-series test since $p = 3/2 > 1$ and $p = \frac{5}{2} > 1$; thus, $\sum_{n=1}^{\infty} \frac{n+1}{n^2\sqrt{n}}$ converges since it is the sum of convergent series

23.) $$\sum_{n=1}^{\infty} \frac{10n+1}{n(n+1)(n+2)}$$; limit compare to $\frac{1}{n^2}$: $$\lim_{n\to\infty} \frac{10n+1}{n(n+1)(n+2)} \cdot \frac{n^2}{1}$$

$$= \lim_{n\to\infty} \frac{10n+1}{n} \cdot \frac{n}{n+1} \cdot \frac{n}{n+2}$$

$$\overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} \frac{10}{1} \cdot \frac{1}{1} \cdot \frac{1}{1} = 10$$; and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by p-series

test $(p=2>1)$, so

$$\sum_{n=1}^{\infty} \frac{10n+1}{n(n+1)(n+2)}$$ converges by limit comparison test

26.) $0 \le \frac{1}{\sqrt{n^3+2}} \le \frac{1}{\sqrt{n^3+0}} = \frac{1}{n^{3/2}}$ and

$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges by p-series test since $p=\frac{3}{2}>1$; so

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+2}}$ converges by comparison test

27.) $\lim_{n\to\infty} \frac{\frac{1}{\ln(\ln n)}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{\ln(\ln n)}$

$\overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} \frac{1}{\frac{1}{\ln n}\cdot\frac{1}{n}} = \lim_{n\to\infty} n \ln n = \infty$;

since $\sum_{n=3}^{\infty} \frac{1}{n}$ diverges by p-series test $(p = 1 \leq 1)$, then $\sum_{n=3}^{\infty} \frac{1}{\ln(\ln n)}$ diverges by limit comparison test

28.) $\lim_{n\to\infty} \frac{\frac{(\ln n)^2}{n^3}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{(\ln n)^2}{n}$

$\overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} \frac{2\cdot\ln n\cdot\frac{1}{n}}{1} = \lim_{n\to\infty} \frac{2\ln n}{n}$

$\overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} \frac{2\cdot\frac{1}{n}}{1} = 0$; since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by p-series test $(p = 2 > 1)$, then $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^3}$ converges by limit comparison test

29.) $$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}\ln n}}{\frac{1}{n^{3/4}}} = \lim_{n\to\infty} \frac{n^{3/4}}{n^{1/2}\ln n}$$

$$= \lim_{n\to\infty} \frac{n^{1/4}}{\ln n} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} \frac{\frac{1}{4} n^{-3/4}}{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{1}{4}\frac{1}{n^{3/4}}\cdot\frac{n}{1} = \lim_{n\to\infty} \frac{1}{4} n^{1/4} = \infty;$$

since $\sum_{n=2}^{\infty} \frac{1}{n^{3/4}}$ diverges by p-series test ($p = \frac{3}{4} \le 1$), then $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}\ln n}$ diverges by limit comparison test.

30.) $$\lim_{n\to\infty} \frac{\frac{(\ln n)^2}{n^{3/2}}}{\frac{1}{n^{5/4}}} = \lim_{n\to\infty} \frac{(\ln n)^2}{n^{1/4}}$$

$$\overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} \frac{2\cdot\ln n\cdot\frac{1}{n}}{\frac{1}{4}n^{-3/4}} = \lim_{n\to\infty} 8\cdot\frac{\ln n}{n^{1/4}}$$

$$\overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} 8\cdot\frac{1/n}{1/4\, n^{-3/4}} = \lim_{n\to\infty} 32\cdot\frac{1}{n^{1/4}} = 0;$$

since $\sum_{n=1}^{\infty} \frac{1}{n^{5/4}}$ converges by p-series test $\left(p = \frac{5}{4} > 1\right)$, then $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^{3/2}}$ converges by the limit comparison test

31.) $\lim_{n\to\infty} \frac{\frac{1}{1+\ln n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{1+\ln n}$

$\overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} \frac{1}{1/n} = \lim_{n\to\infty} n = \infty$;

since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by p-series test $(p = 1 \le 1)$, then $\sum_{n=1}^{\infty} \frac{1}{1+\ln n}$ diverges by limit comparison test

33.) $\lim_{n\to\infty} \frac{\frac{1}{n\sqrt{n^2-1}}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2-1}}$

$= \lim_{n\to\infty} \frac{\sqrt{n^2}}{\sqrt{n^2-1}} = \lim_{n\to\infty} \sqrt{\frac{n^2}{n^2-1}}$

$= \lim_{n\to\infty} \sqrt{\frac{1}{1-\left(\frac{1}{n^2}\right)}} = \sqrt{\frac{1}{1-0}} = 1$;

since $\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges by p-series test $(p = 2 > 1)$, then $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$ converges by limit comparison test

34.) $\lim_{n\to\infty} \dfrac{\frac{\sqrt{n}}{n^2+1}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty} \dfrac{n^2}{n^2+1}$

$= \lim_{n\to\infty} \dfrac{1}{1+\frac{1}{n^2}} = \dfrac{1}{1+0} = 1$; since

$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges by p-series test ($p = \frac{3}{2} > 1$), then $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$ converges by limit comparison test

35.) $\sum_{n=1}^{\infty} \frac{1-n}{n2^n} = \sum_{n=1}^{\infty} \frac{1}{n2^n} - \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$;

$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ converges by geometric series test ; $0 < \frac{1}{n2^n} \le \frac{1}{2^n} = \left(\frac{1}{2}\right)^n$; and so $\sum_{n=1}^{\infty} \frac{1}{n2^n}$ converges by comparison test ; then

$\sum_{n=1}^{\infty} \frac{1-n}{n2^n}$ converges since it is the sum/difference of convergent series.

36.) $\lim_{n\to\infty} \frac{\frac{n+2^n}{n^2 2^n}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n+2^n}{2^n}$

$= \lim_{n\to\infty} \left(\frac{n}{2^n}+1\right) \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} \left(\overset{\nearrow 0}{\frac{1}{2^n \ln 2}}+1\right)$

$= 0+1 = 1$; since $\sum_{n=1}^{\infty} \frac{1}{n^2}$

converges by p-series test ($p = 2 > 1$), ~~then~~ $\sum_{n=1}^{\infty} \frac{n+2^n}{n^2 2^n}$

converges by limit comparison test

37.) $\lim_{n\to\infty} \frac{\frac{1}{3^{n-1}+1}}{\left(\frac{1}{3}\right)^n} = \lim_{n\to\infty} \frac{3^n}{3^{n-1}+1} \cdot \frac{\frac{1}{3^n}}{\frac{1}{3^n}}$

$= \lim_{n\to\infty} \frac{1}{\frac{1}{3}+\frac{1}{3^n}} = \frac{1}{\frac{1}{3}+0} = 3$; since

$\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$ converges by geometric series test ($r = \frac{1}{3}$, $-1 < r < 1$),

~~then~~ $\sum_{n=1}^{\infty} \frac{1}{3^{n-1}+1}$ converges by limit comparison test

38.) $\sum_{n=1}^{\infty} \frac{3^{n-1}+1}{3^n} = \sum_{n=1}^{\infty} \frac{3^{n-1}}{3^n} + \sum_{n=1}^{\infty} \frac{1}{3^n}$

$$= \sum_{n=1}^{\infty} 3^{(n-1)-n} + \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$$

$$= \sum_{n=1}^{\infty} \frac{1}{3} + \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n ;$$

$\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$ converges by geometric series since $r = \frac{1}{3}$, $-1 < r < 1$;

$\sum_{n=1}^{\infty} \frac{1}{3}$ diverges by nth-term test since $\lim_{n \to \infty} \frac{1}{3} = \frac{1}{3} \neq 0$;

then $\sum_{n=1}^{\infty} \frac{3^{n-1}+1}{3^n}$ diverges

since one part converges and the other part diverges

39.) $\sum_{n=1}^{\infty} \frac{n+1}{n^2+3n} \cdot \frac{1}{5n} \longrightarrow$

$$\lim_{n\to\infty} \frac{\frac{n+1}{n^2+3n} \cdot \frac{1}{5n}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n+1}{n^2+3n} \cdot \frac{1}{5\cancel{n}} \cdot \frac{n^{\cancel{2}}}{1}$$

$$= \lim_{n\to\infty} \frac{n^2+n}{5n^2+15n} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}}$$

$$= \lim_{n\to\infty} \frac{1+\frac{1}{n}}{5+\frac{15}{n}} = \frac{1+0}{5+0} = \frac{1}{5};$$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by p-series test $(p=2>1)$, so

$\sum_{n=1}^{\infty} \frac{n+1}{n^2+3n} \cdot \frac{1}{5n}$ <u>converges</u>

by <u>limit comparison</u> test

40.) $\sum_{n=1}^{\infty} \frac{2^n+3^n}{3^n+4^n}$; $0 < \frac{2^n+3^n}{3^n+4^n} \leq \frac{3^n+3^n}{0+4^n} = 2\left(\frac{3}{4}\right)^n$;

$0 < \frac{3}{4} < 1$ so $\sum_{n=1}^{\infty} 2\left(\frac{3}{4}\right)^n$ converges by geom. series test, so $\sum_{n=1}^{\infty} \frac{2^n+3^n}{3^n+4^n}$ converges by comparison test

41.) $\sum_{n=1}^{\infty} \frac{2^n-n}{n2^n} = \sum_{n=1}^{\infty} \left(\frac{2^n}{n2^n} - \frac{n}{n2^n}\right)$

$= \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{2^n}\right) = \sum_{n=1}^{\infty} \frac{1}{n} - \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$

($\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by p-series, $p=1$,

$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ converges by geom. series, $-1 < \frac{1}{2} < 1$), so

$\sum_{n=1}^{\infty} \frac{2^n-n}{n2^n}$ diverges

43.) $\sum_{n=2}^{\infty} \frac{1}{n!}$; $\frac{1}{n!} = \frac{1}{n(n-1)(n-2)\cdots 3\cdot 2\cdot 1} \leq \frac{1}{n(n-1)}$;

$\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges by p-series $(p=2>1)$

so $\sum_{n=2}^{\infty} \frac{1}{n(n-1)}$ converges by limit comp. test since

$$\lim_{n\to\infty} \frac{\frac{1}{n(n-1)}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n(n-1)} = \lim_{n\to\infty} \frac{n}{n-1} = 1;$$

thus $\sum_{n=2}^{\infty} \frac{1}{n!}$ converges by the comparison test.

44.) $\sum_{n=1}^{\infty} \frac{(n-1)!}{(n+2)!}$

$$= \sum_{n=1}^{\infty} \frac{\cancel{(n-1)}\cancel{(n-2)}\cdots \cancel{3}\cdot\cancel{2}\cdot\cancel{1}}{(n+2)(n+1)n\cancel{(n-1)}\cancel{(n-2)}\cdots\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}$$

$$= \sum_{n=1}^{\infty} \frac{1}{(n+2)(n+1)n};$$

$$\lim_{n\to\infty} \frac{\frac{1}{(n+2)(n+1)n}}{1/n^3} = \lim_{n\to\infty} \frac{\cancel{n^3}\, n^2}{(n+2)(n+1)\cancel{n}}$$

$$= \lim_{n\to\infty} \frac{n}{n+2}\cdot\frac{n}{n+1} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} \frac{1}{1}\cdot\frac{1}{1} = 1;$$

$\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges by p-series test $(p = 3 > 1)$, so

$\sum_{n=1}^{\infty} \frac{(n-1)!}{(n+2)!}$ converges by limit comparison test

45.) $\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{n\to\infty} \frac{\cos\left(\frac{1}{n}\right)\cdot \cancel{\frac{-1}{n^2}}}{\cancel{\frac{-1}{n^2}}}$

$= \cos 0 = 1$; since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by p-series test $(p=1\le 1)$, then $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$ diverges by limit comparison test

47.) $\lim_{n\to\infty} \frac{\frac{\arctan n}{n^{1.1}}}{\frac{1}{n^{1.1}}} = \lim_{n\to\infty} \arctan n$

$= \arctan(\infty) = \frac{\pi}{2}$; since $\sum_{n=1}^{\infty} \frac{1}{n^{1.1}}$ converges by p-series test $(p=1.1>1)$, then $\sum_{n=1}^{\infty} \frac{\arctan n}{n^{1.1}}$ converges by limit comparison test

48.) $\sum_{n=1}^{\infty} \frac{\text{arcsec}\, n}{n^{1.3}}$; $\lim_{n\to\infty} \frac{\frac{\text{arcsec}\, n}{n^{1.3}}}{1/n^{1.3}}$

$= \lim_{n\to\infty} \text{arcsec}\, n = \text{"arcsec}\,\infty\text{"} = \frac{\pi}{2}$;

$\sum_{n=1}^{\infty} \frac{1}{n^{1.3}}$ converges by p-series test $(p=1.3>1)$, so

$\sum_{n=1}^{\infty} \frac{\text{arcsec}\, n}{n^{1.3}}$ converges by limit comparison test

51.) $\sum_{n=1}^{\infty} \frac{1}{n \cdot n^{1/n}}$; $\lim_{n\to\infty} n^{1/n} \overset{"\infty^0"}{=} \lim_{n\to\infty} e^{\ln n^{1/n}}$

$= \lim_{n\to\infty} e^{\frac{\ln n}{n}} \overset{"\frac{\infty}{\infty}"}{=} \lim_{n\to\infty} e^{\frac{1/n}{1}} = e^0 = 1$;

$\lim_{n\to\infty} \frac{\frac{1}{n \cdot n^{1/n}}}{1/n} = \lim_{n\to\infty} \frac{\cancel{n}}{\cancel{n} \cdot n^{1/n}} = \frac{1}{1} = 1$

and $\sum_{n=1}^{\infty} \frac{1}{n}$ div. by p-series $(p = 1 \leq 1)$,

so $\sum_{n=1}^{\infty} \frac{1}{n \cdot n^{1/n}}$ div. by lim. comp. test

52.) $\lim_{n\to\infty} \frac{\frac{n^{1/n}}{n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} n^{1/n} = "\infty^0"$ (indeterminate)

$= \lim_{n\to\infty} e^{\ln n^{1/n}} = \lim_{n\to\infty} e^{\frac{1}{n} \ln n}$

$= e^{\lim_{n\to\infty} \frac{\ln n}{n}} \overset{"\frac{\infty}{\infty}"}{=} e^{\lim_{n\to\infty} \frac{\frac{1}{n}}{1}} = e^0 = 1$;

since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by p-series $(p = 2 > 1)$, then $\sum_{n=1}^{\infty} \frac{n^{1/n}}{n^2}$ converges by limit comparison test

54.) $\sum_{n=1}^{\infty} \frac{1}{1^2 + 2^2 + 3^2 + \cdots + n^2}$

$= \sum_{n=1}^{\infty} \frac{1}{\frac{n(n+1)(2n+1)}{6}} = \sum_{n=1}^{\infty} \frac{6}{n(n+1)(2n+1)}$;

$\lim_{n\to\infty} \frac{\frac{6}{n(n+1)(2n+1)}}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{6n^3}{n(n+1)(2n+1)}$

$= \lim_{n\to\infty} 6 \cdot \frac{\cancel{n}}{\cancel{n}} \cdot \frac{n}{n+1} \cdot \frac{n}{2n+1}$

$= \lim_{n\to\infty} 6 \cdot \frac{1}{1+\frac{1}{n}} \cdot \frac{1}{2+\frac{1}{n}} = 6 \cdot \frac{1}{1+0} \cdot \frac{1}{2+0}$

$= \frac{6}{2} = 3$; since $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges by p-series test ($p = 3 > 1$), then $\sum_{n=1}^{\infty} \frac{6}{n(n+1)(2n+1)}$ converges by limit comparison test

56.) If $\sum_{n=1}^{\infty} a_n$ converges and $a_n \geq 0$, then $\sum_{n=1}^{\infty} \frac{a_n}{n}$ converges.

<u>Proof</u>: $0 \leq \frac{a_n}{n} \leq a_n$ and $\sum_{n=1}^{\infty} a_n$ converges, so $\sum_{n=1}^{\infty} \frac{a_n}{n}$ converges by comparison test

57.) on last page.

58.) If $\sum_{n=1}^{\infty} a_n$ converges and $a_n \geq 0$, then $\sum_{n=1}^{\infty} a_n^2$ converges.

<u>Proof</u>: Since $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$ (by contrapositive of nth-term test). Since

$\lim_{n\to\infty} a_n = 0$ there is some integer N so that $0 \le a_n < 1$ for all $n \ge N$. Thus

$0 \le a_n^2 \le a_n$ for all $n \ge N$

and $\sum_{n=N}^{\infty} a_n$ converges, so

$\sum_{n=N}^{\infty} a_n^2$ converges by comparison test. It follows that

$$\sum_{n=1}^{\infty} a_n^2 = \underbrace{\sum_{n=1}^{N-1} a_n^2}_{\text{finite \#}} + \underbrace{\sum_{n=N}^{\infty} a_n^2}_{\text{finite \#}}$$

converges.

60.) $a_n > 0$, $\lim_{n\to\infty} n^2 a_n = 0$; show $\sum_{n=1}^{\infty} a_n < \infty$ (converges):

since $\lim_{n\to\infty} n^2 a_n = 0$ it follows there is # K so that $n^2 a_n < \frac{1}{2}$ for $n \ge K$; then $a_n < \frac{1}{2n^2}$ and $\sum_{n=K}^{\infty} \frac{1}{2n^2}$ converges by p-series test ($p = 2 > 1$), so

$\sum_{n=k}^{\infty} a_n$ converges by comparison test and so does

$$\sum_{n=1}^{\infty} a_n = \sum_{n=k}^{\infty} a_n + (a_1 + a_2 + \cdots + a_{k-1})$$

(added finite # of terms)

57.) assume $a_n > 0$, $b_n > 0$ for $n \geq N$, $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$, and Σa_n converges:

Since $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$, then

$\frac{a_n}{b_n} > 1$ for $n \geq K > N$ for some K.

Then $a_n > b_n > 0$ for $n \geq K$. But $\sum_{n=K}^{\infty} a_n$ converges, so

$\boxed{\sum_{n=K}^{\infty} b_n \text{ converges}}$.