Section 10.5

1.) $\sum_{n=1}^{\infty} \frac{2^n}{n!}$; $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}}$

$= \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} = \lim_{n\to\infty} \frac{2}{n+1} = \frac{2}{\infty} = 0 < 1$,

so series converges by the ratio test

3.) $\sum_{n=1}^{\infty} \frac{(n-1)!}{(n+1)^2}$; $\lim_{n\to\infty} \frac{\frac{((n+1)-1)!}{((n+1)+1)^2}}{\frac{(n-1)!}{(n+1)^2}}$

$= \lim_{n\to\infty} \frac{n!}{(n+2)^2} \cdot \frac{(n+1)^2}{(n-1)!} = \lim_{n\to\infty} n \cdot \left(\frac{n+1}{n+2}\right)^2$

$= \lim_{n\to\infty} n \left(\frac{n+1}{n+2} \cdot \frac{\frac{1}{n}}{\frac{1}{n}}\right)^2 = \lim_{n\to\infty} n \left(\frac{1+\frac{1}{n}}{1+\frac{2}{n}}\right)$

$= \infty \left(\frac{1+0}{1+0}\right) = \infty > 1$, so series diverges

by ratio test

4.) $\sum_{n=1}^{\infty} \frac{2^{n+1}}{n3^{n-1}}$; $\lim_{n\to\infty} \frac{\frac{2^{n+1+1}}{(n+1)3^{n+1-1}}}{\frac{2^{n+1}}{n3^{n-1}}}$

$= \lim_{n\to\infty} \frac{2^{n+2}}{(n+1)3^n} \cdot \frac{n3^{n-1}}{2^{n+1}}$

$= \lim_{n\to\infty} 2 \cdot \frac{n}{n+1} \cdot \frac{1}{3} = \lim_{n\to\infty} \frac{2}{3} \cdot \frac{n}{n+1} \cdot \frac{\frac{1}{n}}{\frac{1}{n}}$

$= \lim_{n\to\infty} \frac{2}{3} \cdot \frac{1}{1+\frac{1}{n}} = \frac{2}{3} \cdot \frac{1}{1+0} = \frac{2}{3} < 1$

so series converges by ratio test

5.) $\sum_{n=1}^{\infty}\left|\frac{n^4}{(-4)^n}\right| = \sum_{n=1}^{\infty}\frac{n^4}{4^n}$; $\lim_{n\to\infty}\frac{\frac{(n+1)^4}{4^{n+1}}}{\frac{n^4}{4^n}} = \lim_{n\to\infty}\frac{(n+1)^4}{4^{n+1}}\cdot\frac{4^n}{n^4}$

$= \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^4\cdot\frac{1}{4} = \lim_{n\to\infty}\frac{1}{4}\left(1+\frac{1}{n}\right)^4$

$= \frac{1}{4}(1+0)^4 = \frac{1}{4} < 1$, so series converges by absolute ratio test

6.) $\sum_{n=2}^{\infty}\frac{3^{n+2}}{\ln n}$;

$$\lim_{n\to\infty}\frac{\frac{3^{(n+1)+2}}{\ln(n+1)}}{\frac{3^{n+2}}{\ln n}} = \lim_{n\to\infty}\frac{3^{n+3}}{\ln(n+1)}\cdot\frac{\ln n}{3^{n+2}}$$

$$= \lim_{n\to\infty} 3\cdot\frac{\ln n}{\ln(n+1)} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} 3\,\frac{1/n}{1/(n+1)}$$

$$= \lim_{n\to\infty} 3\cdot\frac{n+1}{n} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} 3\cdot\frac{1}{1} = 3 > 1,$$

so series diverges by ratio test

7.) $\displaystyle\sum_{n=1}^{\infty} \left| (-1)^n \frac{n^2 (n+2)!}{n!\, 3^{2n}} \right|$

$$= \sum_{n=1}^{\infty} \frac{n^2 (n+2)!}{n!\, 3^{2n}} \; ; \; \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\dfrac{(n+1)^2 (n+3)!}{(n+1)!\, 3^{2(n+1)}}}{\dfrac{n^2 (n+2)!}{n!\, 3^{2n}}}$$

$$= \lim_{n\to\infty} \frac{(n+1)^2 (n+3)!}{(n+1)!\, 3^{2n+2}} \cdot \frac{n!\, 3^{2n}}{n^2 (n+2)!}$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^2 \cdot \frac{1}{3^2} \cdot \frac{n+3}{n+1}$$

$$= \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^2 \cdot \frac{1}{9} \cdot \frac{n+3}{n+1} \cdot \frac{1/n}{1/n}$$

$$= \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^2 \cdot \frac{1}{9} \cdot \frac{1 + 3/n}{1 + 1/n} = \frac{(1+0)^2}{9} \cdot \frac{1+0}{1+0} = \frac{1}{9} < 1,$$

so series converges absolute by ratio test

8.) $\displaystyle\sum_{n=1}^{\infty} \frac{n 5^n}{(2n+3) \ln (n+1)} \; ; \; \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$

$$= \lim_{n\to\infty} \frac{(n+1)5^{n+1}}{(2(n+1)+3)\ln(n+2)} \cdot \frac{(2n+3)\ln(n+1)}{n \cdot 5^n}$$

$$= \lim_{n\to\infty} \frac{n+1}{n} \cdot (5) \cdot \frac{2n+3}{2n+5} \cdot \frac{\ln(n+1)}{\ln(n+2)}$$

use L'Hopital

$$\overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} \frac{1}{1} \cdot (5) \cdot \frac{2}{2} \cdot \frac{\frac{1}{n+1}}{\frac{1}{n+2}} = \lim_{n\to\infty} 5 \cdot \frac{n+2}{n+1}$$

$$\overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} 5 \cdot \frac{1}{1} = 5 > 1,$$ so series diverges by ratio test

9.) $\displaystyle\sum_{n=1}^{\infty} \frac{7}{(2n+5)^n}$; $\displaystyle\lim_{n\to\infty} \left(\frac{7}{(2n+5)^n}\right)^{1/n}$

$$= \lim_{n\to\infty} \frac{7^{1/n}}{2n+5} = \frac{7^0}{\infty} = \frac{1}{\infty} = 0 < 1, \text{ ~~so~~}$$

series converges by root test

11.) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{4n+3}{3n-5}\right)^n$; $\displaystyle\lim_{n\to\infty} \left[\left(\frac{4n+3}{3n-5}\right)^n\right]^{1/n}$

$$= \lim_{n\to\infty} \frac{4n+3}{3n-5} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} \frac{4}{3} = \frac{4}{3} > 1,$$

so series diverges by root test

13.) $\displaystyle\sum_{n=1}^{\infty} \frac{8}{(3+\frac{1}{n})^{2n}}$; $\displaystyle\lim_{n\to\infty} \left[\frac{8}{(3+\frac{1}{n})^{2n}}\right]^{1/n}$

$$= \lim_{n\to\infty} \frac{8^{1/n}}{(3+\frac{1}{n})^2} = \frac{8^0}{(3+0)^2} = \frac{1}{9} < 1, \text{ so}$$

series converges by root test

14.) $\sum_{n=1}^{\infty} \sin^n\left(\frac{1}{\sqrt{n}}\right)$; $\lim_{n\to\infty} \left[\sin^n\left(\frac{1}{\sqrt{n}}\right)\right]^{1/n}$

$= \lim_{n\to\infty} \sin\left(\frac{1}{\sqrt{n}}\right) = \sin(0) = 0 < 1$, so

series converges by root test

15.) $\sum_{n=1}^{\infty} \left(1-\frac{1}{n}\right)^{n^2}$; $\lim_{n\to\infty} \left[\left(1-\frac{1}{n}\right)^{n^2}\right]^{1/n}$

$= \lim_{n\to\infty} \left(1-\frac{1}{n}\right)^{n} = \lim_{n\to\infty} \left[\left(1+\frac{1}{(-n)}\right)^{(-n)}\right]^{(-1)}$

$= e^{-1} = \frac{1}{e} < 1$, so series converges by root test

17.) $\sum_{n=1}^{\infty} \frac{n^{\sqrt{2}}}{2^n}$; $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+1)^{\sqrt{2}}}{2^{n+1}}}{\frac{n^{\sqrt{2}}}{2^n}}$

$= \lim_{n\to\infty} \frac{(n+1)^{\sqrt{2}}}{n^{\sqrt{2}}} \cdot \frac{2^n}{2^{n+1}} = \lim_{n\to\infty} \frac{1}{2} \cdot \left(1+\frac{1}{n}\right)^{\sqrt{2}}$

$= \frac{1}{2} \cdot (1)^{\sqrt{2}} = \frac{1}{2} < 1$, so series converges by ratio test

20.) $\sum_{n=1}^{\infty} \frac{n!}{10^n}$; $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+1)!}{10^{n+1}}}{\frac{n!}{10^n}}$

$= \lim_{n\to\infty} \frac{(n+1)!}{n!} \cdot \frac{10^n}{10^{n+1}} = \lim_{n\to\infty} (n+1) \cdot \frac{1}{10} = \infty > 1$,

so series diverges by ratio test

21.) $\sum_{n=1}^{\infty} \frac{n^{10}}{10^n}$; $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+1)^{10}}{10^{n+1}}}{\frac{n^{10}}{10^n}}$

$= \lim_{n\to\infty} \frac{(n+1)^{10}}{n^{10}} \cdot \frac{10^n}{10^{n+1}} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{10} \cdot \frac{1}{10}$

$= (1)^{10} \cdot \frac{1}{10} = \frac{1}{10} < 1$, so series converges by ratio test.

22.) $\sum_{n=1}^{\infty}\left(\frac{n-2}{n}\right)^n$; $\lim_{n\to\infty}\left(\frac{n-2}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{(-2)}{n}\right)^n$

$= e^{-2} = \frac{1}{e^2} \neq 0$, so series diverges by nth-term test

23.) $\sum_{n=1}^{\infty} \frac{2+(-1)^n}{1.25^n}$; $0 \le \frac{2+(-1)^n}{1.25^n} \le \frac{2+1}{1.25^n} = 3\left(\frac{4}{5}\right)^n$;

$\sum_{n=1}^{\infty} 3\left(\frac{4}{5}\right)^n$ converges by geometric series test ($r=\frac{4}{5}$, $-1<r<1$), so $\sum_{n=1}^{\infty} \frac{2+(-1)^n}{1.25^n}$ converges by comparison test

24.) $\sum_{n=1}^{\infty} \frac{(-2)^n}{3^n} = \sum_{n=1}^{\infty}\left(\frac{-2}{3}\right)^n$; $r = \frac{-2}{3}$,

$-1 < r < 1$, so series converges by geometric series test

26.) $\sum_{n=1}^{\infty}\left(1-\frac{1}{3n}\right)^n$; $\lim_{n\to\infty}\left(1-\frac{1}{3n}\right)^n = \lim_{n\to\infty}\left(1+\frac{(-1/3)}{n}\right)^n$

$= e^{-1/3} = \frac{1}{e^{1/3}} \neq 0$ so series diverges by nth-term test

27.) $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$; $\lim_{n\to\infty} \frac{\frac{\ln n}{n^3}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\ln n}{n}$

$\overset{\frac{\infty}{\infty}}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = 0$; $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by

p-series test ($p = 2 > 1$), so $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$ converges by limit comparison test

28.) $\sum_{n=1}^{\infty} \frac{(\ln n)^n}{n^n}$; $\lim_{n\to\infty}\left[\frac{(\ln n)^n}{n^n}\right]^{1/n} = \lim_{n\to\infty} \frac{\ln n}{n}$

$\stackrel{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = 0 < 1$, so series

converges by root test

29.) $\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n^2}\right) = \sum_{n=1}^{\infty}\frac{1}{n} - \sum_{n=1}^{\infty}\frac{1}{n^2}$

diverges by subtle facts about series, since $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges by p-series test $(p=1\leq 1)$ and $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges by p-series test $(p=2>1)$.

30.) $\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n^2}\right)^n$; $\lim_{n\to\infty}\left[\left(\frac{1}{n} - \frac{1}{n^2}\right)^n\right]^{1/n}$

$= \lim_{n\to\infty}\left(\frac{1}{n} - \frac{1}{n^2}\right) = 0 - 0 = 0 < 1$, so series

converges by root test

31.) $\sum_{n=1}^{\infty} \frac{e^n}{n^e}$;

$$\lim_{n\to\infty} \frac{\frac{e^{n+1}}{(n+1)^e}}{\frac{e^n}{n^e}} = \lim_{n\to\infty} \frac{e^{n+1}}{(n+1)^e} \cdot \frac{n^e}{e^n}$$

$$= \lim_{n\to\infty} e\left(\frac{n}{n+1}\right)^e$$

$$= \lim_{n\to\infty} e\left(\frac{n}{n+1} \cdot \frac{1/n}{1/n}\right)^e$$

$$= \lim_{n\to\infty} e\left(\frac{1}{1+1/n}\right)^e$$

$$= e\left(\frac{1}{1+0}\right)^e = e(1)^e = e(1) = e > 1,$$

so series diverges by ratio test

35.) $\displaystyle\sum_{n=1}^{\infty} \frac{(n+3)!}{3!\, n!\, 3^n}$;

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\dfrac{(n+4)!}{3!(n+1)!\,3^{n+1}}}{\dfrac{(n+3)!}{3!\,n!\,3^n}}$$

$$= \lim_{n\to\infty} \frac{(n+4)!}{(n+3)!} \cdot \cancel{\frac{3!}{3!}} \cdot \frac{n!}{(n+1)!} \cdot \frac{3^n}{3^{n+1}}$$

$$= \lim_{n\to\infty} (n+4) \cdot \frac{1}{(n+1)} \cdot \frac{1}{3}$$

$$= \lim_{n\to\infty} \frac{1+\frac{4}{n}}{1+\frac{1}{n}} \cdot \frac{1}{3} = \frac{1+0}{1+0} \cdot \frac{1}{3} = \frac{1}{3}$$

37.) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{(2n+1)!}$; $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\dfrac{(n+1)!}{(2(n+1)+1)!}}{\dfrac{n!}{(2n+1)!}}$

$$= \lim_{n\to\infty} \frac{(n+1)!}{n!} \cdot \frac{(2n+1)!}{(2n+3)!}$$

$$= \lim_{n\to\infty} (n+1) \cdot \frac{1}{(2n+3)(2n+2)} = \lim_{n\to\infty} \frac{n+1}{4n^2+10n+6}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n}+\frac{1}{n^2}}{4+\frac{10}{n}+\frac{6}{n^2}} = \frac{0+0}{4+0+0} = 0 < 1,$$ so

series converges by ratio test

38.) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$; $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}}$

$= \lim_{n\to\infty} \frac{(n+1)!}{n!} \cdot \frac{n^n}{(n+1)^{n+1}}$

$= \lim_{n\to\infty} \cancel{(n+1)} \cdot \frac{n^n}{\cancel{(n+1)}(n+1)^n} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n$

$= \lim_{n\to\infty} \left(\frac{1}{\frac{n+1}{n}}\right)^n = \lim_{n\to\infty} \frac{1^n}{(1+\frac{1}{n})^n} = \frac{1}{e} < 1,$

so series converges by ratio test

39.) $\sum_{n=2}^{\infty} \frac{n}{(\ln n)^n}$; $\lim_{n\to\infty} \left[\frac{n}{(\ln n)^n}\right]^{1/n}$

$= \lim_{n\to\infty} \frac{n^{1/n}}{\ln n}$ (and $\lim_{n\to\infty} n^{1/n} =$ "∞^0"

(indeterminate) $= \lim_{n\to\infty} e^{\ln n^{1/n}}$

$= \lim_{n\to\infty} e^{\frac{\ln n}{n}} = e^{\lim_{n\to\infty} \frac{\ln n}{n}}$

$\overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} e^{\lim_{n\to\infty} \frac{1/n}{1}} = e^0 = 1.$)

$= \frac{1}{\infty} = 0 < 1$, so series converges by root test

42.) $\sum_{n=1}^{\infty} \left| \frac{(-3)^n}{n^3 2^n} \right| = \sum_{n=1}^{\infty} \frac{3^n}{n^3 2^n}$;

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{3^{n+1}}{(n+1)^3 2^{n+1}} \cdot \frac{n^3 \cdot 2^n}{3^n}$$

$$= \lim_{n\to\infty} \frac{3}{2}\left(\frac{n}{n+1}\right)^3 \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} \frac{3}{2} \cdot \left(\frac{1}{1}\right)^3 = \frac{3}{2} > 1,$$

so series diverges by absolute ratio test

43.) $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$; $\lim_{n\to\infty} \frac{((n+1)!)^2}{(2(n+1))!} \cdot \frac{(2n)!}{(n!)^2}$

$$= \lim_{n\to\infty} \left(\frac{(n+1)!}{n!}\right)^2 \cdot \frac{(2n)!}{(2n+2)!}$$

$$= \lim_{n\to\infty} (n+1)^2 \cdot \frac{1}{(2n+2)(2n+1)}$$

$$= \lim_{n\to\infty} \frac{n^2+2n+1}{4n^2+6n+2} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} \frac{2n+2}{8n+6}$$

$$\overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} \frac{2}{8} = \frac{1}{4} < 1,$$ so series converges by ratio test

44.) $\sum_{n=1}^{\infty} \frac{(2n+3)(2^n+3)}{3^n+2}$; $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$

$$= \lim_{n\to\infty} \frac{(2(n+1)+3)(2^{n+1}+3)}{3^{n+1}+2} \cdot \frac{3^n+2}{(2n+3)(2^n+3)}$$

$$= \lim_{n\to\infty} \frac{2n+5}{2n+3} \cdot \frac{3^n+2}{3^{n+1}+2} \cdot \frac{2^{n+1}+3}{2^n+3}$$

$$= \lim_{n\to\infty} \frac{2+5/n}{2+3/n} \cdot \frac{1+\frac{2}{3}^n}{3+\frac{2}{3}^n} \cdot \frac{2+\frac{3}{2}^n}{1+\frac{3}{2}^n}$$

$$= \frac{2+0}{2+0} \cdot \frac{1+0}{3+0} \cdot \frac{2+0}{1+0} = (1)\left(\frac{1}{3}\right)(2) = \frac{2}{3} < 1,$$

so series converges by ratio test

46.) $a_1 = 1, \quad a_{n+1} = \dfrac{1+\arctan n}{n} \cdot a_n$;

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{1+\arctan n}{n} \cdot \cancel{a_n}}{\cancel{a_n}}$$

$= \dfrac{1+\frac{\pi}{2}}{\infty} = 0 < 1$, so series converges by ratio test

47.) $a_1 = \frac{1}{3}, \quad a_{n+1} = \dfrac{3n-1}{2n+5} \cdot a_n$;

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{3n-1}{2n+5} \cdot \cancel{a_n}}{\cancel{a_n}}$$

$= \lim\limits_{n\to\infty} \dfrac{3 - \frac{1}{n}}{2 + \frac{5}{n}} = \dfrac{3-0}{2+0} = \dfrac{3}{2} > 1$, so series diverges by ratio test.

48.) $a_1 = 3, \quad a_{n+1} = \dfrac{n}{n+1} a_n$;

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{n}{n+1} \cdot \cancel{a_n}}{\cancel{a_n}}$$

$\overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim\limits_{n\to\infty} \dfrac{1}{1} = 1$, so ratio test is inconclusive, so TRY SOMETHING ELSE;

Let's compute the first few terms:

$a_1 = 3 = 6/2, \quad a_2 = \frac{2}{3} a_1 = \frac{2}{3} \cdot \frac{6}{2} = \frac{6}{3},$

$a_3 = \frac{3}{4} \cdot a_2 = \frac{3}{4} \cdot \frac{6}{3} = \frac{6}{4},$

$a_4 = \frac{4}{5} \cdot a_3 = \frac{4}{5} \cdot \frac{6}{4} = \frac{6}{5}, \ldots$, so

$a_n = \frac{6}{n+1}$; then

$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{6}{n+1}$ and $\frac{6}{n+1} > \frac{6}{n+n} = \frac{6}{2n} = \frac{3}{n}$,

where $\sum_{n=1}^{\infty} \frac{3}{n} = 3 \sum_{n=1}^{\infty} \frac{1}{n}$ diverges

by p-series test $(p = 1 \leq 1)$, so

series $\sum_{n=1}^{\infty} \frac{6}{n+1}$ diverges by comparison test

51.) $a_1 = 1$, $a_{n+1} = \frac{1 + \ln n}{n} a_n$;

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{\frac{1+\ln n}{n} \cancel{a_n}}{\cancel{a_n}}$$

$$\overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n \to \infty} \frac{\frac{1}{n}}{1} = 0 < 1,$$ so series

converges by the ratio test

52.) $a_1 = \frac{1}{2}$, $a_{n+1} = \frac{n + \ln n}{n + 10} a_n$;

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{\frac{n+\ln n}{n+10} \cdot \cancel{a_n}}{\cancel{a_n}}$$

$$\overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n \to \infty} \frac{1 + 1/n}{1} = \frac{1+0}{1} = 1,$$ so ratio

test is inconclusive,

so TRY SOMETHING ELSE ;

$a_1 = \frac{1}{2}$ and $\frac{n + \ln n}{n + 10} > \frac{n+10}{n+10} = 1$

for $n > e^{10}$ (try it!), then

$a_{n+1} = \frac{n + \ln n}{n+10} a_n > (1) a_n$ for $n > e^{10}$;

let K be the smallest integer $> e^{10}$, then we can say

$a_{K+1} > a_K$ (a nonzero constant),

$a_{K+2} > a_{K+1} > a_K$,

$a_{K+3} > a_{K+2} > a_K$,

$a_{K+4} > a_{K+3} > a_K$; now we can say that

$\sum_{n=K}^{\infty} a_n \geq \sum_{n=K}^{\infty} a_K$ and $\lim_{n\to\infty} a_K = a_K \neq 0$

so $\sum_{n=K}^{\infty} a_K$ diverges by Nth-term test, so series

$\sum_{n=K}^{\infty} a_n$ diverges by comparison test,

so $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{K-1} a_n + \sum_{n=K}^{\infty} a_n$ diverges.

54.) $a_1 = \frac{1}{2}$, $a_{n+1} = (a_n)^{n+1}$;

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(a_n)^{n+1}}{a_n}$$

$= \lim_{n\to\infty} (a_n)^n = ?!$ ☆ (we don't know!),

so TRY SOMETHING ELSE;

$a_1 = \frac{1}{2}$, $a_2 = a_1^2 = \left(\frac{1}{2}\right)^2$,

$a_3 = (a_2)^3 = \left(\left(\frac{1}{2}\right)^2\right)^3 = \left(\frac{1}{2}\right)^6 < \left(\frac{1}{2}\right)^3$,

$a_4 = (a_3)^4 = \left(\left(\frac{1}{2}\right)^6\right)^4 = \left(\frac{1}{2}\right)^{24} < \left(\frac{1}{2}\right)^4$,

$a_5 = (a_4)^5 = \left(\left(\frac{1}{2}\right)^{24}\right)^5 = \left(\frac{1}{2}\right)^{120} < \left(\frac{1}{2}\right)^5$,

so $a_n \leq \left(\frac{1}{2}\right)^n$ for $n = 1, 2, 3, \ldots$; then

$\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$, which converges

by geometric series test ($r = 1/2$, $-1 < r < 1$), so $\sum_{n=1}^{\infty} a_n$ converges

by comparison test

55.) $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \dfrac{\dfrac{2^{n+1}\cdot(n+1)!\cdot(n+1)!}{(2(n+1))!}}{\dfrac{2^n\cdot n!\cdot n!}{(2n)!}}$

$= \lim_{n\to\infty} \frac{2^{n+1}}{2^n}\cdot\frac{(n+1)!}{n!}\cdot\frac{(n+1)!}{n!}\cdot\frac{(2n)!}{(2n+2)!}$

$= \lim_{n\to\infty} 2\cdot(n+1)\cdot(n+1)\cdot\frac{1}{(2n+2)(2n+1)}$

$= \lim_{n\to\infty} \frac{2\,(n^2+2n+1)}{4n^2+6n+2} = \lim_{n\to\infty} \frac{\cancel{2}}{\cancel{2}}\cdot\frac{n^2+2n+1}{2n^2+3n+1}$

$= \lim_{n\to\infty} \frac{1+\frac{2}{n}+\frac{1}{n^2}}{2+\frac{3}{n}+\frac{1}{n^2}} = \frac{1+0+0}{2+0+0} = \frac{1}{2} < 1,$

so series converges by ratio test

57.) $\sum_{n=1}^{\infty} \frac{(n!)^n}{(n^n)^2} = \sum_{n=1}^{\infty} \frac{(n!)^n}{n^{2n}}$; apply ratio test:

(*) $\lim_{n\to\infty} \left[\frac{(n!)^n}{n^{2n}}\right]^{1/n} = \lim_{n\to\infty} \frac{n!}{n^2} = \boxed{?}$;

this is a sneaky twist: consider the series $\sum_{n=1}^{\infty} \frac{n!}{n^2}$; apply ratio test:

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^2}\cdot\frac{n^2}{n!} = \lim_{n\to\infty} (n+1)\left(\frac{n}{n+1}\right)^2$

$= \lim_{n\to\infty} (n+1)\left(\frac{1}{1+1/n}\right)^2 = \infty\,(1)^2 = \infty > 1,$

thus, $\lim_{n\to\infty} \frac{n!}{n^2} = 0$ (by contrapositive to nth-term test) ; so

(*) $\lim_{n\to\infty}\left[\frac{(n!)^n}{n^{2n}}\right]^{1/n} = \lim_{n\to\infty}\frac{n!}{n^2} = \boxed{0} < 1$,

so series $\sum_{n=1}^{\infty}\frac{(n!)^n}{(n^n)^2}$ converges by the root test

58.) $\sum_{n=1}^{\infty}\frac{(n!)^n}{n^{(n^2)}}$; apply root test :

(*) $\lim_{n\to\infty}\left[\frac{(n!)^n}{n^{(n^2)}}\right]^{1/n} = \lim_{n\to\infty}\frac{n!}{n^n} = \boxed{?}$;

this is a sneaky twist :
consider the series $\sum_{n=1}^{\infty}\frac{n!}{n^n}$;

apply ratio test : (SEE problem 38.)

$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = \cdots = \frac{1}{e} < 1$

so series $\sum_{n=1}^{\infty}\frac{n!}{n^n}$ converges ;

thus, $\lim_{n\to\infty} \frac{n!}{n^n} = 0$ (by contrapositive to nth-term test);

(*) so $\lim_{n\to\infty} \left[\frac{(n!)^n}{n^{(n^2)}}\right]^{1/n} = \lim_{n\to\infty} \frac{n!}{n^n} = \boxed{0} < 1$,

so series $\sum_{n=1}^{\infty} \frac{(n!)^n}{n^{(n^2)}}$ converges by root test

60.) on next page

61.) $\sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{4^n 2^n n!}$;

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\dfrac{1\cdot 3\cdot 5\cdots(2n-1)\cdot(2(n+1)-1)}{4^{n+1}2^{n+1}(n+1)!}}{\dfrac{1\cdot 3\cdot 5\cdots(2n-1)}{4^n 2^n n!}}$$

$$= \lim_{n\to\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)\cdot(2n+1)}{1\cdot 3\cdot 5\cdots(2n-1)} \cdot \frac{4^n}{4^{n+1}} \cdot \frac{2^n}{2^{n+1}} \cdot \frac{n!}{(n+1)!}$$

$$= \lim_{n\to\infty} (2n+1)\cdot\frac{1}{4}\cdot\frac{1}{2}\cdot\frac{1}{n+1}$$

$$= \lim_{n\to\infty} \frac{1}{8}\cdot\frac{2n+1}{n+1} = \lim_{n\to\infty} \frac{1}{8}\cdot\frac{2+\frac{1}{n}}{1+\frac{1}{n}}$$

$= \frac{1}{8}\cdot\frac{2+0}{1+0} = \frac{1}{4} < 1$, so series converges by ratio test

60.) $\sum_{n=1}^{\infty} \frac{n^n}{(2^n)^2} = \sum_{n=1}^{\infty} \frac{n^n}{2^{2n}}$;

$\lim_{n\to\infty} a_n^{1/n} = \lim_{n\to\infty} \left(\frac{n^n}{2^{2n}}\right)^{1/n} = \lim_{n\to\infty} \frac{n}{2^2}$

$= \infty > 1$, so series diverges by root test

66.) $\sum_{n=1}^{\infty} \frac{2^{n^2}}{n!}$; $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$

$= \lim_{n\to\infty} \frac{(2^{n+1})^{n+1}}{(n+1)!} \cdot \frac{n!}{(2^n)^n}$

$= \lim_{n\to\infty} \frac{2^{n^2+2n+1}}{2^{n^2}} \cdot \frac{1}{n+1}$

$= \lim_{n\to\infty} \frac{2^{2n+1}}{n+1}$

$\overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} \frac{2^{2n+1} \cdot \ln 2 \cdot 2}{1} = \infty > 1$,

so series diverges by ratio test.