Section 10.7

1.) a.) $\sum_{n=0}^{\infty} x^n$; $\lim_{n\to\infty} \frac{|x|^{n+1}}{|x|^n} = \lim_{n\to\infty} |x| = |x| < 1$

$\to$ $\underline{-1 < x < 1}$; $\underline{\text{check } x=1}$: $\sum_{n=0}^{\infty} 1^n = \sum_{n=0}^{\infty} 1$
diverges by nth-term test since
$\lim_{n\to\infty} 1 = 1 \neq 0$; $\underline{\text{check } x=-1}$: $\sum_{n=0}^{\infty} (-1)^n$

diverges by nth-term test since
$\lim_{n\to\infty} (-1)^n \neq 0$; so interval of
convergence is $\boxed{-1 < x < 1}$.

4.) a.) $\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n}$; $\lim_{n\to\infty} \frac{|3x-2|^{n+1}}{n+1} \cdot \frac{n}{|3x-2|^n}$

$= \lim_{n\to\infty} |3x-2| \cdot \frac{n}{n+1} = |3x-2| \cdot (1)$

$= |3x-2| < 1 \longrightarrow -1 < 3x-2 < 1 \longrightarrow$
$1 < 3x < 3 \to$ $\underline{1/3 < x < 1}$; $\underline{\text{check } x=1}$:
$\sum_{n=1}^{\infty} \frac{1^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges by p-series

test $(p = 1 \leq 1)$; $\underline{\text{check } x = 1/3}$:
$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges by alternating series test since
$a_n = \frac{1}{n}$ is $+$, $\downarrow$, and $\lim_{n\to\infty} \frac{1}{n} = 0$;
so interval of convergence is
$\boxed{\frac{1}{3} \leq x < 1}$.

6.) a.) $\sum_{n=0}^{\infty} (2x)^n$; $\lim_{n\to\infty} \frac{|2x|^{n+1}}{|2x|^n}$

$= \lim_{n\to\infty} |2x| = |2x| < 1 \to -1 < 2x < 1 \to$

$\underline{\frac{-1}{2} < x < \frac{1}{2}}$; $\underline{\text{check } x = 1/2}$: $\sum_{n=0}^{\infty} 1^n = \sum_{n=0}^{\infty} 1$

diverges by nth-term test

since $\lim_{n\to\infty} 1 = 1 \neq 0$; $\underline{\text{check } x = -1/2}$:

$\sum_{n=0}^{\infty} (-1)^n$ diverges by nth-term

test since $\lim_{n\to\infty} (-1)^n \neq 0$; so

interval of convergence is

$\boxed{\frac{-1}{2} < x < \frac{1}{2}}$.

7.) a.) $\sum_{n=0}^{\infty} \frac{nx^n}{n+2}$; $\lim_{n\to\infty} \frac{(n+1)|x|^{n+1}}{n+3} \cdot \frac{n+2}{n|x|^n}$

$= \lim_{n\to\infty} |x| \cdot \frac{n+1}{n+3} \cdot \frac{n+2}{n} = |x| \cdot (1) \cdot (1) = |x| < 1$

$\rightarrow \underline{-1 < x < 1}$; $\underline{\text{check } x = 1}$: $\sum_{n=0}^{\infty} \frac{n \cdot 1^n}{n+2}$

$= \sum_{n=0}^{\infty} \frac{n}{n+2}$ diverges by nth-term

test since $\lim_{n\to\infty} \frac{n}{n+2} = 1 \neq 0$;

$\underline{\text{check } x = -1}$: $\sum_{n=0}^{\infty} \frac{n(-1)^n}{n+2}$ diverges

by the nth-term test since

$\lim_{n\to\infty} \frac{n}{n+2} = 1$ and so $\lim_{n\to\infty} (-1)^n \cdot \frac{n}{n+2} \neq 0$;

so interval of convergence is

$\boxed{-1 < x < 1}$.

10.) a.) $\sum_{n=1}^{\infty} \frac{(x-1)^n}{\sqrt{n}}$; $\lim_{n\to\infty} \frac{|x-1|^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{|x-1|^n}$

$= \lim_{n\to\infty} |x-1| \cdot \sqrt{\frac{n}{n+1}} = |x-1| \cdot \sqrt{1} = |x-1| < 1$

$\to -1 < x-1 < 1 \to \underline{0 < x < 2}$; <u>check $x=0$</u>:

$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges by the alternating series test since $a_n = \frac{1}{\sqrt{n}}$ is $+$, $\downarrow$, and $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$; <u>check $x=2$</u>:

$\sum_{n=1}^{\infty} \frac{1^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges by the p-series test ($p = \frac{1}{2} \le 1$); so interval of convergence is

$\boxed{0 \le x < 2}$

12.) a.) $\sum_{n=0}^{\infty} \frac{3^n x^n}{n!}$; $\lim_{n\to\infty} \frac{3^{n+1}|x|^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n|x|^n}$

$= \lim_{n\to\infty} 3 \cdot \frac{1}{n+1} |x| = 3 \cdot (0) \cdot |x| = 0 < 1$

for all x-values, so interval of convergence is $\boxed{-\infty < x < \infty}$.

17.) a.) $\sum_{n=0}^{\infty} \frac{n(x+3)^n}{5^n}$; $\lim_{n\to\infty} \frac{(n+1)|x+3|^{n+1}}{5^{n+1}} \cdot \frac{5^n}{n|x+3|^n}$

$= \lim_{n\to\infty} \frac{1}{5} \cdot \left(1 + \frac{1}{n}\right)|x+3| = \frac{1}{5} \cdot (1) \cdot |x+3|$

$= \frac{1}{5}|x+3| < 1 \to |x+3| < 5 \to -5 < x+3 < 5$

$\to \underline{-8 < x < 2}$; <u>check $x=2$</u>:

$\sum_{n=0}^{\infty} n \cdot \frac{\cancel{5^n}}{\cancel{5^n}} = \sum_{n=0}^{\infty} n$ diverges by nth-term test since $\lim_{n\to\infty} n = \infty \neq 0$;

check $x=-8$: $\sum_{n=0}^{\infty} n \cdot \frac{(-5)^n}{5^n} = \sum_{n=0}^{\infty} (-1)^n \cdot n$

diverges by the nth-term test since $\lim_{n\to\infty} (-1)^n \cdot n \neq 0$; so interval of convergence is $\boxed{-8 < x < 2}$.

24.) a.) $\sum_{n=1}^{\infty} (\ln n) x^n$; $\lim_{n\to\infty} \frac{\ln(n+1) \cdot |x|^{n+1}}{\ln n \cdot |x|^n}$

$= \lim_{n\to\infty} |x| \cdot \frac{\ln n+1}{\ln n} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} |x| \cdot \frac{\frac{1}{n+1}}{\frac{1}{n}}$

$= \lim_{n\to\infty} |x| \cdot \frac{n}{n+1} = |x| \cdot (1) = |x| < 1 \longrightarrow$

$-1 < x < 1$; check $x=1$: $\sum_{n=1}^{\infty} (\ln n) \cdot 1^n$

$= \sum_{n=1}^{\infty} (\ln n)$ diverges by nth-term test since $\lim_{n\to\infty} \ln n = \infty \neq 0$;

check $x=-1$: $\sum_{n=1}^{\infty} (\ln n) \cdot (-1)^n$ diverges by nth-term test since $\lim_{n\to\infty} \ln n = \infty$, so $\lim_{n\to\infty} (-1)^n \cdot \ln n \neq 0$;

so interval of convergence is $\boxed{-1 < x < 1}$.

25.) a.) $\sum_{n=1}^{\infty} n^n x^n$; $\lim_{n\to\infty} \frac{(n+1)^{n+1} \cdot |x|^{n+1}}{n^n \cdot |x|^n}$

$= \lim_{n\to\infty} (n+1) \cdot \frac{(n+1)^n}{n^n} \cdot |x|$

$= \lim_{n\to\infty} (n+1) \cdot \left(1+\frac{1}{n}\right)^n \cdot |x| = \begin{cases} \infty(1)\cdot|x| = \infty, \text{ if } x \neq 0 \\ 0<1 \text{ if } x=0 \end{cases}$;

so interval of convergence is $\boxed{x=0}$.

27.) a.) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x+2)^n}{n 2^n}$;

$\lim_{n\to\infty} \frac{|x+2|^{n+1}}{(n+1)\cdot 2^{n+1}} \cdot \frac{n \cdot 2^n}{|x+2|^n} = \lim_{n\to\infty} \frac{n}{n+1} \cdot \frac{1}{2} \cdot |x+2|$

$= (1) \cdot \frac{1}{2} \cdot |x+2| = \frac{1}{2}|x+2| < 1 \longrightarrow$

$|x+2| < 2 \longrightarrow -2 < x+2 < 2 \longrightarrow$

$\underline{-4 < x < 0}$; $\underline{\text{check } x=0}$: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cdot \cancel{2^n}}{n \cdot \cancel{2^n}}$

$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges by the alternating series test since $a_n = \frac{1}{n}$ is +, ↓, and $\lim_{n\to\infty} \frac{1}{n} = 0$; $\underline{\text{check } x=-4}$:

$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(-2)^n}{n 2^n} = \sum_{n=1}^{\infty} (-1)^{n+1} \cdot (-1)^n \cdot \frac{1}{n}$

$= \sum_{n=1}^{\infty} (-1)\cdot(-1)^n (-1)^n \cdot \frac{1}{n} = \sum_{n=1}^{\infty} (-1)(-1)^{2n} \cdot \frac{1}{n}$

$= \sum_{n=1}^{\infty} (-1) \underbrace{((-1)^2)}_{1}{}^n \cdot \frac{1}{n} = -\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by p-series test $(p = 1 \le 1)$; so interval of convergence is $\boxed{-4 < x \le 0}$.

29.) a.) $\displaystyle\sum_{n=2}^{\infty} \frac{x^n}{n(\ln n)^2}$; $\displaystyle\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)(\ln(n+1))^2} \cdot \frac{n(\ln n)^2}{|x|^n}$

$= \displaystyle\lim_{n\to\infty} |x| \cdot \frac{n}{n+1} \cdot \left(\frac{\ln n}{\ln(n+1)}\right)^2$

$\overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \displaystyle\lim_{n\to\infty} |x| \cdot \frac{1}{1} \cdot \left(\frac{\frac{1}{n}}{\frac{1}{n+1}}\right)^2 = \lim_{n\to\infty} |x| \cdot \left(\frac{n+1}{n}\right)^2$

$= |x| \cdot (1)^2 = |x| < 1 \longrightarrow \underline{-1 < x < 1}$;

<u>check $x = 1$</u> : $\displaystyle\sum_{n=2}^{\infty} \frac{1^n}{n(\ln n)^2} = \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$;

$f(x) = \dfrac{1}{x(\ln x)^2}$ is $+$, $\downarrow$, and continuous for $x \ge 2$ and

$\displaystyle\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx = \lim_{A\to\infty} \frac{-1}{\ln x}\Big|_2^A$

$= \displaystyle\lim_{x\to\infty} \left(\frac{-1}{\ln A} - \frac{-1}{\ln 2}\right) = \frac{1}{\ln 2}$ (converges)

so series converges by integral test;

<u>check $x = -1$</u> : $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)^2}$ converges

by the alternating series test since $a_n = \dfrac{1}{n(\ln n)^2}$ is $+$, $\downarrow$, and

$\displaystyle\lim_{n\to\infty} \frac{1}{n(\ln n)^2} = 0$; so interval of convergence is $\boxed{-1 \le x \le 1}$.

33.) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2\cdot 4\cdot 8 \cdots (2n)} x^n$; $\displaystyle\lim_{n\to\infty} \frac{\frac{1}{2\cdot 4\cdot 8\cdots(2n)(2(n+1))}|x|^{n+1}}{\frac{1}{2\cdot 4\cdot 8\cdots(2n)}|x|^n}$

$$= \lim_{n\to\infty} \frac{\cancel{2}\cdot\cancel{4}\cdot\cancel{8}\cdots(2\cancel{n})}{\cancel{2}\cdot\cancel{4}\cdot 8\cdots(2\cancel{n})(2n+2)}|x|$$

$$= \lim_{n\to\infty} \frac{1}{2n+2}|x| = \frac{1}{\infty}|x| = (0)|x| = 0 < 1$$

for all x-values so interval of convergence is $-\infty < x < \infty$.

34.) $\displaystyle\sum_{n=1}^{\infty} \frac{3\cdot 5\cdot 7\cdots(2n+1)}{n^2 2^n} x^{n+1}$;

$$\lim_{n\to\infty} \frac{\cancel{3\cdot 5\cdot 7\cdots(2n+1)}(2(n+1)+1)|x|^{n+2}}{(n+1)^2 2^{n+1}} \cdot \frac{n^2 2^n}{\cancel{3\cdot 5\cdot 7\cdots(2n+1)}|x|^{n+1}}$$

$$= \lim_{n\to\infty} \frac{1}{2}\left(\frac{n}{n+1}\right)^2 (2n+3)|x|$$

$$= \begin{cases} \frac{1}{2}(1)^2(\infty)|x| = \infty, & \text{if } x \neq 0 \\ 0 < 1, & \text{if } x = 0 \end{cases}$$, so

interval of convergence is $x = 0$.

35.) $\displaystyle\sum_{n=1}^{\infty} \frac{1+2+3+\cdots+n}{1^2+2^2+3^2+\cdots+n^2} x^n$

$$= \sum_{n=1}^{\infty} \frac{\frac{1}{2}\cancel{n}(\cancel{n+1})}{\frac{1}{6}\cancel{n}(\cancel{n+1})(2n+1)} x^n = \sum_{n=1}^{\infty} \frac{3}{2n+1} x^n ;$$

$$\lim_{n\to\infty} \frac{\cancel{3}\cdot|x|^{n+1}}{2(n+1)+1} \cdot \frac{2n+1}{\cancel{3}|x|^n}$$

$= \lim_{n\to\infty} \frac{2n+1}{2n+3}|x| \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} \frac{2}{2}|x| = |x| < 1$

$\rightarrow -1 < x < 1$; <u>check $x=1$</u> : $\sum_{n=1}^{\infty} \frac{3}{2n+1}(1)^n$

$= \sum_{n=1}^{\infty} \frac{3}{2n+1}$; $\lim_{n\to\infty} \frac{\frac{3}{2n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{3n}{2n+1}$

$\overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} \frac{3}{2} = \frac{3}{2}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

by p-series test $(p=1\le 1)$, so series $\sum_{n=1}^{\infty} \frac{3}{2n+1}$ diverges by limit comparison test ;

<u>check $x=-1$</u> : $\sum_{n=1}^{\infty} \frac{3}{2n+1}(-1)^n$; let $a_n = \frac{3}{2n+1}$

then a_n is $+$, $\downarrow$, and $\lim_{n\to\infty} \frac{3}{2n+1} = 0$, so series converges by alternating series test ; interval of convergence is $-1 \le x < 1$.

36.) $\sum_{n=1}^{\infty} (\sqrt{n+1} - \sqrt{n})(x-3)^n$;

$\lim_{n\to\infty} \frac{(\sqrt{(n+1)+1} - \sqrt{n+1})|x-3|^{n+1}}{(\sqrt{n+1} - \sqrt{n})|x-3|^n}$

$= \lim_{n\to\infty} \frac{\sqrt{n+2} - \sqrt{n+1}}{\sqrt{n+1} - \sqrt{n}} \cdot \frac{\sqrt{n+2} + \sqrt{n+1}}{\sqrt{n+2} + \sqrt{n+1}} \cdot |x-3|$

$= \lim_{n\to\infty} \frac{(n+2) - (n+1)}{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+2} + \sqrt{n+1})} \cdot |x-3|$

$$= \lim_{n\to\infty} \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} \cdot \frac{1}{(\sqrt{n+1}-\sqrt{n})(\sqrt{n+2}+\sqrt{n+1})} |x-3|$$

$$= \lim_{n\to\infty} \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+2}+\sqrt{n+1}} \cdot \frac{\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n}}} \cdot |x-3|$$

$$= \lim_{n\to\infty} \frac{\sqrt{1+1/n}+1}{\sqrt{1+2/n}+\sqrt{1+1/n}} \cdot |x-3| = \frac{1+1}{1+1}|x-3| = |x-3| < 1$$

$\rightarrow -1 < x-3 < 1 \rightarrow 2 < x < 4$;

<u>check $x=2$</u>: $\sum_{n=1}^{\infty} (\sqrt{n+1}-\sqrt{n})(-1)^n$

$$= \sum_{n=1}^{\infty} (\sqrt{n+1}-\sqrt{n}) \cdot \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} \cdot (-1)^n$$

$$= \sum_{n=1}^{\infty} \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} (-1)^n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}} (-1)^n$$

<u>converges</u> by AST since

$a_n = \frac{1}{\sqrt{n+1}+\sqrt{n}}$ is $+$, $\downarrow$, and $\lim_{n\to\infty} a_n = 0$

<u>check $x=4$</u>: $\sum_{n=1}^{\infty} (\sqrt{n+1}-\sqrt{n})(1)^n$

$$= \sum_{n=1}^{\infty} (\sqrt{n+1}-\sqrt{n}) \quad ;$$

$s_1 = \sqrt{2}-1$,

$s_2 = (\sqrt{3}-\cancel{\sqrt{2}}) + (\cancel{\sqrt{2}}-1) = \sqrt{3}-1$,

$s_3 = (\sqrt{4}-\cancel{\sqrt{3}}) + (\cancel{\sqrt{3}}-\cancel{\sqrt{2}}) + (\cancel{\sqrt{2}}-1) = \sqrt{4}-1$

$\vdots$

$s_n = \sqrt{n+1} - 1$ and

$\lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} (\sqrt{n+1} - 1) = \infty$,

so series diverges by sequence of partial sums test; the interval of convergence is

$2 \le x < 4$.

38.) $\sum\limits_{n=1}^{\infty} \left[\frac{2\cdot4\cdot6\cdots(2n)}{2\cdot5\cdot8\cdots(3n-1)}\right]^2 x^n$;

$$\lim_{n\to\infty} \frac{\left[\frac{2\cdot4\cdot6\cdots(2n)(2(n+1))}{2\cdot5\cdot8\cdots(3n-1)(3(n+1)-1)}\right]^2 |x|^{n+1}}{\left[\frac{2\cdot4\cdot6\cdots(2n)}{2\cdot5\cdot8\cdots(3n-1)}\right]^2 |x|^n}$$

$$= \lim_{n\to\infty} \left[\frac{2\cdot4\cdot6\cdots(2n)(2n+2)}{2\cdot4\cdot6\cdots(2n)} \cdot \frac{2\cdot5\cdot8\cdots(3n-1)}{2\cdot5\cdot8\cdots(3n-1)(3n+2)}\right]^2 |x|$$

$$= \lim_{n\to\infty} \left[\frac{2n+2}{3n+2}\right]^2 |x| = \left(\frac{2}{3}\right)^2 |x| = \frac{4}{9}|x| < 1$$

$\to |x| < \frac{9}{4}$, so radius of convergence is $R = 9/4$.

39.) $\sum\limits_{n=1}^{\infty} \frac{(n!)^2}{2^n(2n)!} x^n$;

$$\lim_{n\to\infty} \frac{((n+1)!)^2 |x|^{n+1}}{2^{n+1}(2(n+1))!} \cdot \frac{2^n \cdot (2n)!}{(n!)^2 |x|^n}$$

$$= \lim_{n\to\infty} \frac{1}{2} \cdot \frac{(2n)!}{(2n+2)!} \cdot \left[\frac{(n+1)!}{n!}\right]^2 \cdot |x|$$

$$= \lim_{n\to\infty} \frac{1}{2} \cdot \frac{1}{(2n+2)(2n+1)} \cdot (n+1)^2 \cdot |x|$$

$$= \lim_{n\to\infty} \frac{1}{2} \cdot \frac{n^2+2n+1}{4n^2+6n+2} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}} \cdot |x|$$

$$= \lim_{n\to\infty} \frac{1}{2} \cdot \frac{1+2/n+1/n^2}{4+6/n+2/n^2} \cdot |x|$$

$$= \frac{1}{2} \cdot \frac{1+0+0}{4+0+0} \cdot |x| = \frac{1}{8}|x| < 1 \rightarrow |x| < 8,$$

so radius of convergence is $R=8$.

40.) $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2} x^n$ (use absolute root test)

$$\lim_{n\to\infty} \left[\left(\frac{n}{n+1}\right)^{n^2} |x|^n\right]^{1/n} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n |x|$$

$$= \lim_{n\to\infty} \left(\frac{n+1-1}{n+1}\right)^n |x| = \lim_{n\to\infty} \left[\left(1+\frac{1}{-(n+1)}\right)^{-(n+1)}\right]^{\frac{n}{-(n+1)}} |x|$$

$$= [e]^{-1}|x| < 1 \longrightarrow |x| < e,$$ so radius

of convergence is $R=e$.

41.) $\sum_{n=0}^{\infty} 3^n x^n = \sum_{n=0}^{\infty} (3x)^n$;

$\lim_{n\to\infty} \frac{|3x|^{n+1}}{|3x|^n} = \lim_{n\to\infty} |3x| = |3x| < 1$

$\to -1 < 3x < 1 \to \frac{-1}{3} < x < \frac{1}{3}$;

if $x = \frac{1}{3}$: $\sum_{n=0}^{\infty} 1^n = \sum_{n=0}^{\infty} 1$ clearly diverges;

if $x = \frac{-1}{3}$: $\sum_{n=0}^{\infty} (-1)^n$ diverges by nth-term test since $\lim_{n\to\infty} (-1)^n \neq 0$; interval of convergence is $\boxed{\frac{-1}{3} < x < \frac{1}{3}}$;

$\sum_{n=0}^{\infty} (3x)^n = 1 + (3x) + (3x)^2 + (3x)^3 + \cdots$

$= \frac{1}{1-(3x)}$.

42.) $\sum_{n=0}^{\infty} (e^x - 4)^n$; $\lim_{n\to\infty} \frac{|e^x - 4|^{n+1}}{|e^x - 4|^n}$

$= \lim_{n\to\infty} |e^x - 4| = |e^x - 4| < 1 \to$

$-1 < e^x - 4 < 1 \to 3 < e^x < 5 \to$

$\ln 3 < \ln e^x < \ln 5 \to \ln 3 < x < \ln 5$;

if $x = \ln 3$: $\sum_{n=0}^{\infty} (e^{\ln 3} - 4)^n = \sum_{n=0}^{\infty} (3-4)^n = \sum_{n=0}^{\infty} (-1)^n$

diverges by nth-term test since

$\lim_{n\to\infty} (-1)^n \neq 0$;

if $x = \ln 5$: $\sum_{n=0}^{\infty} (e^{\ln 5} - 4)^n = \sum_{n=0}^{\infty} (5-4)^n = \sum_{n=0}^{\infty} 1^n$

$= \sum_{n=0}^{\infty} 1$ clearly diverges ; interval of convergence is $\boxed{\ln 3 < x < \ln 5}$;

$$\sum_{n=0}^{\infty} (e^x - 4)^n = 1 + (e^x - 4) + (e^x - 4)^2 + (e^x - 4)^3 + \cdots$$

$$= \frac{1}{1-(e^x-4)} = \frac{1}{5-e^x}$$

44.) $\sum_{n=0}^{\infty} \frac{(x+1)^{2n}}{9^n}$; $\lim_{n\to\infty} \frac{|x+1|^{2(n+1)}}{9^{n+1}} \cdot \frac{9^n}{|x+1|^{2n}}$

$= \lim_{n\to\infty} \frac{1}{9}|x+1|^2 = \frac{1}{9}|x+1|^2 < 1 \longrightarrow$

$|x+1|^2 < 9 \longrightarrow |x+1| < 3 \longrightarrow$

$-3 < x+1 < 3 \longrightarrow \underline{-4 < x < 2}$;

<u>check $x=2$</u>: $\sum_{n=0}^{\infty} \frac{3^{2n}}{9^n} = \sum_{n=0}^{\infty} \frac{9^n}{9^n}$

$= \sum_{n=0}^{\infty} 1$ diverges by nth-term test since $\lim_{n\to\infty} 1 = 1 \neq 0$; <u>check $x=-4$</u>:

$\sum_{n=0}^{\infty} \frac{(-3)^{2n}}{9^n} = \sum_{n=0}^{\infty} \frac{9^n}{9^n} = \sum_{n=0}^{\infty} 1$ diverges

by nth-term test since $\lim_{n\to\infty} 1 = 1 \neq 0$;

so interval of convergence is

$\boxed{-4 < x < 2}$;

$\sum_{n=0}^{\infty} \frac{(x+1)^{2n}}{9^n} = \sum_{n=0}^{\infty} \left(\frac{(x+1)^2}{9}\right)^n$

$= 1 + \left(\frac{(x+1)^2}{9}\right) + \left(\frac{(x+1)^2}{9}\right)^2 + \left(\frac{(x+1)^2}{9}\right)^3 + \cdots$

$= \frac{1}{1 - \frac{(x+1)^2}{9}} \cdot \frac{9}{9} = \frac{9}{9-(x^2+2x+1)} = \frac{9}{8-x^2-2x}$.

46.) $\sum_{n=0}^{\infty} (\ln x)^n$; $\lim_{n\to\infty} \frac{|\ln x|^{n+1}}{|\ln x|^n}$

$= \lim_{n\to\infty} |\ln x| = |\ln x| < 1 \longrightarrow$

$-1 < \ln x < 1 \longrightarrow e^{-1} < x < e^{1} \longrightarrow$

$\underline{\frac{1}{e} < x < e}$; $\underline{\text{check } x = e}$: $\sum_{n=0}^{\infty} (\ln e)^n$

$= \sum_{n=0}^{\infty} 1^n = \sum_{n=0}^{\infty} 1$ diverges by

nth-term test since $\lim_{n\to\infty} 1 = 1 \neq 0$;

$\underline{\text{check } x = \frac{1}{e}}$: $\sum_{n=0}^{\infty} (\ln e^{-1})^n = \sum_{n=0}^{\infty} (-1)^n$

diverges by nth-term test since

$\lim_{n\to\infty} (-1)^n \neq 0$; so interval of

convergence is $\boxed{\frac{1}{e} < x < e}$;

$\sum_{n=0}^{\infty} (\ln x)^n = 1 + (\ln x) + (\ln x)^2 + (\ln x)^3 + \cdots$

$= \frac{1}{1 - \ln x}$.

50.) a.) $f(x) = \frac{5}{3-x} = \frac{5}{3(1-\frac{x}{3})} = \frac{5}{3} \cdot \frac{1}{1-(\frac{x}{3})}$

$= \frac{5}{3}\left(1 + \left(\frac{x}{3}\right) + \left(\frac{x}{3}\right)^2 + \left(\frac{x}{3}\right)^3 + \cdots\right)$

$= \frac{5}{3} \sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n$; $\lim_{n\to\infty} \frac{\left|\frac{x}{3}\right|^{n+1}}{\left|\frac{x}{3}\right|^n}$

$= \lim_{n\to\infty} \left|\frac{x}{3}\right| = \left|\frac{x}{3}\right| < 1 \rightarrow -1 < \frac{x}{3} < 1$

$\rightarrow -3 < x < 3$;

if $x = -3$: $\frac{5}{3} \sum_{n=0}^{\infty} (-1)^n$ diverges by

nth-term test since $\lim_{n\to\infty} (-1)^n \neq 0$;

if $x = 3$: $\frac{5}{3} \sum_{n=0}^{\infty} (1)^n = \frac{5}{3} \sum_{n=0}^{\infty} 1$ clearly

diverges so interval of

convergence is $-3 < x < 3$.

53.) $1 - \frac{1}{2}(x-3) + \frac{1}{4}(x-3)^2 - \frac{1}{8}(x-3)^3 + \cdots$

$$= \sum_{n=0}^{\infty} \left(\frac{-1}{2}\right)^n (x-3)^n = \sum_{n=0}^{\infty} \left(\frac{3-x}{2}\right)^n$$

$$= \frac{1}{1-\left(\frac{3-x}{2}\right)} \cdot \frac{2}{2} = \frac{2}{2-(3-x)} = \boxed{\frac{2}{x-1}} \ ;$$

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|3-x|^{n+1}}{2^{n+1}} \cdot \frac{2^n}{|3-x|^n}$$

$$= \lim_{n\to\infty} \frac{1}{2}|3-x| = \frac{1}{2}|3-x| < 1 \rightarrow |3-x| < 2$$

$$\rightarrow -2 < 3-x < 2 \rightarrow -5 < -x < -1$$

$\rightarrow \boxed{1 < x < 5}$; if $\underline{x = 1}$: $\sum_{n=0}^{\infty} (1)^n$ diverges

by Nth-term test since $\lim_{n\to\infty} (+1)^n = 1 \neq 0$;

if $\underline{x = 5}$: $\sum_{n=0}^{\infty} (-1)^n$ diverges by Nth-term

test since $\lim_{n\to\infty} (-1)^n \neq 0$; differentiate series term by term getting

$$0 - \frac{1}{2} + \frac{1}{2^2}\cdot 2(x-3) - \frac{1}{2^3}\cdot 3(x-3)^2 + \frac{1}{2^4}\cdot 4(x-3)^3 + \cdots$$

$$= \sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)^n \cdot n(x-3)^{n-1} = D\left(\frac{2}{x-1}\right) = \frac{-2}{(x-1)^2} ;$$

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty}\frac{\left(\frac{1}{2}\right)^{n+1}(n+1)\,|x-3|^n}{\left(\frac{1}{2}\right)^n\cdot n\cdot|x-3|^{n-1}}$$

$$= \lim_{n\to\infty}\frac{1}{2}\left(1+\frac{1}{n}\right)|x-3| = \frac{1}{2}|x-3| < 1 \rightarrow$$

$$|x-3| < 2 \rightarrow -2 < x-3 < 2 \rightarrow 1 < x < 5 ,$$

if $\boxed{x=1}$: $\displaystyle\sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)^n n(-2)^{n-1}$

$$= (-2)^{-1}\cdot\sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)^n\cdot n\cdot(-2)^n = \frac{-1}{2}\sum_{n=1}^{\infty}(1)^n n = \frac{-1}{2}\sum_{n=1}^{\infty} n$$

and

$$\lim_{n\to\infty} n = \infty \neq 0$$

so series diverges by nth term test ;

if $\underline{x=5}$: $\displaystyle\sum_{n=1}^{\infty}\left(\frac{-1}{2}\right)^n\cdot n\cdot 2^{n-1}$

$$= 2^{-1}\sum_{n=1}^{\infty}\left(\frac{-1}{2}\right)^n\cdot n\cdot 2^n = \frac{1}{2}\sum_{n=1}^{\infty}(-1)^n n ,$$

which diverges by the Nth-term test since $\lim_{n\to\infty}(-1)^n n \neq 0$.

54.) $1-\frac{1}{2}(x-3)+\frac{1}{4}(x-3)^2-\frac{1}{8}(x-3)^3+\cdots=\frac{2}{x-1}$

$\rightarrow \int_3^x \left(1-\frac{1}{2}(t-3)+\frac{1}{4}(t-3)^2-\frac{1}{8}(t-3)^3+\cdots\right)dx$

$$= \int_3^x \frac{2}{t-1}\,dx \rightarrow$$

$\left[t-\frac{1}{2}\cdot\frac{1}{2}(t-3)^2+\frac{1}{2^2}\cdot\frac{1}{3}(t-3)^3-\frac{1}{2^3}\cdot\frac{1}{4}(t-3)^4+\cdots\right]_3^x$

$$= 2\ln(t-1)\Big|_3^x \rightarrow$$

$\left(x-\frac{1}{2}\cdot\frac{1}{2}(x-3)^2+\frac{1}{2^2}\cdot\frac{1}{3}(x-3)^3-\frac{1}{2^3}\cdot\frac{1}{4}(x-3)^4+\cdots\right)$

$-(3-0+0-0+\cdots)=2\ln(x-1)$

$-2\ln 2 \rightarrow$

$\sum_{n=1}^{\infty}\left(\frac{-1}{2}\right)^{n-1}\cdot\frac{1}{n}(x-3)^n = 2\ln(x-1)-2\ln 2$;

$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}=\lim_{n\to\infty}\frac{\left(\frac{1}{2}\right)^n\cdot\frac{1}{n+1}\cdot|x-3|^{n+1}}{\left(\frac{1}{2}\right)^{n-1}\cdot\frac{1}{n}\cdot|x-3|^n}$

$=\lim_{n\to\infty}\left(\frac{1}{2}\right)\cdot\frac{n}{n+1}\cdot|x-3| \overset{\frac{\infty}{\infty}}{=} \frac{1}{2}\cdot\frac{1}{1}\cdot|x-3|$

$=\frac{1}{2}|x-3|<1 \rightarrow |x-3|<2 \rightarrow$

$-2<x-3<2 \rightarrow 1<x<5$;

if $\underline{x=1}$: $\sum_{n=1}^{\infty}\left(\frac{-1}{2}\right)^{n-1}\cdot\frac{1}{n}\cdot(-2)^n$

$$= \left(\frac{-1}{2}\right)^{-1} \sum_{n=1}^{\infty} \left(\frac{-1}{2}\right)^{n} \cdot \frac{1}{n} \cdot (-2)^n$$

$$= -2 \sum_{n=1}^{\infty} \frac{1}{n}$$, which diverges by p-series test ($p = 1 \leq 1$);

if $\underline{x=5}$: $\displaystyle\sum_{n=1}^{\infty} \left(\frac{-1}{2}\right)^{n-1} \cdot \frac{1}{n} \cdot (2)^n$

$$= \left(\frac{-1}{2}\right)^{-1} \cdot \sum_{n=1}^{\infty} \left(\frac{-1}{2}\right)^{n} \cdot \frac{1}{n} \cdot (2)^n = -2 \sum_{n=1}^{\infty} (-1)^n \frac{1}{n},$$

which converges by the A.S.T. since $a_n = \frac{1}{n}$ is $+$, $\downarrow$, and $\lim_{n\to\infty} a_n = 0$.

60.) $\displaystyle\sum_{n=0}^{\infty} \frac{n^2}{2^n}$; begin with

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \xrightarrow{D}$$

$$\frac{1}{(1-x)^2} = 0 + 1 + 2x + 3x^2 + 4x^3 + \cdots \longrightarrow$$

$$\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + 4x^4 + \cdots \xrightarrow{D}$$

$$\frac{(1-x)^2 \cdot (1) - x \cdot 2(1-x)(-1)}{(1-x)^4} = 1 + 4x + 9x^2 + 16x^3 + \cdots$$

$$\longrightarrow$$

$$\frac{1+x}{(1-x)^3} = 1 + 4x + 9x^2 + 16x^3 + \cdots \longrightarrow$$

Let $x = \frac{1}{2}$

$$\frac{x(1+x)}{(1-x)^3} = x + 4x^2 + 9x^3 + 16x^4 + \cdots \longrightarrow$$

$$\frac{\frac{1}{2}\left(\frac{3}{2}\right)}{\left(\frac{1}{2}\right)^3} = \frac{1}{2} + \frac{4}{2^2} + \frac{9}{2^3} + \frac{16}{2^4} + \cdots$$

$$= \frac{1^2}{2} + \frac{2^2}{2^2} + \frac{3^2}{2^3} + \frac{4^2}{2^4} + \cdots = \sum_{n=0}^{\infty} \frac{n^2}{2^n} \longrightarrow$$

$$\sum_{n=0}^{\infty} \frac{n^2}{2^n} = \frac{3/4}{1/8} = \frac{3}{4} \cdot \frac{8}{1} = 6 .$$