Section 10.8

Taylor polynomial of degree n:

$P_n(x;a) = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3$
$+ \cdots + a_n(x-a)^n$, where

$$a_n = \frac{f^{(n)}(a)}{n!} \text{ for } n = 0, 1, 2, 3, \ldots$$

1.) $f(x) = e^{2x}$, $a = 0$:

$f'(x) = 2e^{2x}$, $f''(x) = 4e^{2x}$, $f'''(x) = 8e^{2x}$, so

$a_0 = \frac{f(0)}{0!} = \frac{e^0}{1} = 1$, $a_1 = \frac{f'(0)}{1!} = \frac{2e^0}{1} = 2$,

$a_2 = \frac{f''(0)}{2!} = \frac{4e^0}{2} = 2$, $a_3 = \frac{f'''(0)}{3!} = \frac{8e^0}{6} = 4/3$; then

$P_0(x;0) = a_0 = 1$,
$P_1(x;0) = a_0 + a_1 x = 1 + 2x$,
$P_2(x;0) = a_0 + a_1 x + a_2 x^2 = 1 + 2x + 2x^2$,
$P_3(x;0) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$
$= 1 + 2x + 2x^2 + \frac{4}{3}x^3$

2.) $f(x) = \sin x$, $a = 0$:

$f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$,

so $a_0 = \frac{f(0)}{0!} = \frac{\sin 0}{1} = 0$,

$a_1 = \frac{f'(0)}{1!} = \frac{\cos 0}{1} = \frac{1}{1} = 1$,

$$a_2 = \frac{f''(0)}{2!} = \frac{-\sin 0}{2} = \frac{0}{2} = 0,$$

$$a_3 = \frac{f'''(0)}{3!} = \frac{-\cos 0}{6} = \frac{-1}{6}; \text{ then}$$

$$P_0(x;0) = a_0 = 0,$$
$$P_1(x;0) = a_0 + a_1 x = 0 + 1\cdot x = x,$$
$$P_2(x;0) = a_0 + a_1 x + a_2 x^2 = 0 + 1\cdot x + 0\cdot x^2 = x,$$
$$P_3(x;0) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$
$$= 0 + 1\cdot x + 0\cdot x^2 + \frac{-1}{6}x^3 = x - \frac{1}{6}x^3$$

3.) $f(x) = \ln x$, $a = 1$:

$$f'(x) = \frac{1}{x}, \quad f''(x) = \frac{-1}{x^2}, \quad f'''(x) = \frac{2}{x^3}, \text{ so}$$

$$a_0 = \frac{f(1)}{0!} = \frac{\ln 1}{1} = \frac{0}{1} = 0,$$

$$a_1 = \frac{f'(1)}{1!} = \frac{1}{1} = 1, \quad a_2 = \frac{f''(1)}{2!} = \frac{-1}{2},$$

$$a_3 = \frac{f'''(1)}{3!} = \frac{2}{6} = \frac{1}{3}; \text{ then}$$

$$P_0(x;1) = a_0 = 0,$$
$$P_1(x;1) = a_0 + a_1(x-1) = 1\cdot(x-1) = x-1,$$
$$P_2(x;1) = a_0 + a_1(x-1) + a_2(x-1)^2$$
$$= (x-1) + \frac{-1}{2}(x-1)^2,$$

$$P_3(x;1) = a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3$$
$$= (x-1) + \frac{-1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$$

4.) $f(x) = \ln(1+x)$, $a = 0$:

$f'(x) = \frac{1}{1+x} = (1+x)^{-1}$, $f''(x) = -(1+x)^{-2} = \frac{-1}{(1+x)^2}$

$f'''(x) = 2(1+x)^{-3} = \frac{2}{(1+x)^3}$; so

$a_0 = \frac{f(0)}{0!} = \frac{\ln 1}{1} = \frac{0}{1} = 0$,

$a_1 = \frac{f'(0)}{1!} = \frac{\frac{1}{1+0}}{1} = \frac{1}{1} = 1$,

$a_2 = \frac{f''(0)}{2!} = \frac{\frac{-1}{(1+0)^2}}{2} = \frac{-1}{2}$,

$a_3 = \frac{f'''(0)}{3!} = \frac{\frac{2}{1^3}}{6} = \frac{2}{6} = \frac{1}{3}$; then

$P_0(x;0) = a_0 = 0$,

$P_1(x;0) = a_0 + a_1 x = 0 + 1\cdot x = x$,

$P_2(x;0) = a_0 + a_1 x + a_2 x^2 = 0 + 1\cdot x + \frac{-1}{2}x^2 = x - \frac{1}{2}x^2$,

$P_3(x;0) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

$= 0 + 1\cdot x + \frac{-1}{2}x^2 + \frac{1}{3}x^3 = x - \frac{1}{2}x^2 + \frac{1}{3}x^3$

6.) $f(x) = \frac{1}{x+2} = (x+2)^{-1}$, $a = 0$:

$f'(x) = -(x+2)^{-2}$, $f''(x) = 2(x+2)^{-3}$,

$f'''(x) = -6(x+2)^{-4}$, so

$a_0 = \frac{f(0)}{0!} = \frac{1/2}{1} = 1/2$, $a_1 = \frac{f'(0)}{1!} = \frac{-1/4}{1} = -1/4$,

$$a_2 = \frac{f''(0)}{2!} = \frac{2 \cdot 1/8}{2} = \frac{1}{8},$$

$$a_3 = \frac{f'''(0)}{3!} = \frac{-6 \cdot 1/16}{6} = \frac{-1}{16}; \text{ then}$$

$$P_0(x;0) = a_0 = 1/2,$$

$$P_1(x;0) = a_0 + a_1 x = \frac{1}{2} - \frac{1}{4}x,$$

$$P_2(x;0) = a_0 + a_1 x + a_2 x^2 = \frac{1}{2} - \frac{1}{4}x + \frac{1}{8}x^2,$$

$$P_3(x;0) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 = \frac{1}{2} - \frac{1}{4}x + \frac{1}{8}x^2 - \frac{1}{16}x^3$$

8.) $f(x) = \tan x$, $a = \frac{\pi}{4}$:

$$f'(x) = \sec^2 x, \quad f''(x) = 2\sec x \cdot \sec x \tan x = 2\sec^2 x \cdot \tan x,$$

$$f'''(x) = 2\sec^2 x \cdot \sec^2 x + 4\sec x \cdot \sec x \tan x \cdot \tan x = 2\sec^4 x + 4\sec^2 x \cdot \tan^2 x, \text{ and}$$

$$\tan\frac{\pi}{4} = 1, \quad \sec\frac{\pi}{4} = \frac{1}{\cos\frac{\pi}{4}} = \frac{1}{\frac{\sqrt{2}}{2}} = \frac{2}{\sqrt{2}} = \sqrt{2};$$

$$a_0 = \frac{f(\frac{\pi}{4})}{0!} = \frac{\tan\frac{\pi}{4}}{1} = \frac{1}{1} = 1,$$

$$a_1 = \frac{f'(\frac{\pi}{4})}{1!} = \frac{\sec^2\frac{\pi}{4}}{1} = (\sqrt{2})^2 = 2,$$

$$a_2 = \frac{f''\left(\frac{\pi}{4}\right)}{2!} = \frac{\cancel{2}\sec^2\frac{\pi}{4}\tan\frac{\pi}{4}}{\cancel{2}} = (\sqrt{2})^2(1) = 2,$$

$$a_3 = \frac{f'''\left(\frac{\pi}{4}\right)}{3!} = \frac{2\sec^4\left(\frac{\pi}{4}\right) + 4\sec^2\left(\frac{\pi}{4}\right)\tan^2\left(\frac{\pi}{4}\right)}{6}$$

$$= \frac{2(\sqrt{2})^4 + 4(\sqrt{2})^2\cdot(1)^2}{6} = \frac{8+8}{6} = \frac{16}{6} = \frac{8}{3};$$

then $P_0\left(x; \frac{\pi}{4}\right) = a_0 = 1,$

$$P_1\left(x; \frac{\pi}{4}\right) = a_0 + a_1\left(x - \frac{\pi}{4}\right) = 1 + 2\left(x - \frac{\pi}{4}\right),$$

$$P_2\left(x; \frac{\pi}{4}\right) = a_0 + a_1\left(x - \frac{\pi}{4}\right) + a_2\left(x - \frac{\pi}{4}\right)^2$$

$$= 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2,$$

$$P_3\left(x; \frac{\pi}{4}\right) = a_0 + a_1\left(x - \frac{\pi}{4}\right) + a_2\left(x - \frac{\pi}{4}\right)^2 + a_3\left(x - \frac{\pi}{4}\right)^3$$

$$= 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3$$

10.) $f(x) = \sqrt{1-x}, \quad a = 0:$

$$f'(x) = -\frac{1}{2}(1-x)^{-1/2}, \quad f''(x) = -\frac{1}{4}(1-x)^{-3/2},$$

$$f'''(x) = -\frac{3}{8}(1-x)^{-5/2}; \text{ so}$$

$$a_0 = \frac{f(0)}{0!} = \frac{\sqrt{1}}{1} = 1,$$

$$a_1 = \frac{f'(0)}{1!} = \frac{-\frac{1}{2}(1)^{-1/2}}{1} = \frac{-1}{2},$$

$$a_2 = \frac{f''(0)}{2!} = \frac{-\frac{1}{4}(1)^{-3/2}}{2} = \frac{-1}{8},$$

$$a_3 = \frac{f'''(0)}{3!} = \frac{-\frac{3}{8}(1)^{-5/2}}{6} = \frac{-3}{48} = \frac{-1}{16}; \text{ then}$$

$$P_0(x;0) = a_0 = 1,$$
$$P_1(x;0) = a_0 + a_1 x = 1 - \tfrac{1}{2}x,$$
$$P_2(x;0) = a_0 + a_1 x + a_2 x^2 = 1 - \tfrac{1}{2}x - \tfrac{1}{8}x^2,$$
$$P_3(x;0) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$
$$= 1 - \tfrac{1}{2}x - \tfrac{1}{8}x^2 - \tfrac{1}{16}x^3$$

11.) $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \dfrac{x^n}{n!}$, so

$$e^{-x} = e^{(-x)} = 1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \cdots$$
$$= 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}$$

12.) $xe^x = x\left(1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots\right)$

$$= x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \cdots = \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!}$$

13.) $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$, so

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 + (-x) + (-x)^2 + (-x)^3 + (-x)^4 + \cdots$$
$$= 1 - x + x^2 - x^3 + x^4 - \cdots \sum_{n=0}^{\infty} (-1)^n \cdot x^n$$

14.) $\dfrac{2+x}{1-x} = (2+x) \cdot \dfrac{1}{1-x}$

$$= (2+x)(1 + x + x^2 + x^3 + \cdots)$$
$$= 2 + 2x + 2x^2 + 2x^3 + 2x^4 + \cdots$$
$$+ x + x^2 + x^3 + x^4 + \cdots$$

$$= 2 + 3x + 3x^2 + 3x^3 + \cdots = 2 + 3\sum_{n=1}^{\infty} x^n$$

15.) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)!}, \text{ then}$$

$$\sin(3x) = (3x) - \frac{(3x)^3}{3!} + \frac{(3x)^5}{5!} - \frac{(3x)^7}{7!} + \cdots$$

$$= 3x - \frac{3^3 x^3}{3!} + \frac{3^5 x^5}{5!} - \frac{3^7 x^7}{7!} + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{3^{2n+1}}{(2n+1)!} x^{2n+1}$$

17.) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \text{ then}$$

$$x^2 \cos(x^3) = x^2\left(1 - \frac{(x^3)^2}{2!} + \frac{(x^3)^4}{4!} - \frac{(x^3)^6}{6!} + \cdots\right)$$

$$= x^2\left(1 - \frac{x^6}{2!} + \frac{x^{12}}{4!} - \frac{x^{18}}{6!} + \cdots\right)$$

$$= x^2 - \frac{x^8}{2!} + \frac{x^{14}}{4!} - \frac{x^{20}}{6!} + \frac{x^{26}}{8!} - \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{6n+2}}{(2n)!}$$

21.) $x^4 - 2x^3 - 5x + 4 = 4 - 5x + 0 \cdot x^2 - 2x^3$
$+ 1 \cdot x^4 + 0 \cdot x^5 + 0 \cdot x^6 + 0 \cdot x^7 + \cdots$

22.) $\dfrac{x^2}{x^3+1} = \dfrac{x^2}{1-(-x^3)} = x^2 \cdot \dfrac{1}{1-(-x^3)}$

$= x^2 \cdot (1 + (-x^3) + (-x^3)^2 + (-x^3)^3 + (-x^3)^4 + \cdots)$

$= x^2 (1 - x^3 + x^6 - x^9 + x^{12} - \cdots)$

$= x^2 - x^5 + x^8 - x^{11} + x^{14} - \cdots$

$= \displaystyle\sum_{n=0}^{\infty} (-1)^n \cdot x^{3n+2}$

25.) $f(x) = x^4 + x^2 + 1$, $a = -2$:

$f'(x) = 4x^3 + 2x$, $f''(x) = 12x^2 + 2$,

$f'''(x) = 24x$, $f^{(4)}(x) = 24$, $f^{(5)}(x) = 0$,

$f^{(6)}(x) = f^{(7)}(x) = f^{(8)}(x) = \cdots = 0$; ~~so~~

$a_0 = \dfrac{f(-2)}{0!} = \dfrac{21}{1} = 21$, $a_1 = \dfrac{f'(-2)}{1!} = -\dfrac{36}{1} = -36$,

$a_2 = \dfrac{f''(-2)}{2!} = \dfrac{50}{2} = 25$, $a_3 = \dfrac{f'''(-2)}{3!} = -\dfrac{48}{6} = -8$,

$a_4 = \dfrac{f^{(4)}(-2)}{4!} = \dfrac{24}{24} = 1$, $a_5 = a_6 = a_7 = \cdots = 0$;

~~then~~ $x^4 + x^2 + 1 = a_0 + a_1(x-(-2)) + a_2(x-(-2))^2 + \cdots$

$= 21 - 36(x+2) + 25(x+2)^2 - 8(x+2)^3 + (x+2)^4 + 0(x+2)^5 + \cdots$

$= 21 - 36(x+2) + 25(x+2)^2 - 8(x+2)^3 + (x+2)^4$

27.) $f(x) = \dfrac{1}{x^2}$, $a = 1$:

$f'(x) = -2x^{-3}$, $f''(x) = 3 \cdot 2\, x^{-4} = 3!\, x^{-4}$,

$f'''(x) = -4 \cdot 3!\, x^{-5} = -4!\, x^{-5}$,

$f^{(4)}(x) = 5 \cdot 4!\, x^{-6} = 5!\, x^{-6}, \ldots,$

$f^{(n)}(x) = (-1)^n (n+1)!\, x^{-(n+2)}$ for $n = 0, 1, 2, 3, \ldots$;

~~then~~

$a_0 = \dfrac{f(1)}{0!} = \dfrac{1}{1} = 1$, $a_1 = \dfrac{f'(1)}{1!} = \dfrac{-2}{1} = -2$,

$a_2 = \dfrac{f''(1)}{2!} = \dfrac{3!}{2!} = 3$, $a_3 = \dfrac{f'''(1)}{3!} = \dfrac{-4!}{3!} = -4$,

$a_4 = \dfrac{f^{(4)}(1)}{4!} = \dfrac{5!}{4!} = 5, \ldots,$ $a_n = (-1)^n \cdot (n+1)$

for $n = 0, 1, 2, 3, \ldots$; then

$$\frac{1}{x^2} = a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3 + \cdots$$
$$= 1 - 2(x-1) + 3(x-1)^2 - 4(x-1)^3 + \cdots$$
$$= \sum_{n=0}^{\infty} (-1)^n \cdot (n+1) \cdot (x-1)^n .$$

29.) $f(x) = e^x$, $a = 2$:
$f'(x) = e^x$, $f^{(n)}(x) = e^x$ for $n = 0, 1, 2, 3, \ldots$;
so $a_0 = \frac{f(2)}{0!} = \frac{e^2}{1} = e^2$, $a_1 = \frac{f'(2)}{1!} = \frac{e^2}{1} = e^2$,
$a_2 = \frac{f''(2)}{2!} = \frac{e^2}{2!}$, $a_3 = \frac{e^2}{3!}$, $\ldots$, $a_n = \frac{e^2}{n!}$ for
$n = 0, 1, 2, 3, \ldots$; then
$$e^x = a_0 + a_1(x-2) + a_2(x-2)^2 + a_3(x-2)^3 + \cdots$$
$$= e^2 + e^2(x-2) + \frac{e^2}{2!}(x-2)^2 + \frac{e^2}{3!}(x-2)^3 + \cdots$$
$$= \sum_{n=0}^{\infty} \frac{e^2}{n!}(x-2)^n$$

33.) $\cos x - \frac{2}{1-x}$
$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) - 2\left(1 + x + x^2 + x^3 + \cdots\right)$$
$$= \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \cdots\right) - 2 - 2x - 2x^2 - 2x^3 \cdots$$
$$= -1 - 2x - \frac{5}{2}x^2 - 2x^3 + \cdots$$

34.) $(1 - x + x^2)e^x$
$$= (1 - x + x^2)\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)$$

$= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots$
$\quad - x - x^2 - \frac{1}{2}x^3 + \cdots$
$\quad + x^2 + x^3 + \cdots$
$= 1 + \frac{1}{2}x^2 + \frac{2}{3}x^3 + \cdots$

36.) $x \sin^2 x = x(\sin x)^2$
$= x\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots\right)^2$
$= x\left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \cdots\right)\left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \cdots\right)$
$= \left(x^2 - \frac{1}{6}x^4 + \frac{1}{120}x^6 - \cdots\right)\left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \cdots\right)$
$= x^3 - \frac{1}{6}x^5 + \frac{1}{120}x^7 + \cdots$
$\quad - \frac{1}{6}x^5 + \frac{1}{36}x^7 + \cdots$
$\quad + \frac{1}{120}x^7 - \cdots$
$= x^3 - \frac{1}{3}x^5 + \frac{2}{45}x^7 - \cdots$

41.) b.) $f(x) = \ln(\cos x) \xrightarrow{D} f'(x) = \frac{1}{\cos x} \cdot -\sin x$
$= -\tan x, \quad f''(x) = -\sec^2 x;$
$a_0 = \frac{f(0)}{0!} = \frac{\ln(\cos 0)}{1} = \ln 1 = 0,$
$a_1 = \frac{f'(0)}{1!} = \frac{-\tan 0}{1} = 0,$
$a_2 = \frac{f''(0)}{2!} = \frac{-\sec^2 0}{2} = \frac{-(1)^2}{2} = \frac{-1}{2},$

then $P_2(x;0) = a_0 + a_1 x + a_2 x^2 = -\frac{1}{2}x^2$.

42.) b.) $f(x) = e^{\sin x} \xrightarrow{D} f'(x) = e^{\sin x} \cdot \cos x$,

$f''(x) = e^{\sin x} \cdot (-\sin x) + e^{\sin x} \cdot \cos x \cdot \cos x$

$= \cos^2 x \cdot e^{\sin x} - \sin x \, e^{\sin x}$;

$a_0 = \frac{f(0)}{0!} = \frac{e^{\sin 0}}{1} = e^0 = 1$,

$a_1 = \frac{f'(0)}{1!} = e^{\sin 0} \cos 0 = e^0 \cdot 1 = 1$,

$a_2 = \frac{f''(0)}{2!} = \frac{\cos^2 0 \cdot e^{\sin 0} - \sin 0 \cdot e^{\sin 0}}{2}$

$= \frac{1 \cdot 1 - 0 \cdot 1}{2} = \frac{1}{2}$, then

$P_2(x;0) = a_0 + a_1 x + a_2 x^2$

$= 1 + x + \frac{1}{2}x^2$

43.) b.) $f(x) = \frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-1/2} \xrightarrow{D}$

$f'(x) = -\frac{1}{2}(1-x^2)^{-3/2} \cdot (-2x) = \frac{x}{(1-x^2)^{3/2}}$,

$f''(x) = \frac{(1-x^2)^{3/2}(1) - x \cdot \frac{3}{2}(1-x^2)^{1/2} \cdot (-2x)}{[(1-x^2)^{3/2}]^2}$

$= \frac{(1-x^2)^{3/2} + 3x^2(1-x^2)^{1/2}}{(1-x^2)^3}$

$$= \frac{(1-x^2)^{1/2} \cdot [(1-x^2) + 3x^2]}{(1-x^2)^3}$$

$$= \frac{1+2x^2}{(1-x^2)^{5/2}}$$; then

$$a_0 = \frac{f(0)}{0!} = \frac{1}{1} = 1,$$

$$a_1 = \frac{f'(0)}{1!} = \frac{0}{1} = 0,$$

$$a_2 = \frac{f''(0)}{2!} = \frac{1}{2},$$ so

$$P_2(x;0) = a_0 + a_1 x + a_2 x^2$$

$$= 1 + 0 \cdot x + \frac{1}{2}x^2$$

$$= 1 + \frac{1}{2}x^2$$