# (Math 21D) Vector Calculus

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"How the FTC extends to vectors $(x,y,z) \in \mathbb{R}^3$"

Instructor: Blake Temple (Distinguished Professor)

Research: General Relativity and Shock Wave Theory

(2023) First nonlinear theory of time-periodic sound waves (w. Robin Young UMass)

(2020-23) Extend Kazden-DeTurck optimal regularity & Uhlenbeck compactness from Riemannian geometry to arbitrary connections (w Moritz Reintjes HK)

(2017-23) A theory of Dark Energy based on an Instability in the Standard Model of Cosmology (w Christopher Alexander and Zeke Vogler)

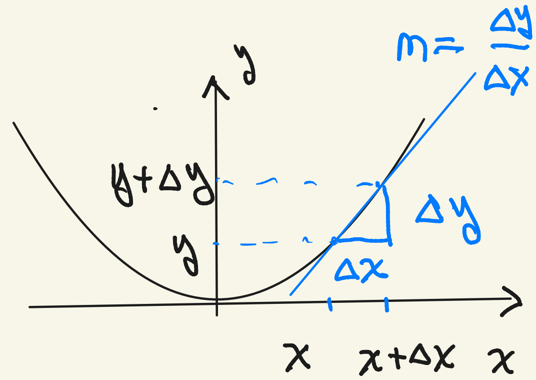
(2020-2023) A causal dissipative Relativistic Navier-Stokes eqn in which all shocks admit profiles (H Freistuehler, Konstanz)

# Introduction

## Math 21D
## Vector Calculus

### Introduction:

- Vector Calculus is the theory of Calculus for functions with more than one variable
- Viewing formulas as <u>functions</u> makes Calculus easy to describe




Eg: $y = x^2$ parabola

View it as a function:

$$\underset{\text{output}}{y} = f(\underset{\text{input}}{x}) = x^2$$

Then we can describe the tangent line:

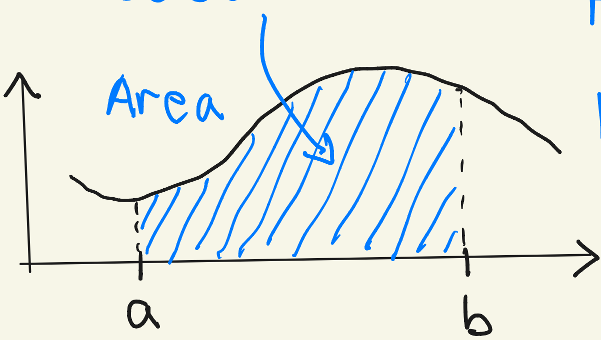
$$\underset{\text{slope}}{m} = f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = 2\underset{\text{input } x}{x}$$

- We call the function $f'(x)$ the **derivative**
- **Fundamental Theorem of Calculus (FTC)**

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

$F'(x) = f(x)$

$F(x) =$ Anti-deriv of $f$




**Area** is **found by** **Anti-differentiation**

(At the start Area appears to have nothing to do with derivatives!)

- **Our Question**: How do you do calculus & what is FTC when inputs/outputs of $f$ are vectors?

- much more complicated
- this is the real world!

$$w = f(\underbrace{x, y, z})$$

output $w = (w_1, w_2, w_3) \in \mathbb{R}^3$

input $\underset{\sim}{x} = (x, y, z) \in \mathbb{R}^3$

Three generalizations of the derivative to vectors form the basis of classical physics —

$\nabla$, Div, Curl

gradiant Divergence Curl

(1) $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$ applies to scalar $f$

$$w = f(x,y,z) \underset{\text{eg}}{=} x^2 + y^2 + xyz$$

"the gradiant points in direction of steepest increase of a function"

(2) $\text{Div}\,\vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} \equiv \nabla \cdot \vec{F}$ "Del dot $\vec{F}$"

$\vec{F}$ is a "vector field"

applies to vector valued functions

$$\vec{F}\underbrace{(x,y,z)}_{\text{input}} = \underbrace{(M(x,y,z), N(x,y,z), P(x,y,z))}_{\text{output}}$$

"input $(x,y,z) \in \mathbb{R}^3$ & output $(M,N,P) \in \mathbb{R}^3$ are vectors"

Thm (Ch 16) $\text{Div}\,\vec{F} = \frac{\text{Flux}}{\text{Vol}}$

④

③ Curl $\vec{F}$ = $\begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial_x & \partial_y & \partial_z \\ M & N & P \end{vmatrix} = (P_y - N_z, M_z - P_x, N_x - M_y)$

$= \nabla \times \vec{F}$

"del cross $\vec{F}$"

$\vec{F}(\underbrace{x,y,z}_{\text{input } (x,y,z)}) = \underbrace{(M(x,y,z), N(x,y,z), P(x,y,z))}_{\text{output } (M,N,P) \in \mathbb{R}^3}$ = Vector Field

Thm: (Ch 16) Curl $\vec{F} = \dfrac{\text{Circulation}}{\text{Area}}$

Conclude: Both Div & Curl apply to Vector Fields

- Math 21D is the mathematics required to understand $\nabla$, Div, Curl
- $\nabla$, Div, Curl are the three first order operators of Classical Physics –

- They were introduced by W. Gibbs to describe Maxwell's Theory of Electro-Magnetism ~ 1880's
- They are the language of Fluid Mechanics

The **meaning** of $\nabla$, Div, Curl come from three generalizations of FTC: (Topics of Math 21D)



## ① $\nabla$ :

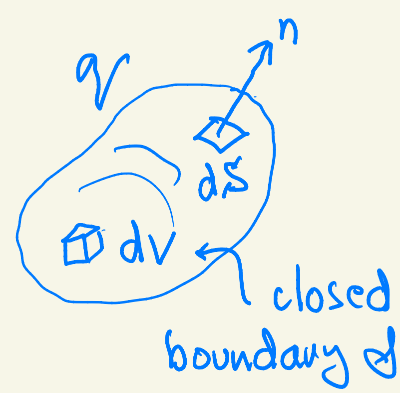
$$\int_{\mathcal{C}} \nabla f \cdot \vec{T}\, ds = f(B) - f(A)$$


$\boxed{\text{Conservation of Energy}}$

"The line integral of the gradiant along a curve is the change in f between the endpoints"

## ② Div :

$$\iiint_{\mathcal{V}} \operatorname{Div} \vec{F}\, dV = \iint_{\partial} \vec{F} \cdot \vec{n}\, dA$$



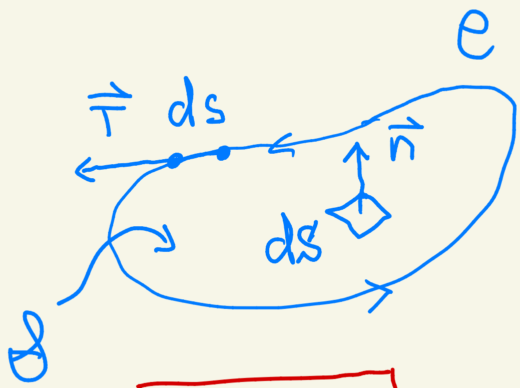

"The integral of the divergence over a volume is equal to the flux thru the boundary"

$\boxed{\text{Divergence Theorem}}$

## 3 Curl:

$$\iint_{S} \text{Curl}\,\vec{F}\cdot\vec{n}\,dS = \oint_{C} \vec{F}\cdot\vec{T}\,ds$$

"The flux of the Curl thru a surface $S$ is equal to the circulation around the boundary"




$\boxed{\text{Stokes Theorem}}$

- From these three theorems we will discover the meanings of $\nabla$, Div, Curl

$$\text{Div}\,\vec{F} = \frac{\text{flux}}{\text{Vol}}$$

$$\text{Curl}\,\vec{F} = \frac{\text{Circulation}}{\text{area}}$$

$\nabla f$ = rate of increase of $f$ in direction of steepest increase

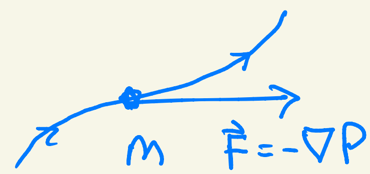
(we already know this from Math 21C)

- **Physics** is "The pursuit of finding the equations which describe physical processes"

**$\nabla$, Div, Curl** appear in the fundamental equations of Physics —

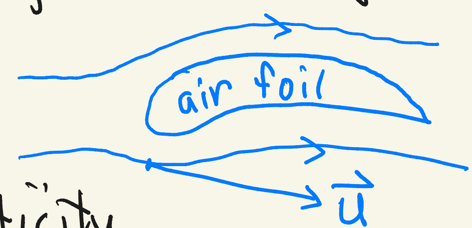
(1) **Newton's Equations** for motion of a particle in a conservative force field —

$$\boxed{m\ddot{\underset{\sim}{x}} = -\nabla P(\underset{\sim}{x})}$$




(2) **Fluid Mechanics**: $\rho$ = density, $\vec{u}$ = velocity

$\rho_t + \mathrm{Div}(\rho u) = 0$ (Cons. of mass)




$\frac{D}{Dt}\omega = 0$, $\omega = \mathrm{Curl}\,\vec{u}$ = Vorticity

(Cons of vorticity)

(3) **Electromagnetism** (Maxwell's Eqn empty space)

$$\boxed{\begin{array}{ll} \mathrm{Div}\,\vec{E} = 0 & \mathrm{Div}\,\vec{B} = 0 \\ -B_t = \mathrm{Curl}\,\vec{E} & \\ E_t = \frac{1}{\mu_0 \varepsilon_0}\mathrm{Curl}\,\vec{B} & \end{array}}$$

$\vec{E}$ = electric field
$\vec{B}$ = magnetic field

Faraday's Law
Ampere/Maxwell Law

The basic 2nd order linear operators of classical physics are also based on $\nabla$, Div, Curl:

① Laplacian: $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \text{Div}\,\nabla$

$$\Delta f = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\text{Div}\,\nabla f = \text{Div}\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$\Delta = \text{Div}\,\nabla$

② Wave Equation: $u_{tt} - c^2 \Delta u = 0$

Heat Equation: $u_t - k^2 \Delta u = 0$

Schroedinger Eqn: $i\hbar \frac{\partial u}{\partial t} = \Delta u$

Conclude: The time rates of change of quantities in Physics always seem to come out to be

$\nabla$, Div, Curl or $\Delta$ !

☒ Math 21D Vector Calculus covers all of the mathematics required to understand $\nabla$, Curl, Div and the versions of the Fundamental Theorems of Calculus which explain them.

(Note: the Laplacian $\Delta$ and the equations which involve $\Delta$ are the topic of a class in partial differential equations (PDE) and will not be studied here.)