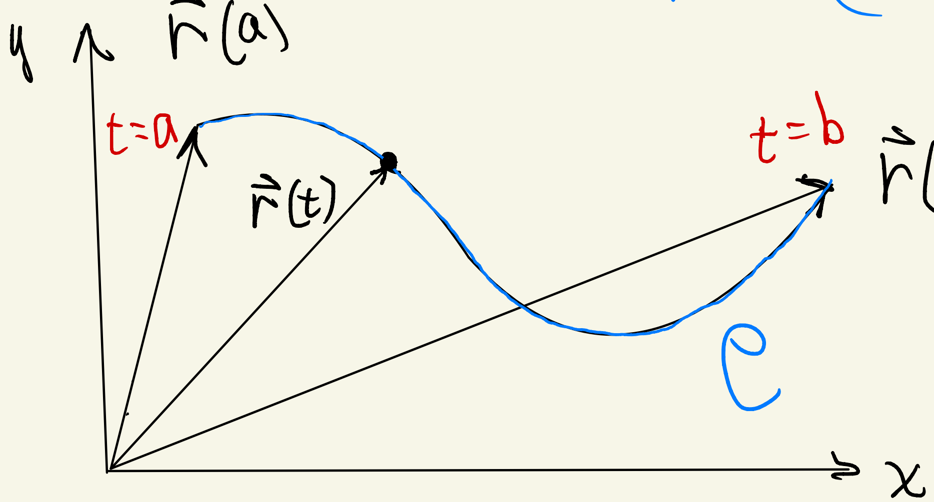
# § 13.3 Arclength and the unit tangent vector (we use $\vec{r}(t)$, book uses $\vec{\sigma}(t)$) ①

- Consider a curve $\mathcal{C}$

$$\vec{r} = \vec{r}(t), \quad a \le t \le b$$

$\vec{r}(t) = (x(t), y(t), z(t))$




$\mathcal{C}$ is the name of the curve

- Let $s$ denote arclength distance along $\mathcal{C}$ starting at $s=0$ when $t=a$.

Q: What is $t = t(s)$ ?

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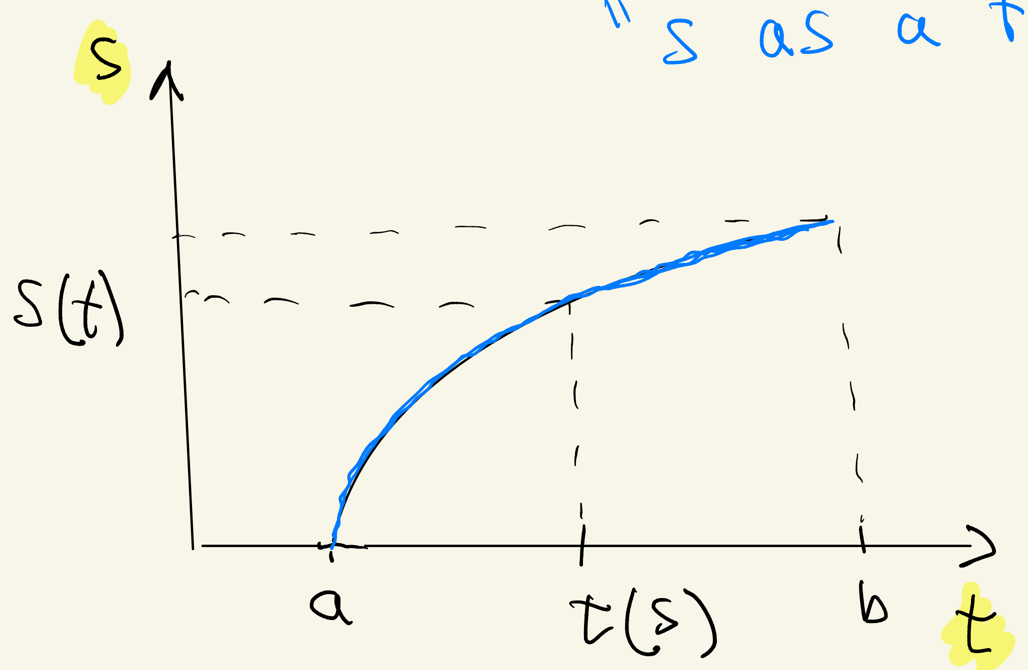
Ans: We know $\frac{ds}{dt} = \|\vec{v}\| = \|\vec{r}'(t)\|$

Thus: $ds = \|\vec{v}(t)\|\,dt$

$$\underbrace{s = \int_a^t \|\vec{v}(\xi)\|\,d\xi}$$

dummy variable of integration

This gives $s = s(t)$
"s as a function of t"




$$s = \varphi(t)$$
$$\Downarrow$$
$$\boxed{\underbrace{\varphi^{-1}(s)}_{t(s)} = t}$$

- By this we can parameterize any curve with respect to arclength –

**Thm**: Assume $\vec{v}(t) = \vec{r}\,'(t) \neq 0$, $a \le t \le b$

Then $s = \int_0^t \|\vec{v}(\xi)\| d\xi$ gives $s = \varphi(t)$, $\varphi'(t) \neq 0$,

and we can solve $s = \varphi(t)$

for $t = t(s) = \varphi^{-1}(s)$, and by this obtain

$$\vec{r} = \vec{r}(t) = \vec{r}(t(s)) \equiv \text{"}\vec{r}(s)\text{"}$$

abuse of notation as not same function $\vec{r}(\cdot)$

"There are many ways to parameterize a curve wrt $t$, but only one way to parameterize wrt arclength, $s$ starting at $s=0$ when $t=a$"

- **Claim**: If $\vec{r} = \vec{r}(s)$ is a curve parameterized by arclength, then $\vec{v}(s) = \vec{T}(s)$ is a unit vector

$$\|\vec{v}\| = 1.$$

**Proof**: If $\vec{r} = \vec{r}(t)$, then

$$\frac{ds}{dt} = \|\vec{v}\| = \|\vec{r}\,'(t)\|$$

When $t = s$, we have $\frac{ds}{ds} = 1$, so

$$\|\vec{v}\| = \|\vec{r}\,'(s)\| = \frac{ds}{ds} = 1 \quad \checkmark$$

Since whenever $\|\vec{r}(t)\| = \text{const}$ we know $\vec{r}(t) \perp \vec{v}(t)$, we can apply this to $\vec{v}(s)$:

Theorem: If $\vec{r}(s)$ is parameterized wrt arclength, then

$$\vec{a}(s) \perp \vec{v}(s)$$

Proof: We know $\|\vec{v}(s)\| = 1$,

so $\dfrac{d}{ds}\left[\vec{v}(s) \cdot \vec{v}(s)\right] = \vec{v}'(s) \cdot \vec{v}(s) + \vec{v}(s) \cdot \vec{v}'(s)$

$$2\vec{v}'(s) \cdot \vec{v}(s) = 0$$

$$\vec{a}(s) \cdot \vec{v}(s) = 0 \Rightarrow \vec{a}(s) \perp \vec{v}(s)$$

✓

# ☒ § 13.4 Introduction to the Theory of Curves

- We have: a curve in $\mathbb{R}^3$ is described by a position vector

$$\vec{r} = \vec{r}(t) = x(t)\,\underset{\sim}{i} + y(t)\,\underset{\sim}{j} + z(t)\,\underset{\sim}{k}$$

We know:

$$\vec{v} = \vec{v}(t) = x'(t)\,\underset{\sim}{i} + y'(t)\,\underset{\sim}{j} + z'(t)\,k$$

points tangent to the curve, with speed

$$\frac{ds}{dt} = \|\vec{v}(t)\| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$$

We also have:

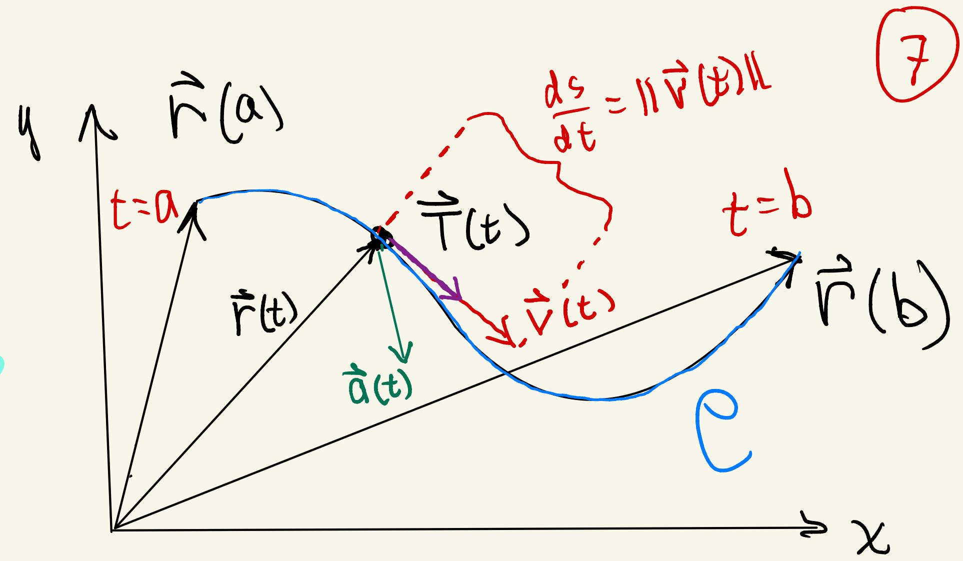
$$\vec{a} = \frac{d\vec{v}}{dt} = \vec{r}\,''(t) = x''(t)\,\underset{\sim}{i} + y''(t)\,\underset{\sim}{j} + z''(t)\,\underset{\sim}{k}$$

We know where $\vec{v}(t)$ points relative to the curve, but $\vec{a}$ seems to point anywhere

Q: What can we say about where the acceleration vector $\vec{a}$ points in general?

## Picture:

Q: What can we say about where $\vec{a}(t)$ must point?




$\vec{r}(t)$: $a \le t \le b$

- We know a lot about $\vec{v}$:

$$\frac{ds}{dt} = \|\vec{v}(t)\| \iff ds = \|\vec{v}(t)\|\,dt$$

$$s - s_0 = \int_{s_0}^{s} ds = \int_{t_0}^{t} \|\vec{v}(\xi)\|\,d\xi$$

Arclength starting at $t_0$ $\quad s = \int_{t_0}^{t} \|\vec{v}(\xi)\|\,d\xi$
(take $s_0 = 0$)

Letting $\vec{T}(s)$ denote the unit tangent vector

$$\vec{v} = \frac{ds}{dt}\vec{T}$$

unit tangent vector

speed $\frac{ds}{dt} = \|\vec{v}\|$ $\qquad \vec{T}(t) = \frac{\vec{v}(t)}{\|\vec{v}(t)\|}$

- Since $\vec{a} = \frac{d}{dt}\vec{v}$, we try to understand $\vec{a}$ by differentiating

$$\vec{a} = \frac{d}{dt}\vec{v} = \frac{d}{dt}\underbrace{\left(\|\vec{v}\|\vec{T}\right)}_{\text{product of scalar } \|\vec{v}(t)\| = \frac{ds}{dt} \text{ and vector } \vec{T}(t)}$$

Liebniz product rule holds!

$$= \underbrace{\left(\frac{d}{dt}\|\vec{v}\|\right)}_{\frac{d}{dt}\frac{ds}{dt} = \frac{d^2s}{dt^2}}\vec{T} + \underbrace{\|\vec{v}\|}_{v = \frac{ds}{dt}}\underbrace{\left(\frac{d\vec{T}}{dt}\right)}$$

Q: What direction does $\frac{d\vec{T}}{dt}$ point?

- But $\|\vec{T}(t)\| = 1$ (Const. Length $\Rightarrow \vec{T}'(t) \perp \vec{T}(t)$)

Thus: $\frac{d\vec{T}(t)}{dt} \cdot \vec{T}(t) = 0$ at every $t \in [a,b]$

↖ dot product $= \|\frac{d\vec{T}}{dt}\| \, \underbrace{\|\vec{T}\|}_{1} \, \underbrace{\cos\theta}$

$\cos\theta = 0$

$\theta = 90^\circ$

Conclude $\frac{d\vec{T}}{dt} = \vec{T}'(t) \perp \vec{v}(t)$!

**We have:**

$$\vec{a} = \frac{d}{dt}\left(\|\vec{v}\|\,\vec{T}\right) = \Big(\underbrace{\frac{d}{dt}\|\vec{v}\|}_{\frac{d^2s}{dt^2}}\Big)\vec{T} + \underbrace{\|\vec{v}\|}_{\frac{ds}{dt}}\Big(\underbrace{\frac{d\vec{T}}{dt}}_{\perp \vec{v}}\Big)$$

[ **Recall:** $\|\vec{T}(t)\| = 1 \Rightarrow 1 = \vec{T}(t)\cdot\vec{T}(t) = \|\vec{T}(t)\|^2$

$\Rightarrow 0 = \vec{T}'\cdot\vec{T} + \vec{T}\cdot\vec{T}' = 2\vec{T}'\cdot\vec{T}$ ✓ ]

**Conclude:** Since $\frac{d\vec{T}}{dt} \perp \vec{T}$ we must have

$$\frac{d\vec{T}}{dt} = \underbrace{\left\|\frac{d\vec{T}}{dt}\right\|}_{\text{positive}}\vec{N}$$

← unit vector $\perp \vec{T}$ and $\vec{v}$

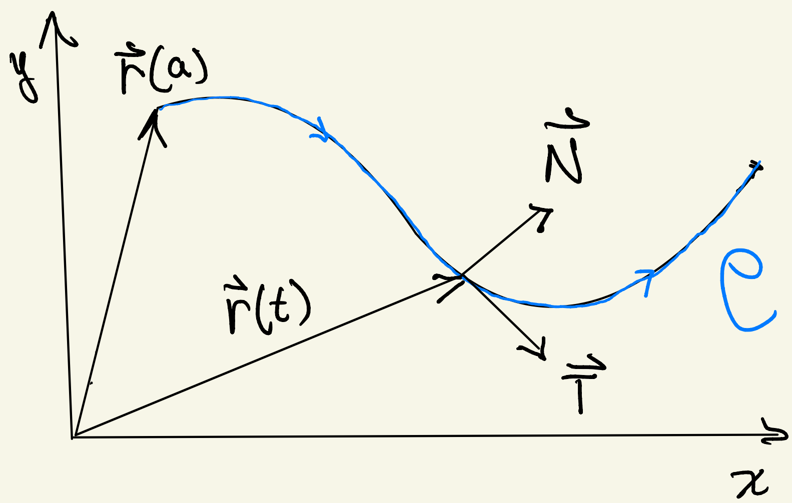
**Defn:** If $\frac{d\vec{T}}{dt} \neq 0$, then we define

$$\vec{N} \equiv \vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|} \approx \frac{d\vec{T}/dt}{\|d\vec{T}/dt\|}$$

to be the **Principle Normal Vector**

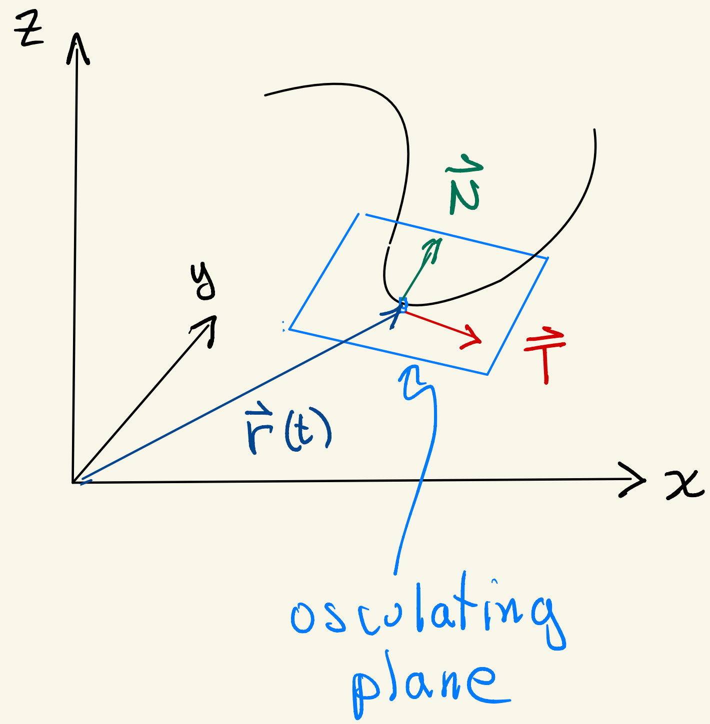
**Picture:** In the plane $\mathbb{R}^2$:

... the unit normal $\vec{N}$ points orthogonal to $\vec{T}$ in the direction in which $\mathcal{C}$ is curving




More interestingly, in $\mathbb{R}^3$, the plane spanned by $\vec{T}$ and $\vec{N}$ (the osculating plane) is the plane in which the curve most closely lies @ $\vec{r}(t)$

**Summary:** The Principal Unit Normal $\vec{N}$ gives the direction and plane into which the curve is "curving away from $\vec{T}$"




$$\vec{N} = \frac{\left(\frac{d\vec{T}}{dt}\right)}{\left\| \frac{d\vec{T}}{dt} \right\|}$$

☒ Collecting what we have:

$$\vec{a} = \frac{d}{dt}\left(\|\vec{v}\|\vec{T}\right) = \underbrace{\frac{d}{dt}\|\vec{v}\|}_{\frac{d}{dt}\frac{ds}{dt}}\vec{T} + \underbrace{\|\vec{v}\|\frac{d\vec{T}}{dt}}_{\frac{ds}{dt}\left\|\frac{d\vec{T}}{dt}\right\|\vec{N}}$$

So

$$\boxed{\vec{a} = \frac{d^2s}{dt^2}\vec{T} + \frac{ds}{dt}\left\|\frac{d\vec{T}}{dt}\right\|\vec{N}}$$

the scalar acceleration

speed $\frac{ds}{dt} = v$

a measure of the "curvature" at $\vec{r} = \vec{r}(t)$ ...

$\kappa = \left\|\frac{d\vec{T}}{dt}\right\|$ is called the Curvature

Said differently:

$$\boxed{\vec{a} = a_T\vec{T} + a_N\vec{N}}$$

$a_T = \frac{d^2s}{dt^2}$ $\quad a_N = v\left\|\frac{d\vec{T}}{dt}\right\|$

Eg: $\vec{a}\cdot\vec{T} = \left(a_T\vec{T} + a_N\vec{N}\right)\cdot\vec{T}$

$$= a_T\underbrace{\vec{T}\cdot\vec{T}}_{1} + a_N\underbrace{\vec{N}\cdot\vec{T}}_{0} = a_T = \underbrace{\frac{d^2s}{dt^2}}_{\text{scalar acceleration}}$$

**Summary:** $\vec{a} = a_T \vec{T} + a_N \vec{N}$

$a_T$: component of $\vec{a}$ in direction $\vec{T}$

$a_N$: component of $\vec{a}$ in direction $\vec{N}$

where $a_T = \frac{dv}{dt}$, $a_N = v \left\| \frac{d\vec{T}}{dt} \right\|$ $\quad v = \frac{ds}{dt}$

So we have proven:

**Theorem:** $\vec{a} \cdot \vec{T} = \frac{dv}{dt}$, $\vec{a} \cdot \vec{N} = v \left\| \frac{d\vec{T}}{dt} \right\|$

- It remains to understand $\left\| \frac{d\vec{T}}{dt} \right\| = \left\| \frac{d\vec{T}}{ds} \frac{ds}{dt} \right\|$

$$= \underbrace{\left\| \frac{d\vec{T}}{ds} \right\|}_{K} \underbrace{\left| \frac{ds}{dt} \right|}_{V}$$

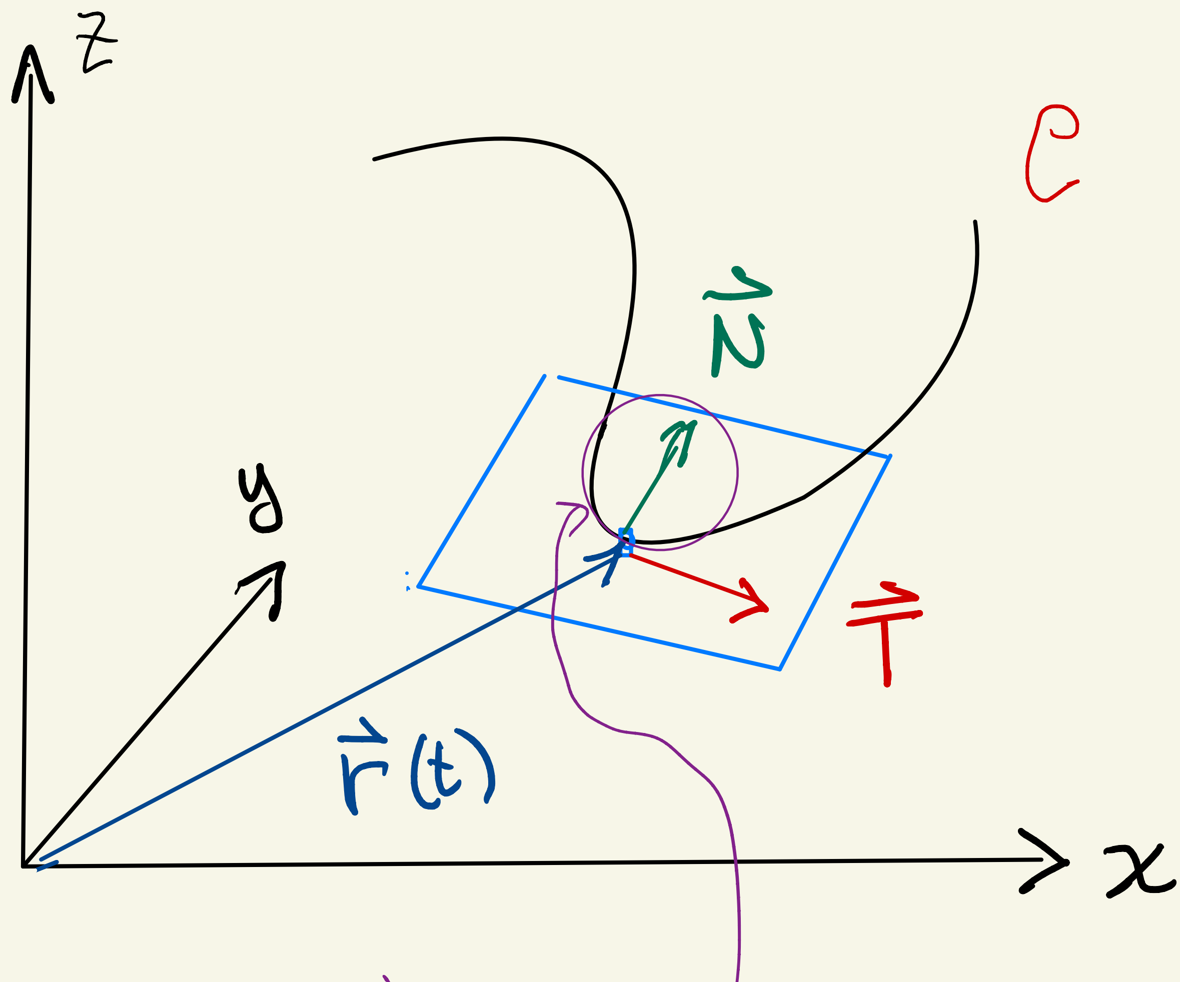
**Theorem:** $\left\| \frac{dT}{ds} \right\| = K$

where $K = \frac{1}{r} = \frac{1}{\text{radius of curvature}}$

$r$ = "radius of the circle that best fits the curve at point $\vec{r}(t)$"

**Defn:** $K = K(t) = \underline{\text{curvature}}$ of $\mathcal{C}$ at $\vec{r}(t)$

# Picture




"osculating circle"
lies in the osculating plane
r = radius of curvature

$$\kappa = \frac{1}{r}$$

**Conclude**: Geometrical Interpretation of the acceleration vector:

$$\vec{a} = a_T \vec{T} + a_N \vec{N}$$

$$= \frac{d^2s}{dt^2} \vec{T} + K v^2 \vec{N}$$

$$v = \frac{ds}{dt}$$

$a_T = \frac{d^2s}{dt^2}$ is the scalar acceleration

$r$ = radius of curvature

$$a_N = v^2 K = \frac{v^2}{r}$$

$v$ = velocity

$$\vec{T} = \frac{\vec{v}}{\|\vec{v}\|} \qquad \vec{N} = \frac{\frac{d\vec{T}}{dt}}{\left\|\frac{d\vec{T}}{dt}\right\|}$$

(assume $\vec{v} \neq 0$)

$\left( \text{or } \vec{N} = 0 \text{ if } \frac{d\vec{T}}{dt} = 0 \right)$

This is the theory – we now do some examples –