Kepler To Newton (Supplementary) ①

Understanding the acceleration vector is key to discovering Newton's Law of Gravity from Kepler's Three Laws.

- In the early 1600's (17'th century), Kepler proposed 3 Laws of planetary motion, which he deduced from careful observations of Tycho Brahe:

1st Law (1609): The planets are orbiting the sun in elliptical orbits with the sun at one focus of the ellipse.

2nd Law (1609): Planets sweep out equal area in equal time.

3rd Law (1619): T^2/a^3 = constant, the same constant for every planet:
a = major axis of ellipse, T = period

- In 1660, Robert Hooke discovered "Hooke's Law" for springs, and proposed to Newton that planetary motion might be due to an "inverse square force" emanating from the sun, exerted on the planets – something like a "spring".

- Newton proposed his Universal Law of Gravitation in 1687 (Principia)

(1) $\vec{F} = m\vec{a}$ (Giving the meaing of "force")

(2) $\vec{F} = M_p \vec{a} = -G \dfrac{M_s M_p}{r^2} \dfrac{\vec{r}}{r}$ (Giving the gravitational force exerted by the sun on planet)

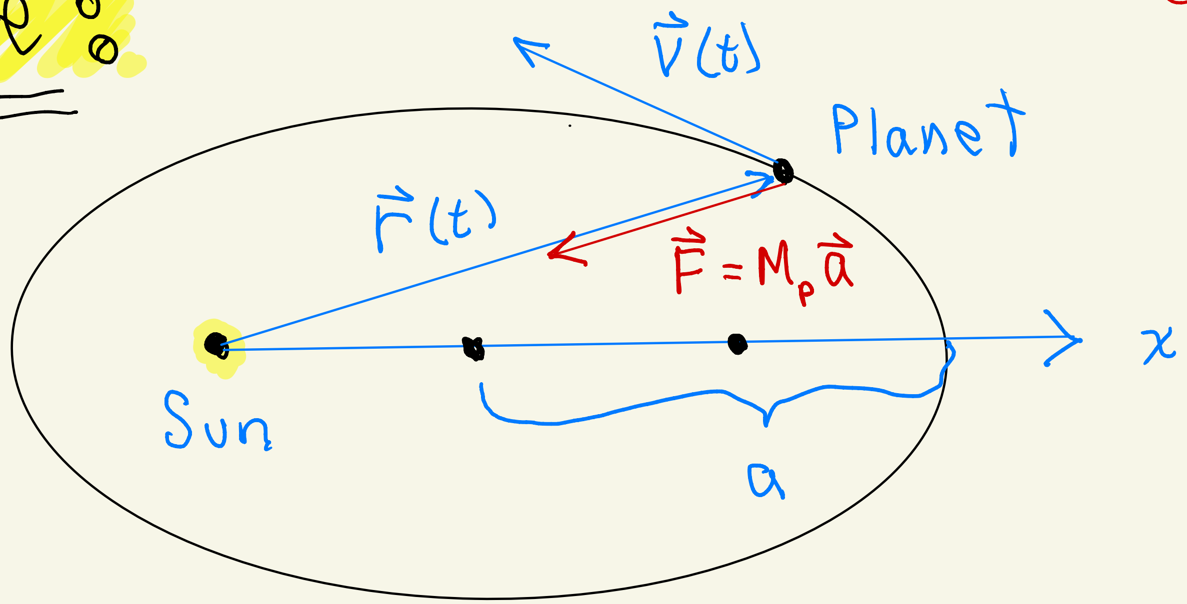
M_p = mass of planet

M_s = mass of sun

$\vec{r}$ = position vector of planet w origin @ sun

$r = \|\vec{r}\|$ = distance from planet to sun

Picture:



Newton Proposed: $\vec{F} = \cancel{M_p} \vec{a} = -G \underbrace{\frac{M_s \cancel{M_p}}{r^2}}_{\substack{\text{magnitude} \\ \text{(inverse square!)}}} \underbrace{\frac{\vec{r}}{r}}_{\text{direction}}$

$$\boxed{\vec{a} = -G \frac{M_s}{r^3} \vec{r}}$$

Million Dollar Question: Could this explain Kepler's 3 Laws? If so it would answer the age old question of why the planets move the way they do in the sky — and it would imply the sun, not the earth, is the center of everything!!

- Newton's Law of Gravitation gave a unified explanation for Kepler's Three Laws, thereby unifying all the laws of (planetary) physics known in his lifetime.

- The essence of Newton's argument is to show that, if a planet moves in an elliptical orbit with the sun at a focus, and its rotation rate is given by "equal area in equal time", then the acceleration points back toward the sun ⇒ "Everything is coming from the Sun!"

He then shows that the magnitude of the acceleration assuming elliptical motion and equal area in equal time must be inverse square!

The third law then gives the final miracle — His force law is independent of planet ⇒ Universal!

Theorem ①: If the planet sweeps out "equal area in equal time", then the acceleration vector points in direction from planet to sun.

Theorem ②: If further, the planet traverses an ellipse, then the magnitude of force is inverse square, with a constant which appears to depend on the planet.

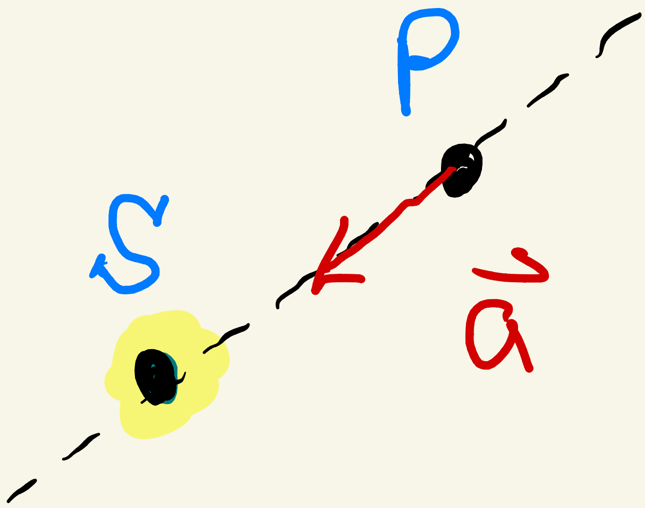
Theorem ③: If further, $\frac{T^2}{a^3}$ = constant independent of planet, then

$$(3) \qquad \boxed{\vec{a} = -\hat{G}\,\frac{1}{r^2}\,\frac{\vec{r}}{r}}$$

with $\hat{G}$ a constant independent of planet. (I.e., you get the same $\hat{G}$ for every ellipse!)

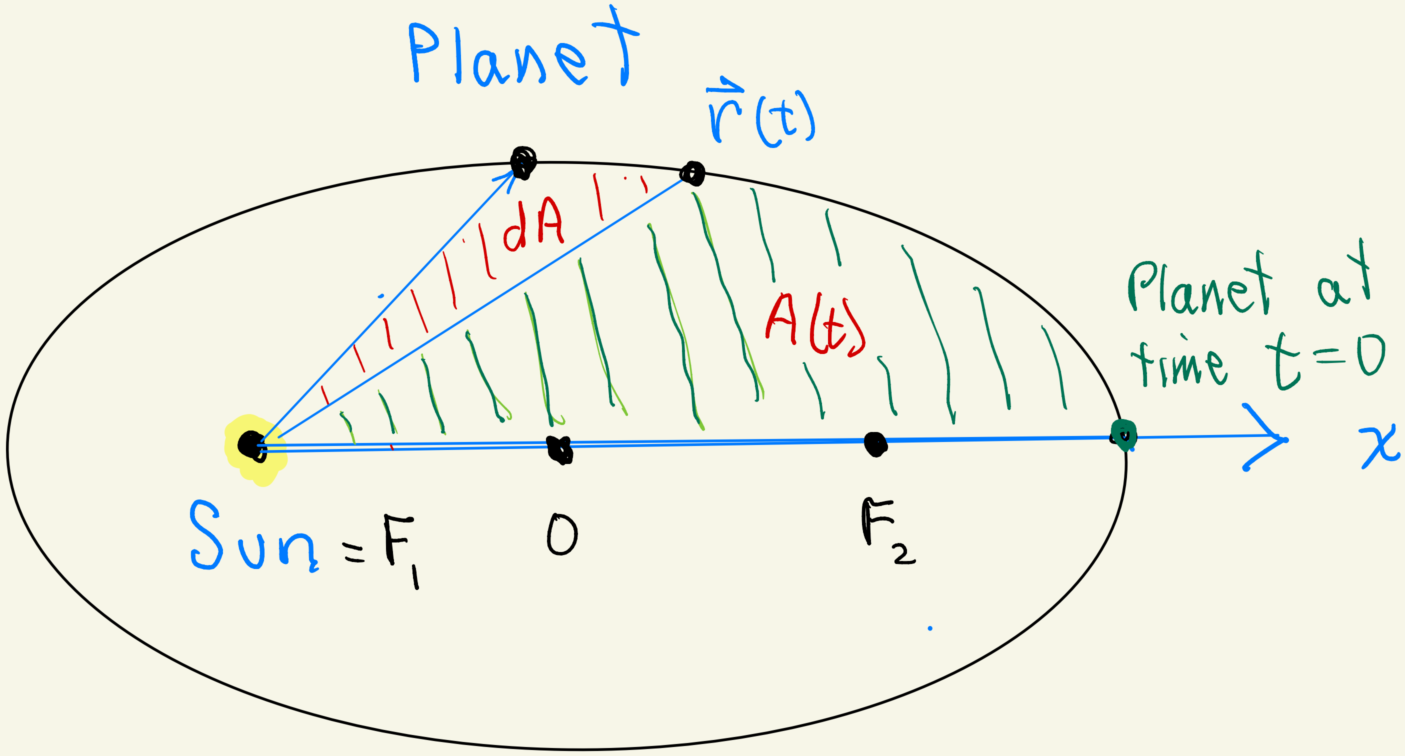
Taking $\hat{G} = GM_s$ then gives Newton's Force law.

- Proof of Theorem (1): Show that Kepler's 2nd Law, that planets are sweeping out "equal area in equal time", alone implies that the acceleration vector must point in the direction of the Sun



Note: This is the main step in making the leap to the idea that the motion of the planets is due to a force emanating from the Sun – I.e., $\vec{F} = m_p \vec{a}$ is coming from the Sun!

Meaning of "Equal Area in Equal Time"



$$\boxed{\frac{dA}{dt} = A'(t) = H = \text{constant}}$$

Constant depends on the planet's chosen elliptical orbit

Solution: Assume a planet P of mass M_p moves along a trajectory $\vec{r}(t) = x(t)\,\underset{\sim}{i} + y(t)\,\underset{\sim}{j}$ where $\vec{r}$ is the position vector that points from the sun to the planet. (Kepler knew the planets moved in a plane containing the sun, the planet and the ellipse.)

We show: "equal area in equal time" implies

$$(4) \qquad \vec{a} = \ddot{\vec{r}} = -\left(\frac{H}{r^3} - \ddot{r}\right)\frac{\vec{r}}{r}$$

Here: We have cancelled M_p from both sides of (2), $r = \|\vec{r}\| = \sqrt{x(t)^2 + y(t)^2}$ and H will be the constant associated with "equal area in equal time". To verify (4)

Notation: In physics, derivatives wrt time are denoted by a "dot" I.e.

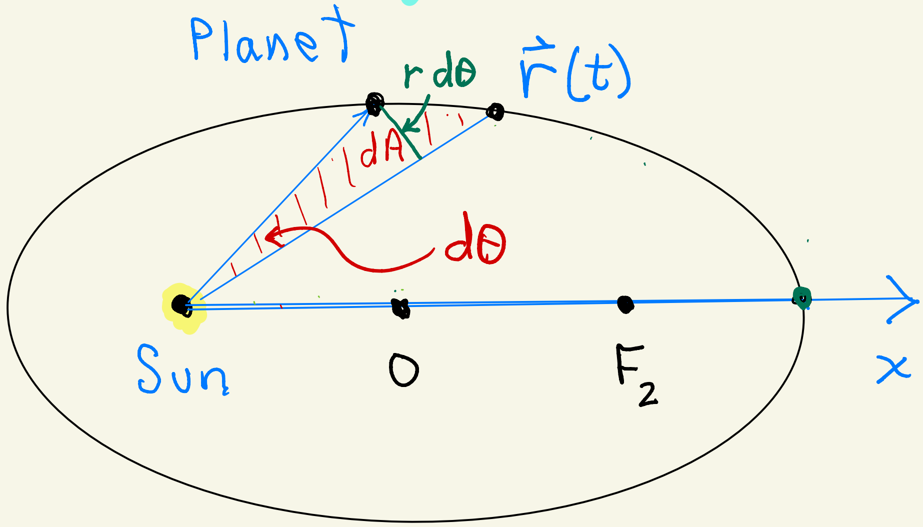
$$\dot{\vec{r}} \equiv \dot{\vec{r}}(t) \equiv \vec{r}\,'(t) = \frac{d\vec{r}}{dt}$$

So Assume: "equal area in equal time"

$\vec{r}(t) = x(t)\,\underset{\sim}{i} + y(t)\,\underset{\sim}{j}$

$r = \|\vec{r}\| = \sqrt{x^2+y^2}$

Then:



(5) $$dA = \frac{1}{2} r^2 \, d\theta$$

(This the triangular area, neglecting higher order errors which $\longrightarrow 0$ as $dA \to 0$)

Thus

(6) $$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{1}{2} r^2 \dot{\theta}$$

So "equal area in equal time" means

(7) $$\frac{dA}{dt} = \text{const.}$$ (which at this stage could depend on planet)

so

(8) $$r^2 \dot{\theta} = H$$ (We take this H as the constant)

Differentiating gives:

(9) $$0 = \frac{d}{dt}\left(r^2 \dot{\theta}\right) = 2 r \dot{r} \dot{\theta} + r^2 \ddot{\theta}$$

So "equal area in equal time" means:

(10) $$\boxed{\dot{\theta} = \frac{H}{r^2}}$$

(11) $$\boxed{2\dot{r}\dot{\theta} + r\ddot{\theta} = 0}$$

Since these conditions are given in terms of (r, θ), it makes sense to change to polar coordinates —

(12) $$x = r\cos\theta \qquad r = \sqrt{x^2+y^2}$$
$$y = r\sin\theta$$

Here: x, y, r, θ all depend on time t, and are determined by the planets position $\bar{r}(t)$ at time t.

Looking to get the direction of $\vec{a} = \ddot{\vec{r}}$, we obtain $\ddot{x}$ and $\ddot{y}$;

Differentiating (12) gives

(13)
$$\dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$$
$$\dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$$

Differentiating (13) gives

(14) $\ddot{x} = \ddot{r}\cos\theta - 2\dot{r}\dot{\theta}\sin\theta - r\dot{\theta}^2\cos\theta - r\ddot{\theta}\sin\theta$

(15) $\ddot{y} = \ddot{r}\sin\theta + 2\dot{r}\dot{\theta}\cos\theta - r\dot{\theta}^2\sin\theta + r\ddot{\theta}\cos\theta$

Multiply (14) by $\cos\theta$ & (15) by $\sin\theta$ and add:

(16) $$\boxed{\ddot{x}\cos\theta + \ddot{y}\sin\theta = \ddot{r} - r\dot{\theta}^2}$$

Multiply (14) by $\sin\theta$ & (15) by $\cos\theta$ and subtr.:

(17) $$\boxed{-\ddot{x}\sin\theta + \ddot{y}\cos\theta = 2\dot{r}\dot{\theta} + r\ddot{\theta}}$$

• Now "area in time" says:

$$r^2\dot{\theta} = H$$

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = 0$$

using these on RHS of (16) & (17) gives

(18) $$\ddot{x}\cos\theta + \ddot{y}\sin\theta = \ddot{r} - H^2/r^3$$

(19) $$-\ddot{x}\sin\theta + \ddot{y}\cos\theta = 0$$

Now use (18), (19) to solve for $\ddot{x}$ & $\ddot{y}$;

Multiply (18) by $\cos\theta$ & (19) by $\sin\theta$ and subtr:

(20) $$\boxed{\ddot{x} = \left(\ddot{r} - \frac{H^2}{r^3}\right)\cos\theta = \left(\ddot{r} - \frac{H^2}{r^3}\right)\frac{x}{r}}$$

Multiply (18) by $\sin\theta$ & (19) by $\cos\theta$ and add:

(21) $$\boxed{\ddot{y} = \left(\ddot{r} - \frac{H^2}{r^3}\right)\sin\theta = \left(\ddot{r} - \frac{H^2}{r^3}\right)\frac{y}{r}}$$

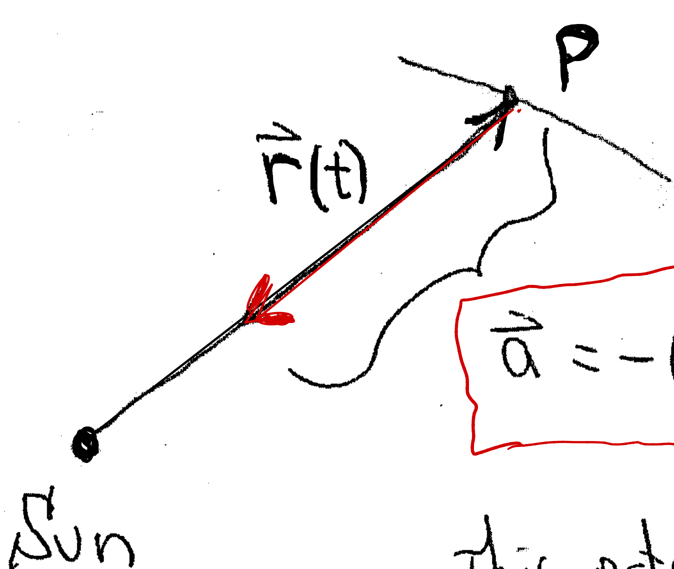
Conclude: "= area in = time" alone implies

$$\vec{a} = \ddot{x}\,\underset{\sim}{i} + \ddot{y}\,\underset{\sim}{j} = -\left(\frac{H^2}{r^3} - \ddot{r}\right)\frac{1}{r}\left(x\,\underset{\sim}{i} + y\,\underset{\sim}{j}\right)$$

or

$$\vec{a} = -\left(\frac{H^2}{r^3} - \ddot{r}\right)\frac{\vec{r}}{r} = \underbrace{-\left(\frac{H^2}{r^3} - \ddot{r}\right)}_{\text{magnitude}}\underbrace{\frac{\vec{r}}{\|\vec{r}\|}}_{\text{unit vector}} \tag{22}$$

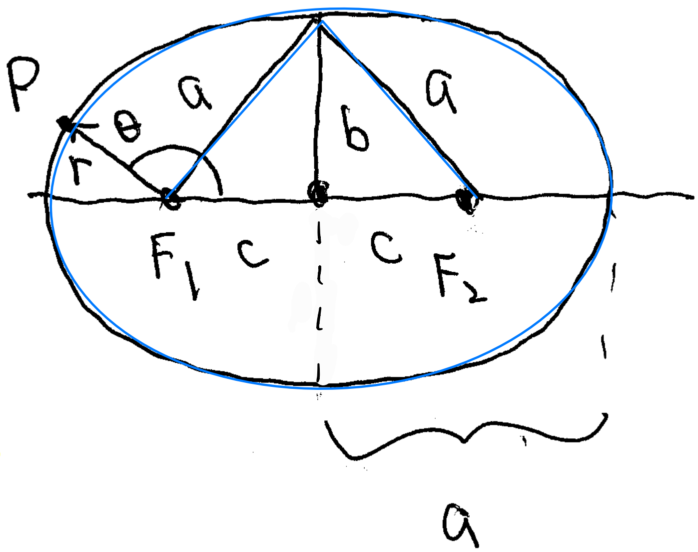
"The acceleration points back toward the sun in direction of position vector $\vec{r}$"



$$\boxed{\vec{a} = -\left(\frac{H^2}{r^3} - \ddot{r}\right)\frac{1}{r}\,\vec{r}}$$

This establishes Thm ①

a To prove theorems (2) & (3), we begin by recalling what we need to know about ellipses —



- An ellipse is defined as the set of points P such that

$$\text{Dist } PF_1 + \text{Dist } PF_2 = 2a$$

This implies $b^2 + c^2 = a^2$

a = length of major axis
b = length of minor axis
c = distance from Focus to center.

- Fact: Taking F_1 to be the origin, P is on the ellipse iff

$$r = \frac{1}{A - B\cos\theta}$$

$r = \text{Dist } PF_1$
θ = angle with line thru F_1F_2

- Using $r = \dfrac{1}{A - B\cos\theta}$, we can find A, B in terms of a, b, c as follows:

If $\theta = 0$, $r(0) = \dfrac{1}{A-B} = a + c$

$$\boxed{A - B = \frac{1}{a+c}}$$

If $\theta = \pi$, $r = \dfrac{1}{A+B} = a - c$

$$\boxed{A + B = \frac{1}{a-c}}$$

Adding gives

$$2A = \frac{1}{a-c} + \frac{1}{a+c} = \frac{\cancel{a+c} + \cancel{a-c}}{a^2 - c^2}$$

$$= \frac{2a}{a^2 - c^2} = 2\frac{a}{b^2}$$

$$\boxed{A = \frac{a}{b^2}}$$

Similarly $B = \dfrac{c}{b^2}$

☐ We can now give proof of theorem ②:

By Thm ① we have: "= area in = time" $\Rightarrow$

$$\vec{a} = -\underbrace{\left(\frac{H^2}{r^3} - \ddot{r}\right)}_{\text{magnitude}} \underbrace{\frac{\vec{r}}{r}}_{\substack{\text{direction} \\ \text{(towards sun)}}} \quad , \quad \dot{\theta} = \frac{H}{r^2}$$

we now show that assuming $r(t)$ is an ellipse traversing

$$(*) \qquad r(t) = \frac{1}{A - B\cos\theta(t)} \qquad \left(\begin{matrix}\text{giving } r \text{ as a} \\ \text{fn of } \theta\end{matrix}\right)$$

then $\dfrac{H^2}{r^3} - \ddot{r} = \dfrac{\text{Const}}{r^2}$.

First we find $\ddot{r}$. By (*),

$$r^{-1} = A - B\cos\theta \qquad (r = r(t),\ \theta = \theta(t))$$

Diff both sides wrt t:

$$\dot{\theta} = \frac{H}{r^2}$$

$$-\frac{1}{\cancel{r^2}}\dot{r} = B\sin\theta\,\dot{\theta} = B\sin\theta\,\frac{H}{\cancel{r^2}}$$

$$\dot{r} = -BH\sin\theta$$

Diff:

$$\ddot{r} = -BH\cos\theta\,\dot{\theta} = -BH\cos\theta\,\frac{H}{r^2}$$

or

$$\ddot{r} = -\frac{H^2}{r^2}\,B\cos\theta \qquad r = \frac{1}{A - B\cos\theta}$$

$$B\cos\theta = A - \frac{1}{r}$$

$$\ddot{r} = -\frac{H^2}{r^2}\left(A - \frac{1}{r}\right)$$

So

$$\ddot{r} = -\frac{H^2 A}{r^2} + \frac{H^2}{r^3}.$$

$$A = \frac{a}{b^2}$$

Thus:

$$\frac{H^2}{r^3} - \ddot{r} = +\frac{H^2 A}{r^2} = +H^2\,\frac{a}{b^2}\,\frac{1}{r^2}$$

We conclude that "= area in = time" plus $r(t) = \dfrac{1}{A - B\cos\theta(t)}$ imply

$$\vec{a} = -\left(\frac{H^2}{r^3} - \ddot{r}\right)\frac{\vec{r}}{r} = -\frac{H^2 A}{r^2}\frac{\vec{r}}{r}$$

$$\boxed{\vec{a} = -\underbrace{\frac{H^2 a}{b^2}}_{\text{constant depending on planet}}\ \underbrace{\frac{1}{r^2}}_{\text{inverse square force}}\ \frac{\vec{r}}{r}}$$

This completes the proof of Thm ② ✓

Proof of Thm 3:

Newton's idea that the force is coming from the sun, and the acceleration depends only on the distance from the sun, requires

$$AH^2 = \hat{G}$$ a universal constant independent of planet.

How do we confirm this?

The final "miracle" — this follows from Kepler's third law —

$$\frac{T^2}{a^3} = \text{constant indep}^{t} \text{ of planet}$$

T = time of one complete orbit

a = major axis of the ellipse

- To see this: Recall "= area in = time" means

$$\frac{dA}{dt} = \text{const} = 2^{-1}H \left(= 2^{-1}r^2\dot{\theta}\right)$$

$$\therefore \int_0^T \frac{dA}{dt}\,dt = \int_0^T 2^{-1}H\,dt$$

$$A(T) = 2^{-1}HT$$

$A(T)$ = area of whole ellipse $= \pi ab$

$$\pi ab = 2^{-1}HT$$

$$\boxed{T = \frac{2\pi ab}{H}}$$

That's all we need –

(1) Kepler's Third Law : $\boxed{\dfrac{T^2}{a^3} = \left(\begin{array}{c}\text{A} \\ \text{Constant} \\ \text{Indep't of} \\ \text{Planet}\end{array}\right)}$

(2) "= area in = time" + ellipse $\Rightarrow$

$$\vec{a} = -AH^2 \frac{1}{r^2} \frac{\vec{r}}{r}$$

where $A = \dfrac{a}{b^2} \Rightarrow$

$$\boxed{\vec{a} = -\frac{aH^2}{b^2} \frac{\vec{r}}{r^3}}$$

(3) "$\dfrac{dA}{dt}$ = Const." $\Rightarrow$ $\boxed{T = \dfrac{2\pi ab}{H}}$

Putting (3) into (1) gives

$$\frac{T^2}{a^3} = \frac{4\pi^2 \cancel{a^2} b^2}{H^2} \frac{1}{\cancel{a^3}_1} = \frac{4\pi^2}{1}\frac{b^2}{aH^2} = \frac{4\pi^2}{1}\frac{1}{\left(\frac{aH^2}{b^2}\right)}$$

$$\Rightarrow \left(\frac{aH^2}{b^2}\right) = 4\pi^2\left(\frac{T^2}{a^3}\right)^{-1} = \text{constant independent of the planet}$$

$$\boxed{\frac{aH^2}{b^2} = \hat{G}}$$

$\hat{G}$ indept of planet

Putting this into (2) gives:

$$\boxed{\vec{a} = -\hat{G}\frac{1}{r^2}\frac{\vec{r}}{r}}$$

<u>Conclude</u>: Newton's Law $\vec{a} = -\hat{G}\frac{1}{r^2}\frac{\vec{r}}{r}$ is consistent with all three of Kepler's Laws . . . with $\hat{G}$ a constant independent of the planet.

<u>Summary</u>: Newton unified all of the laws of planetary motion known in his time, by showing that the orbits of the planets were all explained by a gravitational force pointing toward (emanating from) the sun, with magnitude depending only on the position, and independent of the planet. It doesn't take much from here to postulate $\hat{G} = G M_S$, so

$$\boxed{\vec{F} = M_p \vec{a} = -G \frac{M_S M_p}{r^2} \frac{\vec{r}}{r}}$$

(i.e., the simplest way to make it symmetric in $M_S M_p$)

Note: We <u>never</u> had to solve a differential equation!