

## Motion on the Unit Circle

For the counterclockwise motion

$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$$

around the unit circle,

$$\mathbf{v} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}$$

is already a unit vector, so  $\mathbf{T} = \mathbf{v}$  (Figure 13.18).

The motion  $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$  (Example 5).

In Exercises 1–8, find the curve's unit tangent vector. Also, find the length of the indicated portion of the curve.

- $\mathbf{r}(t) = (2 \cos t)\mathbf{i} + (2 \sin t)\mathbf{j} + \sqrt{5}t\mathbf{k}, \quad 0 \leq t \leq \pi$
- $\mathbf{r}(t) = (6 \sin 2t)\mathbf{i} + (6 \cos 2t)\mathbf{j} + 5t\mathbf{k}, \quad 0 \leq t \leq \pi$
- $\mathbf{r}(t) = t\mathbf{i} + (2/3)t^{3/2}\mathbf{k}, \quad 0 \leq t \leq 8$
- $\mathbf{r}(t) = (2 + t)\mathbf{i} - (t + 1)\mathbf{j} + t\mathbf{k}, \quad 0 \leq t \leq 3$
- $\mathbf{r}(t) = (\cos^3 t)\mathbf{j} + (\sin^3 t)\mathbf{k}, \quad 0 \leq t \leq \pi/2$
- $\mathbf{r}(t) = 6t^3\mathbf{i} - 2t^3\mathbf{j} - 3t^3\mathbf{k}, \quad 1 \leq t \leq 2$
- $\mathbf{r}(t) = (t \cos t)\mathbf{i} + (t \sin t)\mathbf{j} + (2\sqrt{2}/3)t^{3/2}\mathbf{k}, \quad 0 \leq t \leq \pi$
- $\mathbf{r}(t) = (t \sin t + \cos t)\mathbf{i} + (t \cos t - \sin t)\mathbf{j}, \quad \sqrt{2} \leq t \leq 2$
- Find the point on the curve

$$\mathbf{r}(t) = (5 \sin t)\mathbf{i} + (5 \cos t)\mathbf{j} + 12t\mathbf{k}$$

at a distance  $26\pi$  units along the curve from the origin in the direction of increasing arc length.

- Find the point on the curve

$$\mathbf{r}(t) = (12 \sin t)\mathbf{i} - (12 \cos t)\mathbf{j} + 5t\mathbf{k}$$

at a distance  $13\pi$  units along the curve from the origin in the direction opposite to the direction of increasing arc length.

In Exercises 11–14, find the arc length parameter along the curve from the point where  $t = 0$  by evaluating the integral

$$s = \int_0^t |\mathbf{v}(\tau)| d\tau$$

from Equation (3). Then find the length of the indicated portion of the curve.

- $\mathbf{r}(t) = (4 \cos t)\mathbf{i} + (4 \sin t)\mathbf{j} + 3t\mathbf{k}, \quad 0 \leq t \leq \pi/2$
- $\mathbf{r}(t) = (\cos t + t \sin t)\mathbf{i} + (\sin t - t \cos t)\mathbf{j}, \quad \pi/2 \leq t \leq \pi$
- $\mathbf{r}(t) = (e^t \cos t)\mathbf{i} + (e^t \sin t)\mathbf{j} + e^t\mathbf{k}, \quad -\ln 4 \leq t \leq 0$
- $\mathbf{r}(t) = (1 + 2t)\mathbf{i} + (1 + 3t)\mathbf{j} + (6 - 6t)\mathbf{k}, \quad -1 \leq t \leq 0$

- Arc length** Find the length of the curve

$$\mathbf{r}(t) = (\sqrt{2}t)\mathbf{i} + (\sqrt{2}t)\mathbf{j} + (1 - t^2)\mathbf{k}$$

from  $(0, 0, 1)$  to  $(\sqrt{2}, \sqrt{2}, 0)$ .

- Length of helix** The length  $2\pi\sqrt{2}$  of the turn of the helix in Example 1 is also the length of the diagonal of a square  $2\pi$  units on a side. Show how to obtain this square by cutting away and flattening a portion of the cylinder around which the helix winds.

- Ellipse**

a. Show that the curve  $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + (1 - \cos t)\mathbf{k}$ ,  $0 \leq t \leq 2\pi$ , is an ellipse by showing that it is the intersection of a right circular cylinder and a plane. Find equations for the cylinder and plane.

b. Sketch the ellipse on the cylinder. Add to your sketch the unit tangent vectors at  $t = 0, \pi/2, \pi$ , and  $3\pi/2$ .

c. Show that the acceleration vector always lies parallel to the plane (orthogonal to a vector normal to the plane). Thus, if you draw the acceleration as a vector attached to the ellipse, it will lie in the plane of the ellipse. Add the acceleration vectors for  $t = 0, \pi/2, \pi$ , and  $3\pi/2$  to your sketch.

d. Write an integral for the length of the ellipse. Do not try to evaluate the integral; it is nonelementary.

e. **Numerical integrator** Estimate the length of the ellipse to two decimal places.

- Length is independent of parametrization** To illustrate that the length of a smooth space curve does not depend on

the parametrization you use to compute it, calculate the length of one turn of the helix in Example 1 with the following parametrizations.

a.  $\mathbf{r}(t) = (\cos 4t)\mathbf{i} + (\sin 4t)\mathbf{j} + 4t\mathbf{k}, \quad 0 \leq t \leq \pi/2$

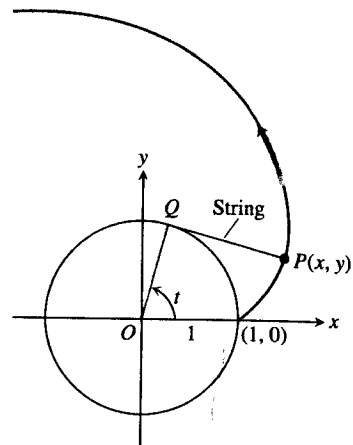
b.  $\mathbf{r}(t) = [\cos(t/2)]\mathbf{i} + [\sin(t/2)]\mathbf{j} + (t/2)\mathbf{k}, \quad 0 \leq t \leq 4\pi$

c.  $\mathbf{r}(t) = (\cos t)\mathbf{i} - (\sin t)\mathbf{j} - t\mathbf{k}, \quad -2\pi \leq t \leq 0$

19. **The involute of a circle** If a string wound around a fixed circle is unwound while held taut in the plane of the circle, its end  $P$  traces an *involute* of the circle. In the accompanying figure, the circle in question is the circle  $x^2 + y^2 = 1$  and the tracing point starts at  $(1, 0)$ . The unwound portion of the string is tangent to the circle at  $Q$ , and  $t$  is the radian measure of the angle from the positive  $x$ -axis to segment  $OQ$ . Derive the parametric equations

$$x = \cos t + t \sin t, \quad y = \sin t - t \cos t, \quad t > 0$$

of the point  $P(x, y)$  for the involute.



20. (Continuation of Exercise 19.) Find the unit tangent vector to the involute of the circle at the point  $P(x, y)$ .

## 13.4

### Curvature and the Unit Normal Vector N

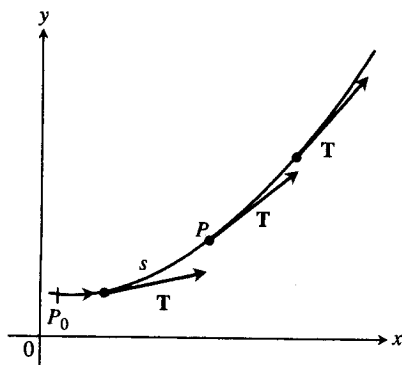


FIGURE 13.19 As  $P$  moves along the curve in the direction of increasing arc length, the unit tangent vector turns. The value of  $|d\mathbf{T}/ds|$  at  $P$  is called the *curvature* of the curve at  $P$ .

In this section we study how a curve turns or bends. We look first at curves in the coordinate plane, and then at curves in space.

#### Curvature of a Plane Curve

As a particle moves along a smooth curve in the plane,  $\mathbf{T} = d\mathbf{r}/ds$  turns as the curve bends. Since  $\mathbf{T}$  is a unit vector, its length remains constant and only its direction changes as the particle moves along the curve. The rate at which  $\mathbf{T}$  turns per unit of length along the curve is called the *curvature* (Figure 13.19). The traditional symbol for the curvature function is the Greek letter  $\kappa$  ("kappa").

#### DEFINITION Curvature

If  $\mathbf{T}$  is the unit vector of a smooth curve, the **curvature** function of the curve is

$$\kappa = \left| \frac{d\mathbf{T}}{ds} \right|.$$

If  $|d\mathbf{T}/ds|$  is large,  $\mathbf{T}$  turns sharply as the particle passes through  $P$ , and the curvature at  $P$  is large. If  $|d\mathbf{T}/ds|$  is close to zero,  $\mathbf{T}$  turns more slowly and the curvature at  $P$  is smaller.

If a smooth curve  $\mathbf{r}(t)$  is already given in terms of some parameter  $t$  other than the arc length parameter  $s$ , we can calculate the curvature as

$$\begin{aligned} \kappa &= \left| \frac{d\mathbf{T}}{ds} \right| = \left| \frac{d\mathbf{T}}{dt} \frac{dt}{ds} \right| && \text{Chain Rule} \\ &= \frac{1}{|ds/dt|} \left| \frac{d\mathbf{T}}{dt} \right| \\ &= \frac{1}{|\mathbf{v}|} \left| \frac{d\mathbf{T}}{dt} \right|. && \frac{ds}{dt} = |\mathbf{v}| \end{aligned}$$