

From these values of M , M_x , and M_y , we find

$$\bar{x} = \frac{M_y}{M} = \frac{1/12}{1/6} = \frac{1}{2} \quad \text{and} \quad \bar{y} = \frac{M_x}{M} = \frac{1/15}{1/6} = \frac{2}{5}.$$

The centroid is the point $(1/2, 2/5)$.

In Exercises 1–8, sketch the region bounded by the given lines and curves. Then express the region's area as an iterated double integral and evaluate the integral.

- The coordinate axes and the line $x + y = 2$
- The lines $x = 0$, $y = 2x$, and $y = 4$
- The parabola $x = -y^2$ and the line $y = x + 2$
- The parabola $x = y - y^2$ and the line $y = -x$
- The curve $y = e^x$ and the lines $y = 0$, $x = 0$, and $x = \ln 2$
- The curves $y = \ln x$ and $y = 2 \ln x$ and the line $x = e$, in the first quadrant
- The parabolas $x = y^2$ and $x = 2y - y^2$
- The parabolas $x = y^2 - 1$ and $x = 2y^2 - 2$

The integrals and sums of integrals in Exercises 9–14 give the areas of regions in the xy -plane. Sketch each region, label each bounding curve with its equation, and give the coordinates of the points where the curves intersect. Then find the area of the region.

- $\int_0^e \int_{y^2/3}^{2y} dx dy$
- $\int_0^3 \int_{-y}^{y(2-x)} dy dx$
- $\int_0^{\pi/4} \int_{\sin x}^{\cos x} dy dx$
- $\int_{-1}^2 \int_{y^2}^{y+2} dx dy$
- $\int_{-2}^0 \int_{-2x}^{1-x} dy dx + \int_0^2 \int_{-x/2}^{1-x} dy dx$
- $\int_0^2 \int_{y^2-4}^0 dy dx + \int_0^4 \int_0^{\sqrt{x}} dy dx$

- Find the average value of $f(x, y) = \sin(x + y)$ over
 - the rectangle $0 \leq x \leq \pi$, $0 \leq y \leq \pi$
 - the rectangle $0 \leq x \leq \pi$, $0 \leq y \leq \pi/2$
- Which do you think will be larger, the average value of $f(x, y) = xy$ over the square $0 \leq x \leq 1$, $0 \leq y \leq 1$, or the average value of f over the quarter circle $x^2 + y^2 \leq 1$ in the first quadrant? Calculate them to find out.

- Find the average height of the paraboloid $z = x^2 + y^2$ over the square $0 \leq x \leq 2$, $0 \leq y \leq 2$.

- Find the average value of $f(x, y) = 1/(xy)$ over the square $\ln 2 \leq x \leq 2 \ln 2$, $\ln 2 \leq y \leq 2 \ln 2$.

- Finding center of mass** Find a center of mass of a thin plate of density $\delta = 3$ bounded by the lines $x = 0$, $y = x$, and the parabola $y = 2 - x^2$ in the first quadrant.

- Finding moments of inertia and radii of gyration** Find the moments of inertia and radii of gyration about the coordinate axes of a thin rectangular plate of constant density δ bounded by the lines $x = 3$ and $y = 3$ in the first quadrant.

- Finding a centroid** Find the centroid of the region in the first quadrant bounded by the x -axis, the parabola $y^2 = 2x$, and the line $x + y = 4$.

- Finding a centroid** Find the centroid of the triangular region cut from the first quadrant by the line $x + y = 3$.

- Finding a centroid** Find the centroid of the semicircular region bounded by the x -axis and the curve $y = \sqrt{1 - x^2}$.

- Finding a centroid** The area of the region in the first quadrant bounded by the parabola $y = 6x - x^2$ and the line $y = x$ is $125/6$ square units. Find the centroid.

- Finding a centroid** Find the centroid of the region cut from the first quadrant by the circle $x^2 + y^2 = a^2$.

- Finding a centroid** Find the centroid of the region between the x -axis and the arch $y = \sin x$, $0 \leq x \leq \pi$.

- Finding moments of inertia** Find the moment of inertia about the x -axis of a thin plate of density $\delta = 1$ bounded by the circle $x^2 + y^2 = 4$. Then use your result to find I_y and I_0 for the plate.

- Finding a moment of inertia** Find the moment of inertia with respect to the y -axis of a thin sheet of constant density $\delta = 1$ bounded by the curve $y = (\sin^2 x)/x^2$ and the interval $\pi \leq x \leq 2\pi$ of the x -axis.

- The centroid of an infinite region** Find the centroid of the infinite region in the second quadrant enclosed by the coordinate axes and the curve $y = e^x$. (Use improper integrals in the mass-moment formulas.)

30. **The first moment of an infinite plate** Find the first moment about the y -axis of a thin plate of density $\delta(x, y) = 1$ covering the infinite region under the curve $y = e^{-x^2/2}$ in the first quadrant.

Variable Density

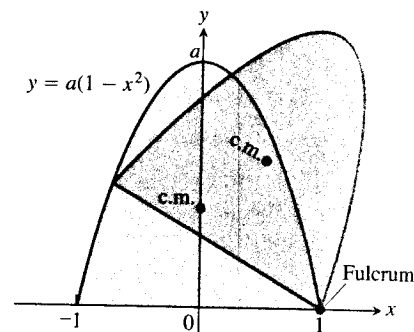
31. **Finding a moment of inertia and radius of gyration** Find the moment of inertia and radius of gyration about the x -axis of a thin plate bounded by the parabola $x = y - y^2$ and the line $x + y = 0$ if $\delta(x, y) = x + y$.
32. **Finding mass** Find the mass of a thin plate occupying the smaller region cut from the ellipse $x^2 + 4y^2 = 12$ by the parabola $x = 4y^2$ if $\delta(x, y) = 5x$.
33. **Finding a center of mass** Find the center of mass of a thin triangular plate bounded by the y -axis and the lines $y = x$ and $y = 2 - x$ if $\delta(x, y) = 6x + 3y + 3$.
34. **Finding a center of mass and moment of inertia** Find the center of mass and moment of inertia about the x -axis of a thin plate bounded by the curves $x = y^2$ and $x = 2y - y^2$ if the density at the point (x, y) is $\delta(x, y) = y + 1$.
35. **Center of mass, moment of inertia, and radius of gyration** Find the center of mass and the moment of inertia and radius of gyration about the y -axis of a thin rectangular plate cut from the first quadrant by the lines $x = 6$ and $y = 1$ if $\delta(x, y) = x + y + 1$.
36. **Center of mass, moment of inertia, and radius of gyration** Find the center of mass and the moment of inertia and radius of gyration about the y -axis of a thin plate bounded by the line $y = 1$ and the parabola $y = x^2$ if the density is $\delta(x, y) = y + 1$.
37. **Center of mass, moment of inertia, and radius of gyration** Find the center of mass and the moment of inertia and radius of gyration about the y -axis of a thin plate bounded by the x -axis, the lines $x = \pm 1$, and the parabola $y = x^2$ if $\delta(x, y) = 7y + 1$.
38. **Center of mass, moment of inertia, and radius of gyration** Find the center of mass and the moment of inertia and radius of gyration about the x -axis of a thin rectangular plate bounded by the lines $x = 0, x = 20, y = -1$, and $y = 1$ if $\delta(x, y) = 1 + (x/20)$.
39. **Center of mass, moments of inertia, and radii of gyration** Find the center of mass, the moment of inertia and radii of gyration about the coordinate axes, and the polar moment of inertia and radius of gyration of a thin triangular plate bounded by the lines $y = x, y = -x$, and $y = 1$ if $\delta(x, y) = y + 1$.
40. **Center of mass, moments of inertia, and radii of gyration** Repeat Exercise 39 for $\delta(x, y) = 3x^2 + 1$.

Theory and Examples

41. **Bacterium population** If $f(x, y) = (10,000e^y)/(1 + |x|/2)$ represents the "population density" of a certain bacterium on the xy -plane, where x and y are measured in centimeters, find the total population of bacteria within the rectangle $-5 \leq x \leq 5$ and $-2 \leq y \leq 0$.

42. **Regional population** If $f(x, y) = 100(y + 1)$ represents the population density of a planar region on Earth, where x and y are measured in miles, find the number of people in the region bounded by the curves $x = y^2$ and $x = 2y - y^2$.

43. **Appliance design** When we design an appliance, one of the concerns is how hard the appliance will be to tip over. When tipped, it will right itself as long as its center of mass lies on the correct side of the *fulcrum*, the point on which the appliance is riding as it tips. Suppose that the profile of an appliance of approximately constant density is parabolic, like an old-fashioned radio. It fills the region $0 \leq y \leq a(1 - x^2)$, $-1 \leq x \leq 1$, in the xy -plane (see accompanying figure). What values of a will guarantee that the appliance will have to be tipped more than 45° to fall over?



44. **Minimizing a moment of inertia** A rectangular plate of constant density $\delta(x, y) = 1$ occupies the region bounded by the lines $x = 4$ and $y = 2$ in the first quadrant. The moment of inertia I_a of the rectangle about the line $y = a$ is given by the integral

$$I_a = \int_0^4 \int_0^2 (y - a)^2 dy dx.$$

Find the value of a that minimizes I_a .

45. **Centroid of unbounded region** Find the centroid of the infinite region in the xy -plane bounded by the curves $y = 1/\sqrt{1 - x^2}$, $y = -1/\sqrt{1 - x^2}$, and the lines $x = 0, x = 1$.
46. **Radius of gyration of slender rod** Find the radius of gyration of a slender rod of constant linear density δ gm/cm and length L cm with respect to an axis
- through the rod's center of mass perpendicular to the rod's axis.
 - perpendicular to the rod's axis at one end of the rod.
47. (Continuation of Exercise 34.) A thin plate of now constant density δ occupies the region R in the xy -plane bounded by the curves $x = y^2$ and $x = 2y - y^2$.
- Constant density** Find δ such that the plate has the same mass as the plate in Exercise 34.
 - Average value** Compare the value of δ found in part (a) with the average value of $\delta(x, y) = y + 1$ over R .