Properties of Triple Integrals

If $F = F(x, y, z)$ and $G = G(x, y, z)$ are continuous, then

1. *Constant Multiple:* $\displaystyle\iiint_D kF\, dV = k\iiint_D F\, dV$ (any number k)

2. *Sum and Difference:* $\displaystyle\iiint_D (F \pm G)\, dV = \iiint_D F\, dV \pm \iiint_D G\, dV$

3. *Domination:*

   (a) $\displaystyle\iiint_D F\, dV \geq 0$ if $F \geq 0$ on D

   (b) $\displaystyle\iiint_D F\, dV \geq \iiint_D G\, dV$ if $F \geq G$ on D

4. *Additivity:* $\displaystyle\iiint_D F\, dV = \iiint_{D_1} E\, dV + \iiint_{D_2} F\, dV$

   if D is the union of two nonoverlapping regions D_1 and D_2.

EXERCISES 15.4

## Evaluating Triple Integrals in Different Iterations

1. Evaluate the integral in Example 2 taking $F(x, y, z) = 1$ to find the volume of the tetrahedron.

2. **Volume of rectangular solid** Write six different iterated triple integrals for the volume of the rectangular solid in the first octant bounded by the coordinate planes and the planes $x = 1, y = 2,$ and $z = 3$. Evaluate one of the integrals.

3. **Volume of tetrahedron** Write six different iterated triple integrals for the volume of the tetrahedron cut from the first octant by the plane $6x + 3y + 2z = 6$. Evaluate one of the integrals.

4. **Volume of solid** Write six different iterated triple integrals for the volume of the region in the first octant enclosed by the cylinder $x^2 + z^2 = 4$ and the plane $y = 3$. Evaluate one of the integrals.

5. **Volume enclosed by paraboloids** Let D be the region bounded by the paraboloids $z = 8 - x^2 - y^2$ and $z = x^2 + y^2$. Write six different triple iterated integrals for the volume of D. Evaluate one of the integrals.

6. **Volume inside paraboloid beneath a plane** Let D be the region bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 2y$. Write triple iterated integrals in the order $dz\, dx\, dy$ and $dz\, dy\, dx$ that give the volume of D. Do not evaluate either integral.

## Evaluating Triple Iterated Integrals

Evaluate the integrals in Exercises 7–20.

7. $\displaystyle\int_0^1\int_0^1\int_0^1 (x^2 + y^2 + z^2)\, dz\, dy\, dx$

8. $\displaystyle\int_0^{\sqrt{2}}\int_0^{3y}\int_{x^2+3y^2}^{8-x^2-y^2} dz\, dx\, dy$

9. $\displaystyle\int_1^e\int_1^e\int_1^e \frac{1}{xyz}\, dx\, dy\, dz$

10. $\displaystyle\int_0^1\int_0^{3-3x}\int_0^{3-3x-y} dz\, dy\, dx$

11. $\displaystyle\int_0^1\int_0^\pi\int_0^\pi y\sin z\, dx\, dy\, dz$

12. $\displaystyle\int_{-1}^1\int_{-1}^1\int_{-1}^1 (x + y + z)\, dy\, dx\, dz$

13. $\displaystyle\int_0^3\int_0^{\sqrt{9-x^2}}\int_0^{\sqrt{9-x^2}} dz\, dy\, dx$

14. $\displaystyle\int_0^2\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_0^{2x+y} dz\, dx\, dy$

15. $\displaystyle\int_0^1\int_0^{2-x}\int_0^{2-x-y} dz\, dy\, dx$

16. $\displaystyle\int_0^1\int_0^{1-x^2}\int_3^{4-x^2-y} x\, dz\, dy\, dx$

17. $\displaystyle\int_0^\pi\int_0^\pi\int_0^\pi \cos(u + v + w)\, du\, dv\, dw$ (uvw-space)

18. $\displaystyle\int_1^e\int_1^e\int_1^e \ln r \ln s \ln t\, dt\, dr\, ds$ (rst-space)

19. $\displaystyle\int_0^{\pi/4}\int_0^{\ln \sec v}\int_{-\infty}^{2t} e^x\, dx\, dt\, dv$ (tvx-space)
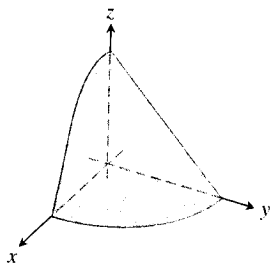
28. The region in the first octant bounded by the coordinate planes, the plane $y = 1 - x$, and the surface $z = \cos(\pi x/2)$, $0 \le x \le 1$
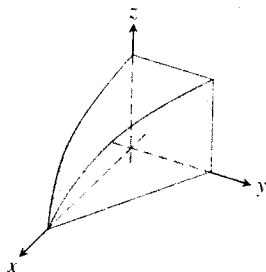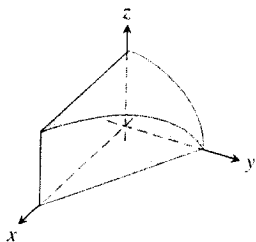

29. The region common to the interiors of the cylinders $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$, one-eighth of which is shown in the accompanying figure.
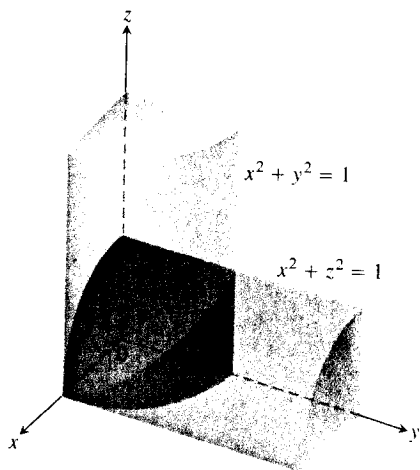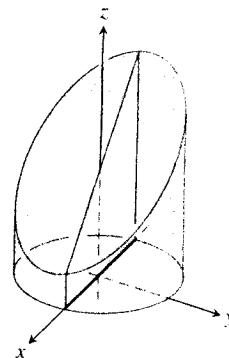

30. The region in the first octant bounded by the coordinate planes and the surface $z = 4 - x^2 - y$

31. The region in the first octant bounded by the coordinate planes, the plane $x + y = 4$, and the cylinder $y^2 + 4z^2 = 16$

32. The region cut from the cylinder $x^2 + y^2 = 4$ by the plane $z = 0$ and the plane $x + z = 3$

33. The region between the planes $x + y + 2z = 2$ and $2x + 2y + z = 4$ in the first octant

34. The finite region bounded by the planes $z = x$, $x + z = 8$, $z = y$, $y = 8$, and $z = 0$.

35. The region cut from the solid elliptical cylinder $x^2 + 4y^2 \le 4$ by the xy-plane and the plane $z = x + 2$

36. The region bounded in back by the plane $x = 0$, on the front and sides by the parabolic cylinder $x = 1 - y^2$, on the top by the paraboloid $z = x^2 + y^2$, and on the bottom by the xy-plane

Average Values

In Exercises 37–40, find the average value of $F(x, y, z)$ over the given region.

37. $F(x, y, z) = x^2 + 9$ over the cube in the first octant bounded by the coordinate planes and the planes $x = 2$, $y = 2$, and $z = 2$

38. $F(x, y, z) = x + y - z$ over the rectangular solid in the first octant bounded by the coordinate planes and the planes $x = 1$, $y = 1$, and $z = 2$

39. $F(x, y, z) = x^2 + y^2 + z^2$ over the cube in the first octant bounded by the coordinate planes and the planes $x = 1$, $y = 1$, and $z = 1$

40. $F(x, y, z) = xyz$ over the cube in the first octant bounded by the coordinate planes and the planes $x = 2$, $y = 2$, and $z = 2$

Changing the Order of Integration

Evaluate the integrals in Exercises 41–44 by changing the order of integration in an appropriate way.

41. $\displaystyle\int_0^4 \int_0^1 \int_{2y}^2 \frac{4\cos(x^2)}{2\sqrt{z}}\, dx\, dy\, dz$

42. $\displaystyle\int_0^1 \int_0^1 \int_{x^2}^1 12xze^{zy^2}\, dy\, dx\, dz$

43. $\displaystyle\int_0^1 \int_{\sqrt[3]{z}}^1 \int_0^{\ln 3} \frac{\pi e^{2x}\sin \pi y^2}{y^2}\, dx\, dy\, dz$

44. $\displaystyle\int_0^2 \int_0^{4-x^2} \int_0^x \frac{\sin 2z}{4 - z}\, dy\, dz\, dx$