Section 13.3

1.) $\vec{r}(t) = (2\cos t)\vec{i} + (2\sin t)\vec{j} + \sqrt{5}\cdot t\,\vec{k} \xrightarrow{D}$

$\vec{v}(t) = (-2\sin t)\vec{i} + (2\cos t)\vec{j} + \sqrt{5}\cdot\vec{k}$ and

$$|\vec{v}(t)| = \sqrt{(-2\sin t)^2 + (2\cos t)^2 + (\sqrt{5})^2}$$

$$= \sqrt{4\sin^2 t + 4\cos^2 t + 5}$$

$$= \sqrt{4(\underbrace{\sin^2 t + \cos^2 t}_{1}) + 5}$$

$$= \sqrt{9} = 3 \text{ , so}$$

$$\vec{T}(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|} = \left(\frac{-2}{3}\sin t\right)\vec{i} + \left(\frac{2}{3}\cos t\right)\vec{j} + \frac{\sqrt{5}}{3}\cdot\vec{k} \text{ ;}$$

$$ARC = \int_0^{\pi} \sqrt{(-2\sin t)^2 + (2\cos t)^2 + 5}\, dt$$

$$= \int_0^{\pi} |\vec{v}(t)|\, dt = \int_0^{\pi} 3\, dt = 3t \Big|_0^{\pi} = 3\pi .$$

3.) $\vec{r}(t) = t\cdot\vec{i} + 0\cdot\vec{j} + \frac{2}{3}t^{3/2}\cdot\vec{k} \xrightarrow{D}$

$\vec{v}(t) = (1)\vec{i} + (0)\vec{j} + (t^{1/2})\vec{k}$ and

$$|\vec{v}(t)| = \sqrt{(1)^2 + (0)^2 + (t^{1/2})^2} = \sqrt{1+t} \text{, so}$$

$$\vec{T}(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|} = \frac{t}{\sqrt{1+t}}\cdot\vec{i} + \frac{0}{\sqrt{1+t}}\vec{j} + \frac{\frac{2}{3}t^{3/2}}{\sqrt{1+t}}\cdot\vec{k} \text{ ;}$$

$$ARC = \int_0^8 |\vec{v}(t)|\,dt = \int_0^8 \sqrt{1+t}\,dt$$

$$= \frac{2}{3}(1+t)^{3/2}\Big|_0^8 = \frac{2}{3}(9)^{3/2} - \frac{2}{3}(1)^{3/2}$$

$$= \frac{2}{3}(27) - \frac{2}{3}(1) = \frac{52}{3}$$

6.) $\vec{r}(t) = 6t^3 \cdot \vec{i} + (-2t^3)\cdot\vec{j} + (-3t^3)\vec{k} \xrightarrow{D}$

$\vec{v}(t) = 18t^2 \cdot \vec{i} + (-6t^2)\cdot\vec{j} + (-9t^2)\vec{k}$ and

$$|\vec{v}(t)| = \sqrt{(18t^2)^2 + (-6t^2)^2 + (-9t^2)^2}$$

$$= \sqrt{324t^4 + 36t^4 + 81t^4}$$

$$= \sqrt{441t^4} = 21t^2 \text{ , so}$$

$$\vec{T}(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|} = \frac{18t^2}{21t^2}\cdot\vec{i} + \frac{-6t^2}{21t^2}\cdot\vec{j} + \frac{-9t^2}{21t^2}\vec{k}$$

$$= \frac{6}{7}\vec{i} + \frac{-2}{7}\vec{j} + \frac{-3}{7}\vec{k} \text{ ;}$$

$$ARC = \int_1^2 |\vec{v}(t)|\,dt = \int_1^2 21t^2\,dt$$

$$= 7t^3\Big|_1^2 = 7(8) - 7(1) = 49$$

7.) $\vec{r}(t) = (t\cos t)\vec{i} + (t\sin t)\vec{j} + \frac{2\sqrt{2}}{3}t^{3/2}\vec{k} \xrightarrow{D}$

$\vec{v}(t) = (\cos t - t\sin t)\vec{i} + (t\cos t + \sin t)\vec{j} + \sqrt{2}\,t^{1/2}\cdot\vec{k}$ and

$$|\vec{v}(t)| = \sqrt{(\cos t - t\sin t)^2 + (t\cos t + \sin t)^2 + 2t}$$

$$= \sqrt{\begin{array}{l}\cos^2 t - 2t\cancel{\cos t}\sin t + t^2\sin^2 t \\ \quad + t^2\cos^2 t + 2t\cancel{\cos t}\sin t + \sin^2 t + 2t\end{array}}$$

$$= \sqrt{(\underbrace{\cos^2 t + \sin^2 t}_{1}) + t^2(\underbrace{\cos^2 t + \sin^2 t}_{1}) + 2t}$$

$$= \sqrt{t^2 + 2t + 1} = \sqrt{(t+1)^2} = t+1, \text{ so}$$

$$\vec{T}(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|} = \frac{\cos t - t\sin t}{t+1}\cdot\vec{i} + \frac{t\cos t + \sin t}{t+1}\cdot\vec{j}$$

$$+ \frac{\sqrt{2}\cdot t^{1/2}}{t+1}\cdot\vec{k}$$

9.) $\vec{r}(t) = (5\sin t)\vec{i} + (5\cos t)\vec{j} + 12t\cdot\vec{k}$ for $t \geq 0$

$\xrightarrow{D}$ $\vec{v}(t) = (5\cos t)\vec{i} + (-5\sin t)\vec{j} + (12)\vec{k}$, then

$$|\vec{v}(t)| = \sqrt{(5\cos t)^2 + (-5\sin t)^2 + (12)^2}$$

$$= \sqrt{25\cos^2 t + 25\sin^2 t + 144}$$

$$= \sqrt{25(\underbrace{\cos^2 t + \sin^2 t}_{1}) + 144}$$

$$= \sqrt{169} = 13 \text{ ; thus}$$

$$ARC = 26\pi = \int_0^A |\vec{v}(t)|\,dt = \int_0^A 13\,dt$$

$= 13t \Big|_0^A = 13A \rightarrow \quad 26\pi = 13A \rightarrow$

$A = 2\pi$, so that point on curve is

$(5\sin \overset{0}{2\pi},\ 5\cos \overset{1}{2\pi},\ 12(2\pi)) = (0, 5, 24\pi)$

12.) $\vec{r}(t) = (\cos t + t\sin t)\vec{i} + (\sin t - t\cos t)\vec{j} \xrightarrow{D}$

$\vec{v}(t) = (-\cancel{\sin t} + t\cos t + \cancel{\sin t})\vec{i}$

$+ (\cancel{\cos t} - (-t\sin t + \cancel{\cos t}))\vec{j}$

$= t\cos t \cdot \vec{i} + t\sin t \cdot \vec{j}$, so

$|\vec{v}(t)| = \sqrt{(t\cos t)^2 + (t\sin t)^2}$

$= \sqrt{t^2\cos^2 t + t^2\sin^2 t} = \sqrt{t^2(\cos^2 t + \sin^2 t)}$

$= \sqrt{t^2(1)} = \sqrt{t^2} = t$; then

arc length

$S = \int_0^t |\vec{v}(\tau)|\, d\tau = \int_0^t \tau\, d\tau = \frac{1}{2}\tau^2 \Big|_0^t = \frac{1}{2}t^2$

or $\underbrace{S = \frac{1}{2}t^2}$ and $\underbrace{t = \sqrt{2S}}$, so that

$\vec{r}(t(S)) = (\cos\sqrt{2S} + \sqrt{2S}\sin\sqrt{2S})\vec{i}$

$+ (\sin\sqrt{2S} - \sqrt{2S}\cos\sqrt{2S})\vec{j}$;

arc length for $\frac{\pi}{2} \le t \le \pi$ is

$S(\pi) - S(\frac{\pi}{2}) = \frac{1}{2}(\pi)^2 - \frac{1}{2}(\frac{\pi}{2})^2$

$$= \frac{\pi^2}{2} - \frac{\pi^2}{8} = \frac{3}{8}\pi^2$$

13.) $\vec{r}(t) = (e^t\cos t)\vec{i} + (e^t\sin t)\vec{j} + e^t\vec{k} \xrightarrow{D}$

$$\vec{v}(t) = (e^t\cos t - e^t\sin t)\vec{i} + (e^t\sin t + e^t\cos t)\vec{j} + e^t\cdot\vec{k}, \text{ so}$$

$$|\vec{v}(t)| = \sqrt{(e^t\cos t - e^t\sin t)^2 + (e^t\sin t + e^t\cos t)^2 + (e^t)^2}$$

$$= \sqrt{e^{2t}\cos^2 t - \cancel{2e^{2t}\cos t\sin t} + e^{2t}\sin^2 t + e^{2t}\sin^2 t + \cancel{2e^{2t}\cos t\sin t} + e^{2t}\cos^2 t + e^{2t}}$$

$$= \sqrt{2e^{2t}\cos^2 t + 2e^{2t}\sin^2 t + e^{2t}}$$

$$= \sqrt{2e^{2t}(\underbrace{\cos^2 t + \sin^2 t}_{1}) + e^{2t}}$$

$$= \sqrt{3e^{2t}} = \sqrt{3}\cdot e^t; \text{ then arc length}$$

$$s = \int_{-\ln 4}^{t} |\vec{v}(\tau)|\,d\tau = \int_{-\ln 4}^{t} \sqrt{3}\cdot e^{\tau}\,d\tau$$

$$= \sqrt{3}\,e^{\tau}\Big|_{-\ln 4}^{t} = \sqrt{3}\,e^t - \sqrt{3}\,e^{-\ln 4}$$

$$= \sqrt{3}\,e^t - \sqrt{3}\,e^{\ln 4^{-1}} = \sqrt{3}\cdot e^t - \sqrt{3}\cdot\frac{1}{4} \rightarrow$$

$$\boxed{s = \sqrt{3}\,e^t - \frac{\sqrt{3}}{4}} \rightarrow s + \frac{\sqrt{3}}{4} = \sqrt{3}\,e^t \rightarrow$$

$$\frac{s}{\sqrt{3}} + \frac{1}{4} = e^t \rightarrow \ln\left(\frac{s}{\sqrt{3}} + \frac{1}{4}\right) = \ln e^t \rightarrow$$

$$\boxed{t = \ln\left(\frac{s}{\sqrt{3}} + \frac{1}{4}\right)}$$ so that

$$\vec{r}(t(s)) = \left(e^{\ln\left(\frac{s}{\sqrt{3}}+\frac{1}{4}\right)} \cos\left(\ln\left(\frac{s}{\sqrt{3}} + \frac{1}{4}\right)\right)\right)\vec{i}$$
$$+ \left(e^{\ln\left(\frac{s}{\sqrt{3}}+\frac{1}{4}\right)} \sin\left(\ln\left(\frac{s}{\sqrt{3}} + \frac{1}{4}\right)\right)\right)\vec{j}$$
$$+ e^{\ln\left(\frac{s}{\sqrt{3}}+\frac{1}{4}\right)}\vec{k} \longrightarrow$$

$$\vec{r}(t(s)) = \left(\frac{s}{\sqrt{3}} + \frac{1}{4}\right)\cos\left(\ln\left(\frac{s}{\sqrt{3}} + \frac{1}{4}\right)\right)\cdot \vec{i}$$
$$+ \left(\frac{s}{\sqrt{3}} + \frac{1}{4}\right)\sin\left(\ln\left(\frac{s}{\sqrt{3}} + \frac{1}{4}\right)\right)\cdot \vec{j} + \left(\frac{s}{\sqrt{3}} + \frac{1}{4}\right)\cdot\vec{k};$$

arc length for $-\ln 4 \le t \le 0$ is

$$s(0) - s(-\ln 4) = \left(\sqrt{3}\cdot e^0 - \cancel{\frac{\sqrt{3}}{4}}\right) - \left(\sqrt{3}\, e^{-\ln 4} - \cancel{\frac{\sqrt{3}}{4}}\right)$$
$$= \sqrt{3} - \sqrt{3}\cdot\frac{1}{4} = \frac{3\sqrt{3}}{4}.$$

14.) $\vec{r}(t) = (1+2t)\vec{i} + (1+3t)\vec{j} + (6-6t)\vec{k} \xrightarrow{D}$

$\vec{v}(t) = 2\vec{i} + 3\vec{j} + (-6)\vec{k}$ so

$|\vec{v}(t)| = \sqrt{2^2 + 3^2 + (-6)^2} = \sqrt{49} = 7;$

then arc length

$$s = \int_{-1}^{t} |\vec{v}(\tau)|\, d\tau = \int_{-1}^{t} 7\, d\tau = 7\tau\Big|_{-1}^{t}$$

$= 7t - 7(-1)$ or $\boxed{S = 7t + 7}$ and

~~$7t = S - 7 \rightarrow$~~ $\boxed{t = \frac{1}{7}(S-7)}$ ~~so that~~

$$\vec{r}(t(S)) = \left(1 + \tfrac{2}{7}(S-7)\right)\vec{i} + \left(1 + \tfrac{3}{7}(S-7)\right)\vec{j} + \left(6 - \tfrac{6}{7}(S-7)\right)\vec{k}$$

$$= \left(\tfrac{2}{7}S - 1\right)\vec{i} + \left(\tfrac{3}{7}S - 2\right)\vec{j} + \left(12 - \tfrac{6}{7}S\right)\vec{k} \; ;$$

arc length for $-1 \le t \le 0$ is

~~$S(0) - S(-1) = 7 - (0) = 7$~~

17.) $\vec{r}(t) = \cos t \cdot \vec{i} + \sin t \cdot \vec{j} + (1 - \cos t)\vec{k} \longrightarrow$

$$\begin{cases} x = \cos t \\ y = \sin t \\ z = 1 - \cos t \end{cases} \quad \text{for } 0 \le t \le 2\pi \longrightarrow$$

a.) $x^2 + y^2 = \cos^2 t + \sin^2 t = 1 \longrightarrow$

$x^2 + y^2 = 1$ is cylinder centered on z-axis ; and $z = 1 - x$ is plane ; $x + z = 1 \rightarrow (1)x + (0)y + (1)z = 1$ so vector normal to plane is

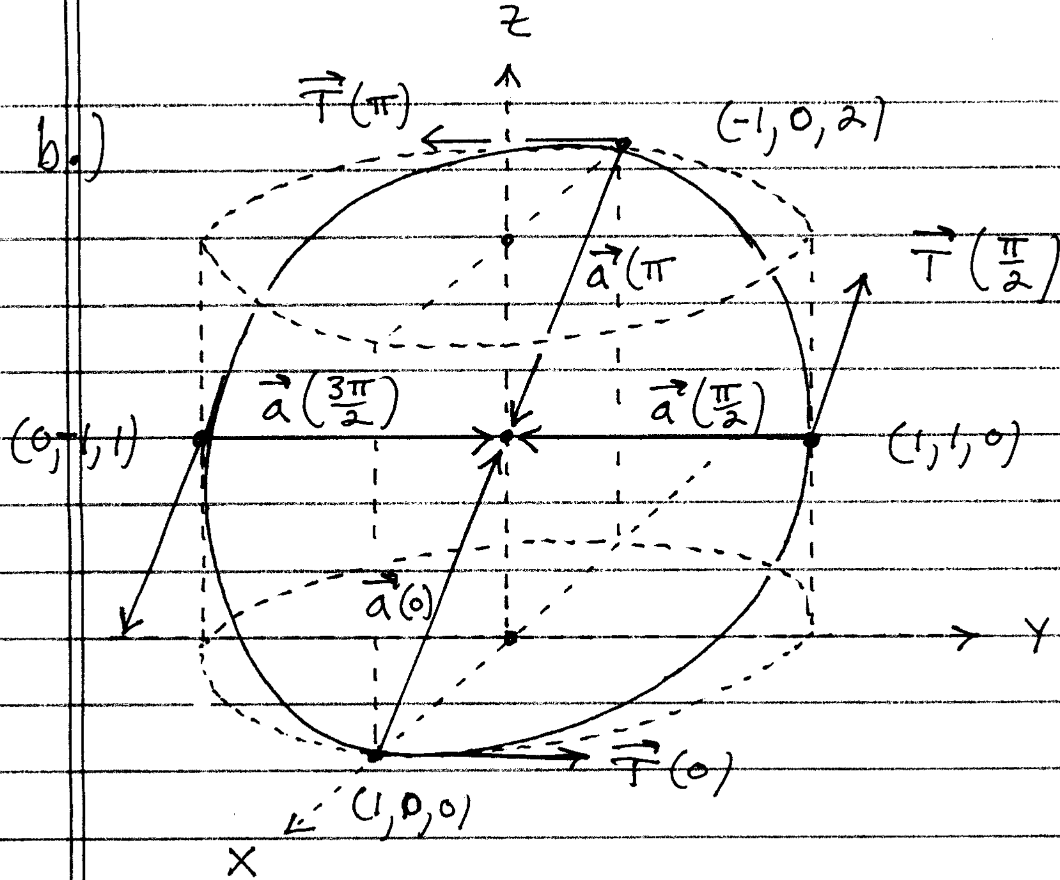
$\vec{n} = 1 \cdot \vec{i} + 0 \cdot \vec{j} + 1 \cdot \vec{k}$; $\xrightarrow{D}$

$\vec{v}(t) = (-\sin t)\vec{i} + (\cos t)\vec{j} + (\sin t)\vec{k}$ and

$$|\vec{v}(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2 + (\sin t)^2}$$

$$= \sqrt{1 + \sin^2 t} \; .$$

b.)



$$\vec{T}(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|} = \frac{-\sin t}{\sqrt{1+\sin^2 t}}\,\vec{i} + \frac{\cos t}{\sqrt{1+\sin^2 t}}\,\vec{j} + \frac{\sin t}{\sqrt{1+\sin^2 t}}\,\vec{k} \;;$$

$t=0$: pt. $(1,0,0)$ and $\vec{T}(0) = \vec{j}$,

$t=\frac{\pi}{2}$: pt. $(0,1,1)$ and $\vec{T}(\frac{\pi}{2}) = -\vec{i} + \vec{k}$

$t=\pi$: pt. $(-1,0,2)$ and $\vec{T}(\pi) = -\vec{j}$

$t=\frac{3\pi}{2}$: pt $(0,-1,1)$ and $\vec{T}(\frac{3\pi}{2}) = \vec{i} - \vec{k}$

c.) $\vec{a}(t) = \vec{v}\,'(t) = (-\cos t)\vec{i} + (-\sin t)\vec{j} + (\cos t)\vec{k}$;

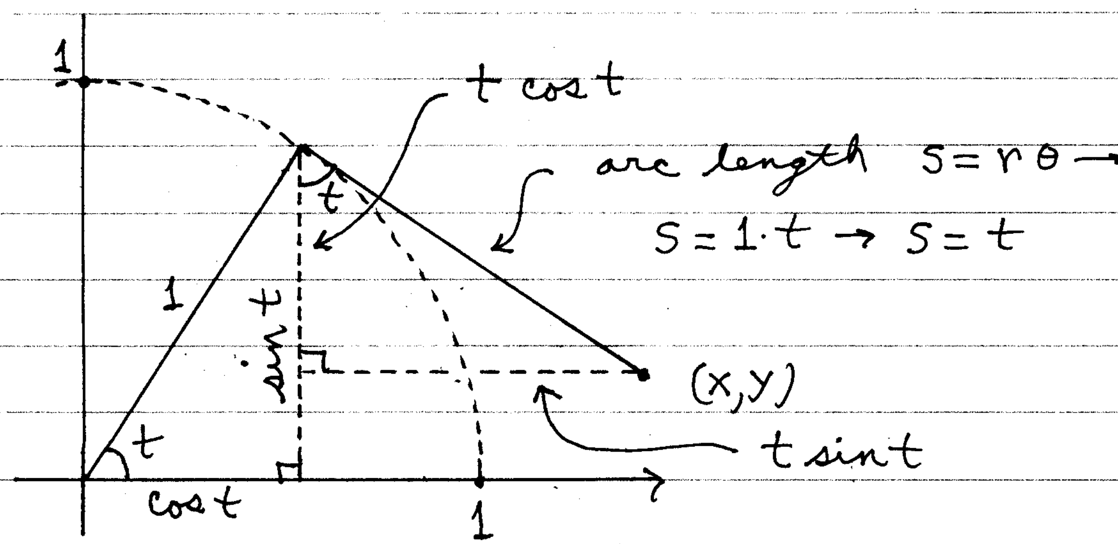
$$\vec{n}\cdot\vec{a}(t) = (1\cdot\vec{i} + 0\cdot\vec{j} + 1\cdot\vec{k})\cdot\vec{a}(t)$$
$$= -\cos t + \cos t = 0$$

so all acceleration vectors lie in plane $z = 1 - x$;

$\vec{a}(0) = -\vec{i} + \vec{k}$, $\vec{a}\left(\frac{\pi}{2}\right) = -\vec{j}$,
$\vec{a}(\pi) = \vec{i} - \vec{k}$, $\vec{a}\left(\frac{3\pi}{2}\right) = \vec{j}$

d.) $ARC = \int_0^{2\pi} |\vec{v}(t)| dt = \int_0^{2\pi} \sqrt{1+\sin^2 t}\, dt$

19.)



$$\begin{cases} x = \cos t + t \sin t \\ y = \sin t - t \cos t \end{cases}$$

20.) $\vec{r}(t) = (\cos t + t \sin t)\,\vec{i}$
$\qquad + (\sin t - t \cos t)\,\vec{j} \xrightarrow{D}$

$\vec{v}(t) = (-\sin t + t \cos t + \sin t)\,\vec{i}$
$\qquad + (\cos t - (-t \sin t + \cos t))\,\vec{j}$

$\rightarrow \vec{v}(t) = (t\sin t)\,\vec{i} + (t\cos t)\,\vec{j}$ and

$$|\vec{v}(t)| = \sqrt{(t\sin t)^2 + (t\cos t)^2}$$

$$= \sqrt{t^2\sin^2 t + t^2\cos^2 t}$$

$$= \sqrt{t^2(\underbrace{\sin^2 t + \cos^2 t}_{1})}$$

$$= \sqrt{t^2} = t \; ; \; \text{~~then~~}$$

$$\vec{T}(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|} = \frac{\vec{v}(t)}{t}$$

$$= \frac{\cancel{t}\sin t}{\cancel{t}}\,\vec{i} + \frac{\cancel{t}\cos t}{\cancel{t}}\,\vec{j}$$

$$= (\sin t)\,\vec{i} + (\cos t)\,\vec{j}$$