Section 15.4

7.) $\int_0^1 \int_0^1 \int_0^1 (x^2+y^2+z^2)\,dz\,dy\,dx$

$= \int_0^1 \int_0^1 \left(x^2z+y^2z+\frac{1}{3}z^3\right)\Big|_{z=0}^{z=1} dy\,dx$

$= \int_0^1 \int_0^1 \left(x^2+y^2+\frac{1}{3}\right) dy\,dx$

$= \int_0^1 \left(x^2y+\frac{1}{3}y^3+\frac{1}{3}y\right)\Big|_{y=0}^{y=1} dx$

$= \int_0^1 \left(x^2+\frac{1}{3}+\frac{1}{3}\right) dx = \int_0^1 \left(x^2+\frac{2}{3}\right) dx$

$= \left(\frac{1}{3}x^3+\frac{2}{3}x\right)\Big|_0^1 = \frac{1}{3}+\frac{2}{3} = 1$

8.) $\int_0^{\sqrt{2}} \int_0^{3y} \int_{x^2+3y^2}^{8-x^2-y^2} dz\,dx\,dy$

$= \int_0^{\sqrt{2}} \int_0^{3y} \left(z\Big|_{z=x^2+3y^2}^{z=8-x^2-y^2}\right) dx\,dy$

$= \int_0^{\sqrt{2}} \int_0^{3y} \left[(8-x^2-y^2)-(x^2+3y^2)\right] dx\,dy$

$= \int_0^{\sqrt{2}} \int_0^{3y} (8-2x^2-4y^2)\,dx\,dy$

$= \int_0^{\sqrt{2}} \left(8x-\frac{2}{3}x^3-4y^2x\right)\Big|_{x=0}^{x=3y} dy$

$= \int_0^{\sqrt{2}} \left(8(3y)-\frac{2}{3}(3y)^3-4y^2(3y)\right) dy$

$= \int_0^{\sqrt{2}} (24y-18y^3-12y^3)\,dy = \int_0^{\sqrt{2}} (24y-30y^3)\,dy$

$= \left(12y^2-\frac{15}{2}y^4\right)\Big|_0^{\sqrt{2}} = 24-\frac{15}{2}(4) = -6$

11.) $\int_0^1 \int_0^\pi \int_0^\pi y \sin z \, dx \, dy \, dz$

$= \int_0^1 \int_0^\pi \left(y \sin z \cdot x \Big|_{x=0}^{x=\pi} \right) dy \, dz$

$= \int_0^1 \int_0^\pi y \sin z \cdot \pi \, dy \, dz$

$= \int_0^1 \left(\pi \sin z \cdot \frac{1}{2} y^2 \Big|_{y=0}^{y=\pi} \right) dz$

$= \int_0^1 \pi \sin z \cdot \frac{1}{2}\pi^2 \, dz = \int_0^1 \frac{1}{2}\pi^3 \sin z \, dz$

$= \frac{1}{2}\pi^3 \cdot -\cos z \Big|_0^1 = \frac{-1}{2}\pi^3(\cos 1 - \cos 0)$

$= \frac{1}{2}\pi^3(1 - \cos 1)$

12.) $\int_{-1}^1 \int_{-1}^1 \int_{-1}^1 (x+y+z) \, dy \, dx \, dz$

$= \int_{-1}^1 \int_{-1}^1 (xy + \frac{1}{2}y^2 + zy) \Big|_{y=-1}^{y=1} \, dx \, dz$

$= \int_{-1}^1 \int_{-1}^1 \left[(x + \frac{1}{2} + z) - (-x + \frac{1}{2} - z) \right] dx \, dz$

$= \int_{-1}^1 \int_{-1}^1 (2x + 2z) \, dx \, dz = \int_{-1}^1 (x^2 + 2zx) \Big|_{x=-1}^{x=1} \, dz$

$= \int_{-1}^1 \left[(1+2z) - (1-2z) \right] dz = \int_{-1}^1 4z \, dz$

$= 2z^2 \Big|_{-1}^1 = 2(1)^2 - 2(-1)^2 = 0$

16.) $\int_0^1 \int_0^{1-x^2} \int_3^{4-x^2-y} x \, dz \, dy \, dx$

$$= \int_0^1 \int_0^{1-x^2} \left(xz \Big|_{z=3}^{z=4-x^2-y} \right) dy\, dx$$

$$= \int_0^1 \int_0^{1-x^2} [x(4-x^2-y) - x(3)]\, dy\, dx$$

$$= \int_0^1 \int_0^{1-x^2} (x - x^3 - xy)\, dy\, dx$$

$$= \int_0^1 (xy - x^3 y - x \cdot \tfrac{1}{2} y^2) \Big|_{y=0}^{y=1-x^2} dx$$

$$= \int_0^1 [(x(1-x^2) - x^3(1-x^2) - \tfrac{1}{2} x (1-x^2)^2]\, dx$$

$$= \int_0^1 [x - x^3 - x^3 + x^5 - \tfrac{1}{2} x (1 - 2x^2 + x^4)]\, dx$$

$$= \int_0^1 [x - 2x^3 + x^5 - \tfrac{1}{2} x + x^3 - \tfrac{1}{2} x^5]\, dx$$

$$= \int_0^1 (\tfrac{1}{2} x^5 - x^3 + \tfrac{1}{2} x)\, dx = (\tfrac{1}{12} x^6 - \tfrac{1}{4} x^4 + \tfrac{1}{4} x^2) \Big|_0^1$$

$$= \frac{1}{12} - \frac{1}{4} + \frac{1}{4} = \frac{1}{12}$$

17.) $\int_0^\pi \int_0^\pi \int_0^\pi \cos(u+v+w)\, du\, dv\, dw$

$$= \int_0^\pi \int_0^\pi \sin(u+v+w) \Big|_{u=0}^{u=\pi} dv\, dw$$

$$= \int_0^\pi \int_0^\pi (\sin(\pi+v+w) - \sin(v+w))\, dv\, dw$$

$$= \int_0^\pi [-\cos(\pi+v+w) + \cos(v+w)] \Big|_{v=0}^{v=\pi} dw$$

$$= \int_0^\pi [(-\cos(2\pi+w) + \cos(\pi+w))$$

$$- (-\cos(\pi+w) + \cos w)]\, dw$$

$$= \int_0^{\pi} \left(2\cos(\pi+w) - \cos(2\pi+w) - \cos w\right) dw$$

$$= \left(2\sin(\pi+w) - \sin(2\pi+w) - \sin w\right)\Big|_0^{\pi}$$

$$= \left(2\underbrace{\cancel{\sin} 2\pi}_{0} - \underbrace{\cancel{\sin} 3\pi}_{0} - \underbrace{\cancel{\sin}\pi}_{0}\right) - \left(2\underbrace{\cancel{\sin}\pi}_{0} - \underbrace{\cancel{\sin} 2\pi}_{0} - \underbrace{\cancel{\sin} 0}_{0}\right) = 0$$

19.) $\displaystyle\int_0^{\frac{\pi}{4}} \int_0^{\ln(\sec V)} \int_{-\infty}^{2t} e^x \, dx \, dt \, dV$

$$= \int_0^{\frac{\pi}{4}} \int_0^{\ln(\sec V)} \left[\lim_{A \to -\infty} \int_A^{2t} e^x \, dx\right] dt \, dV$$

$$= \int_0^{\frac{\pi}{4}} \int_0^{\ln(\sec V)} \left[\lim_{A \to -\infty} e^x \Big|_{x=A}^{x=2t}\right] dt \, dV$$

$$= \int_0^{\frac{\pi}{4}} \int_0^{\ln(\sec V)} \left[\lim_{A \to -\infty} \left(e^{2t} - e^A\right)\right] dt \, dV$$

$$= \int_0^{\frac{\pi}{4}} \int_0^{\ln(\sec V)} \left(e^{2t} - \underbrace{\cancel{e^{-\infty}}}_{0}\right) dt \, dV$$

$$= \int_0^{\frac{\pi}{4}} \left(\frac{1}{2} e^{2t} \Big|_{t=0}^{t=\ln(\sec V)}\right) dV$$

$$= \int_0^{\frac{\pi}{4}} \left(\frac{1}{2} e^{2(\ln(\sec V))} - \frac{1}{2}\underbrace{\cancel{e^0}}_{1}\right) dV$$

$$= \int_0^{\frac{\pi}{4}} \left(\frac{1}{2} e^{\ln(\sec V)^2} - \frac{1}{2}\right) dV$$

$$= \int_0^{\frac{\pi}{4}} \left(\frac{1}{2} \cdot \sec^2 V - \frac{1}{2}\right) dV$$

$$= \left(\frac{1}{2}\tan V - \frac{1}{2} V\right)\Big|_0^{\frac{\pi}{4}} = \left(\frac{1}{2}\underbrace{\cancel{\tan}\frac{\pi}{4}}_{1} - \frac{1}{2}\cdot\frac{\pi}{4}\right) - \left(\frac{1}{2}\underbrace{\cancel{\tan} 0}_{0} - \frac{1}{2}(0)\right)$$

$$= \frac{1}{2} - \frac{\pi}{8}$$

20.) $\int_0^7 \int_0^2 \int_0^{\sqrt{4-q^2}} \frac{q}{r+1} \, dp \, dq \, dr$

$$= \int_0^7 \int_0^2 \left(\frac{q}{r+1} \cdot p \Big|_{p=0}^{p=\sqrt{4-q^2}} \right) dq \, dr$$

$$= \int_0^7 \int_0^2 \frac{1}{r+1} \cdot q\sqrt{4-q^2} \, dq \, dr$$

$$= \int_0^7 \left(\frac{1}{r+1} \cdot \frac{-1}{3}(4-q^2)^{3/2} \Big|_{q=0}^{q=2} \right) dr$$

$$= \int_0^7 \frac{-1}{3} \cdot \frac{1}{r+1} \cdot \left((0)^{3/2} - (4)^{3/2} \right) dr$$

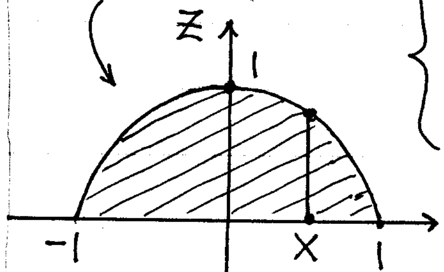
$$= \int_0^7 \frac{8}{3} \cdot \frac{1}{r+1} \, dr = \frac{8}{3} \ln|r+1| \Big|_0^7$$

$$= \frac{8}{3} \ln 8 - \frac{8}{3} \cancel{\ln 1}^{\,0} = \frac{8}{3} \ln 8$$

21.) $\int_{-1}^1 \int_{x^2}^1 \int_0^{1-y} dz \, dy \, dx$ (SEE GRAPH)

a.) $y = x^2$ and $y + z = 1$ then projection onto <u>XZ-plane</u> is : $x^2 + z = 1 \rightarrow$

$z = 1 - x^2$

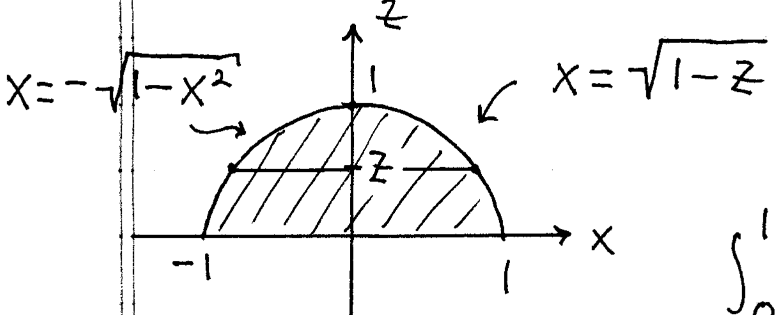


$$\begin{cases} -1 \le x \le 1, \\ 0 \le z \le 1 - x^2, \\ x^2 \le y \le 1 - z \end{cases}$$

$$\int_{-1}^1 \int_0^{1-x^2} \int_{x^2}^{1-z} dy \, dz \, dx$$

b.) $y = x^2$ and $y + z = 1$ then projection onto <u>xz-plane</u> is: $x^2 + z = 1 \rightarrow$

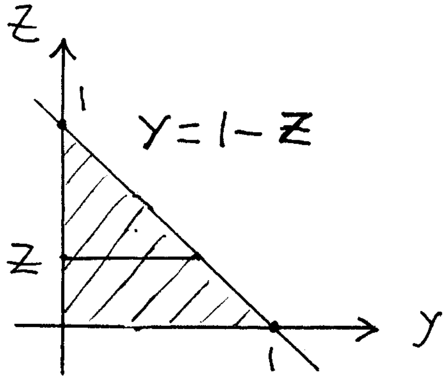
$z = 1 - x^2$



$$\begin{cases} 0 \le z \le 1, \\ -\sqrt{1-x^2} \le x \le \sqrt{1-x^2}, \\ x^2 \le y \le 1 - z \end{cases} \longrightarrow$$

$$\int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{x^2}^{1-z} dy\, dx\, dz$$

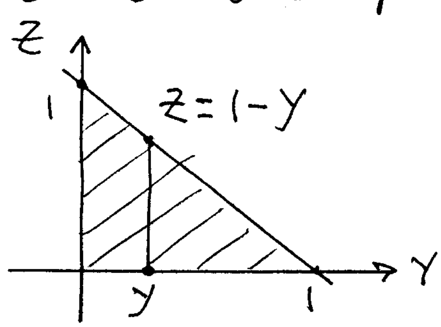
c.) $y = x^2$ and $y + z = 1$ then projection onto yz-plane : $y + z = 1$



$$\begin{cases} 0 \le z \le 1, \\ 0 \le y \le 1 - z \quad \text{and } y = x^2 \\ \qquad \text{or } x = \pm\sqrt{y} \rightarrow \\ -\sqrt{y} \le x \le +\sqrt{y} \end{cases} \longrightarrow$$

$$\int_0^1 \int_0^{1-z} \int_{-\sqrt{y}}^{+\sqrt{y}} dx\, dy\, dz$$

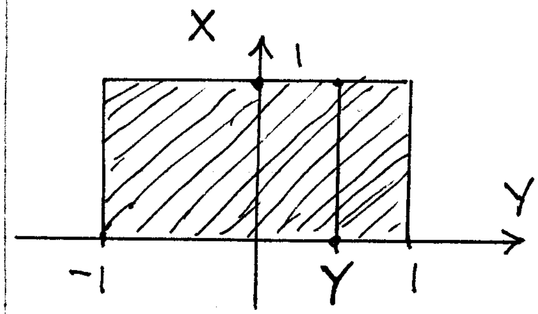
d.) $y = x^2$ and $y + z = 1$ then projection onto yz-plane : $y + z = 1$



$$\begin{cases} 0 \le y \le 1, \\ 0 \le z \le 1 - y, \\ -\sqrt{y} \le x \le +\sqrt{y} \end{cases}$$

$$\int_0^1 \int_0^{1-y} \int_{-\sqrt{y}}^{\sqrt{y}} dx\, dz\, dy$$

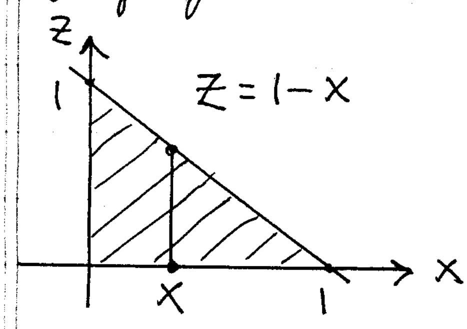
23.) projection onto xy-plane:



$$\begin{cases} -1 \le y \le 1, \\ 0 \le x \le 1, \\ 0 \le z \le y^2 \end{cases} \rightarrow$$

$$\text{Vol} = \int_{-1}^{1} \int_0^1 \int_0^{y^2} 1\, dz\, dx\, dy$$

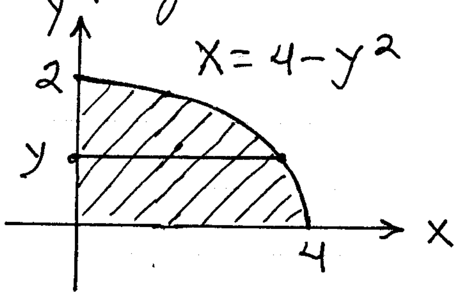
24.) projection onto xz-plane:



$$\begin{cases} 0 \le x \le 1, \\ 0 \le z \le 1 - x, \\ y + 2z = 2 \rightarrow y = 2 - 2z \rightarrow \\ 0 \le y \le 2 - 2z \rightarrow \end{cases}$$

$$\text{Vol} = \int_0^1 \int_0^{1-x} \int_0^{2-2z} 1\, dy\, dz\, dx$$

25.) projection onto xy-plane:



$$\begin{cases} 0 \le y \le 2, \\ 0 \le x \le 4 - y^2, \\ y + z = 2 \rightarrow z = 2 - y \rightarrow \\ 0 \le z \le 2 - y \end{cases}$$

$$\text{Vol} = \int_0^2 \int_0^{4-y^2} \int_0^{2-y} 1 \, dz \, dy \, dx$$

27.) Find equation of plane:

$ax + by + cz = 1$ and $(1,0,0) \rightarrow$

$a(1) + b(0) + c(0) = 1 \rightarrow \underline{a = 1}$; $(0,2,0) \rightarrow$

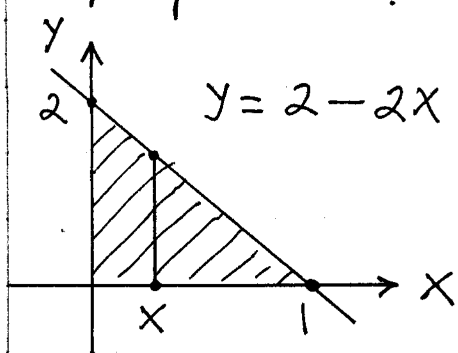
$1\cdot(0) + b(2) + c(0) = 1 \rightarrow 2b = 1 \rightarrow \underline{b = \tfrac{1}{2}}$; $(0,0,3) \rightarrow$

$1\cdot(0) + \frac{1}{2}(0) + c(3) = 1 \rightarrow 3c = 1 \rightarrow \underline{c = \tfrac{1}{3}}$; then

$x + \frac{1}{2}y + \frac{1}{3}z = 1 \rightarrow 3x + \frac{3}{2}y + z = 3 \rightarrow$

$\boxed{z = 3 - 3x - \frac{3}{2}y}$; projection onto

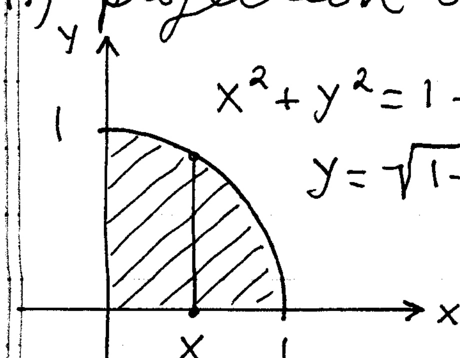
xy-plane:



$$\begin{cases} 0 \le x \le 1, \\ 0 \le y \le 2 - 2x, \\ 0 \le z \le 3 - 3x - \frac{3}{2}y \end{cases} \rightarrow$$

$$\text{Vol} = \int_0^1 \int_0^{2-2x} \int_0^{3-3x-\frac{3}{2}y} 1 \, dz \, dy \, dx$$

29.) projection onto xy-plane:



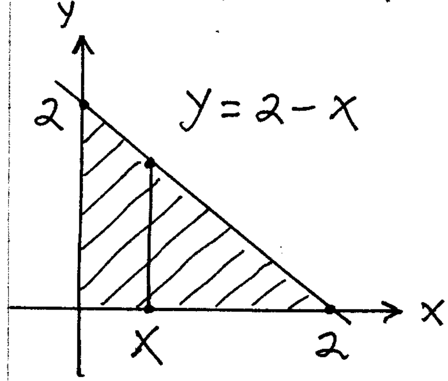
$x^2 + y^2 = 1 \rightarrow$

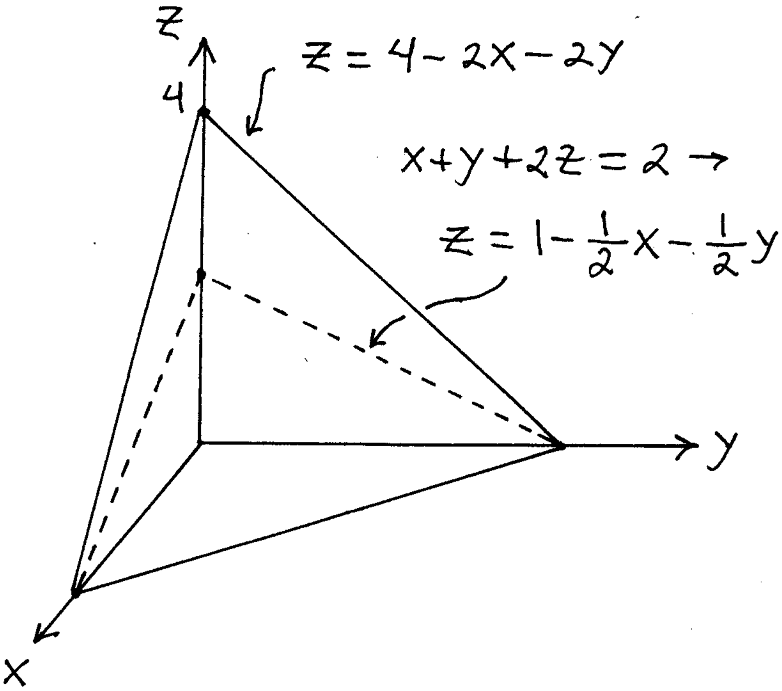
$y = \sqrt{1-x^2}$

$$\begin{cases} 0 \le x \le 1, \\ 0 \le y \le \sqrt{1-x^2}, \\ \quad x^2 + z^2 = 1 \rightarrow z = \sqrt{1-x^2} \rightarrow \\ 0 \le z \le \sqrt{1-x^2} \end{cases} \rightarrow$$

$$\text{Vol} = 8 \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2}} 1 \, dz \, dy \, dx$$

33.) $x+y+2z=2$ and $2x+2y+z=4 \to$
$z=4-2x-2y$ so $x+y+2(4-2x-2y)=2 \to$
$x+y+8-4x-4y=2 \to -3x-3y=-6 \to$
$\underline{x+y=2}$; projection onto xy-plane :

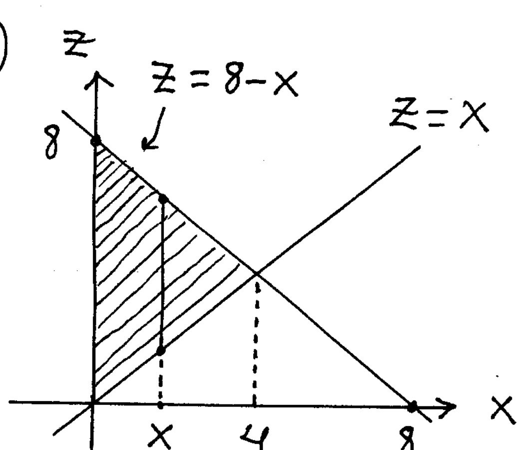


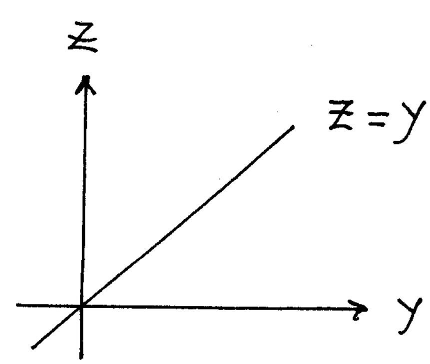


$$\begin{cases} 0 \le x \le 2, \\ 0 \le y \le 2-x, \\ 1-\frac{1}{2}x-\frac{1}{2}y \le z \le 4-2x-2y \to \end{cases}$$

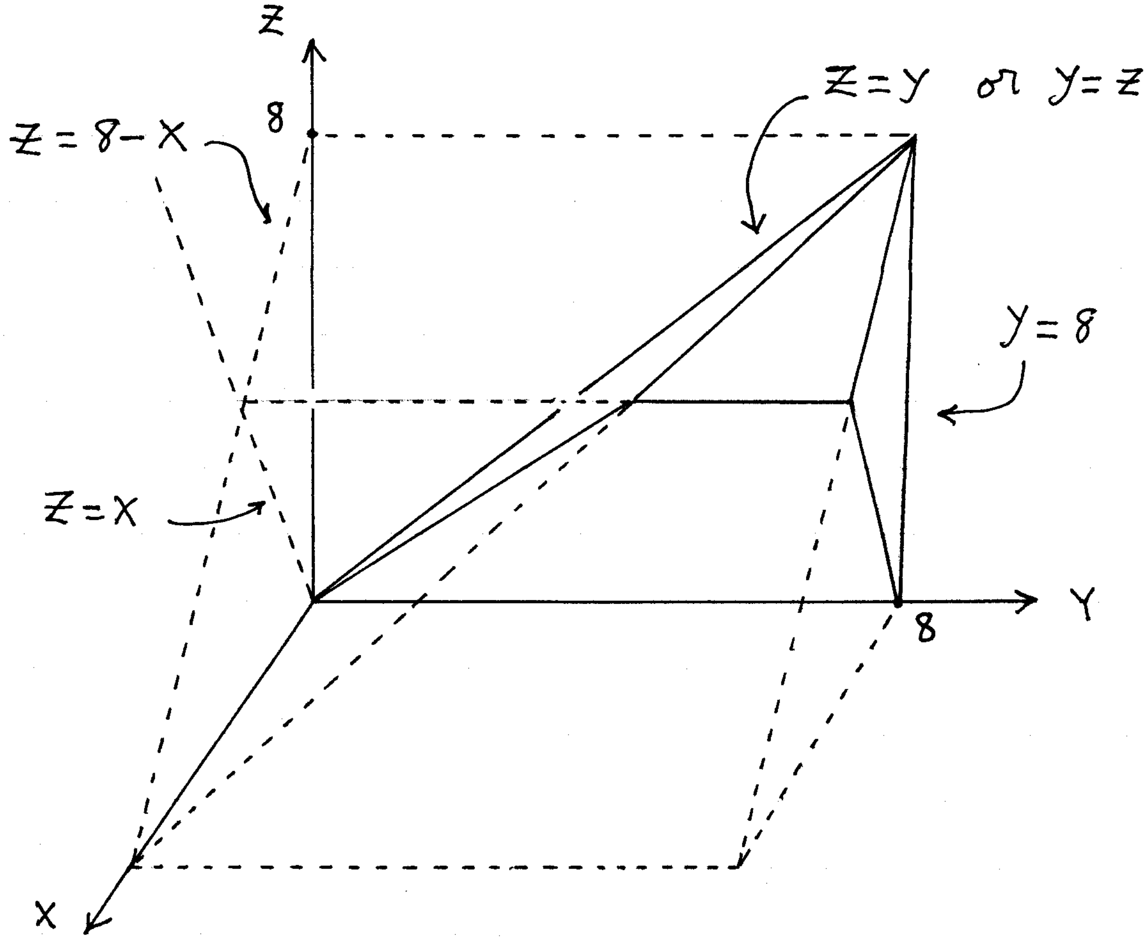
$$\text{Vol} = \int_0^2 \int_0^{2-x} \int_{1-\frac{1}{2}x-\frac{1}{2}y}^{4-2x-2y} 1 \, dz \, dy \, dx$$

34.)



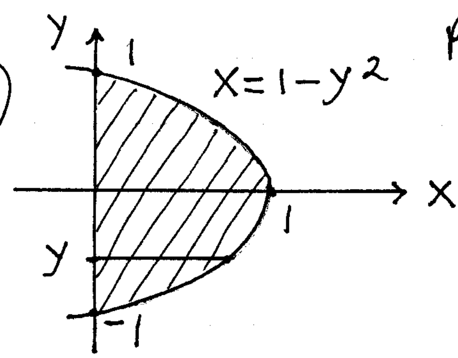


← projection onto xz-plane

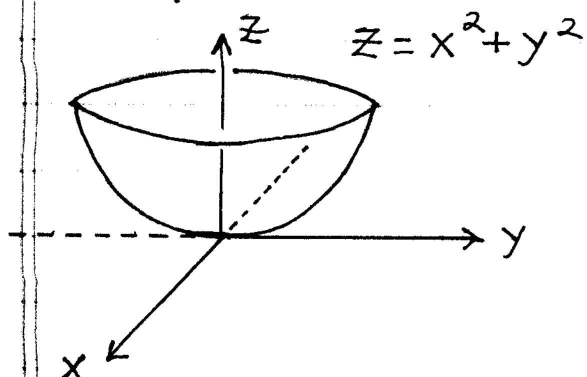


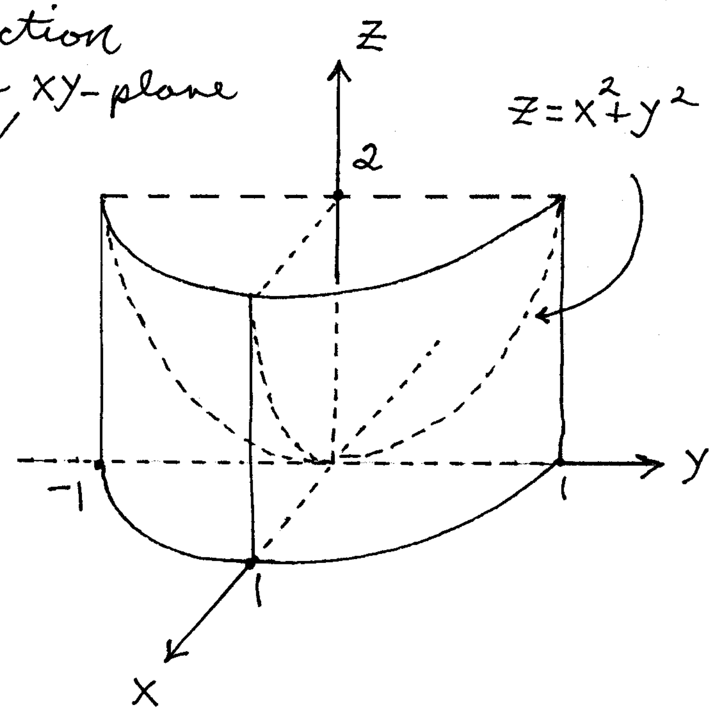
$$\begin{cases} 0 \le x \le 4, \\ x \le z \le 8 - x, \\ z \le y \le 8 \end{cases} \quad Vol = \int_0^4 \int_x^{8-x} \int_z^8 1 \, dy \, dz \, dx$$

36.)



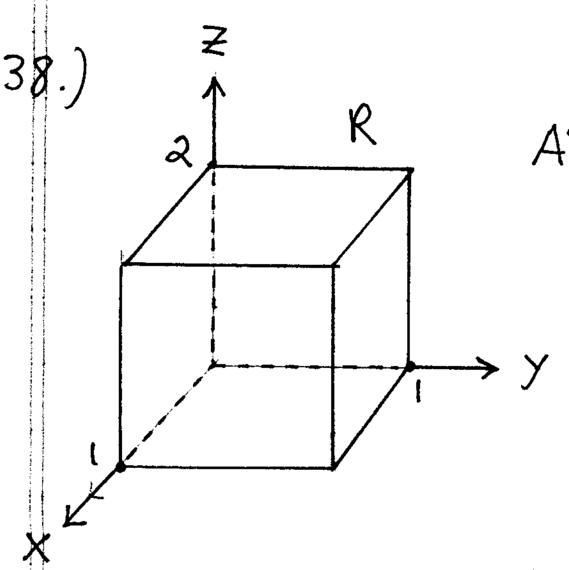
projection onto xy-plane





$$\left\{\begin{array}{l} -1 \le y \le 1, \\ 0 \le x \le 1 - y^2, \\ 0 \le z \le x^2 + y^2 \end{array}\right\} \rightarrow \text{Vol} = \int_{-1}^{1} \int_{0}^{1-y^2} \int_{0}^{x^2+y^2} 1 \, dz \, dx \, dy$$

38.)



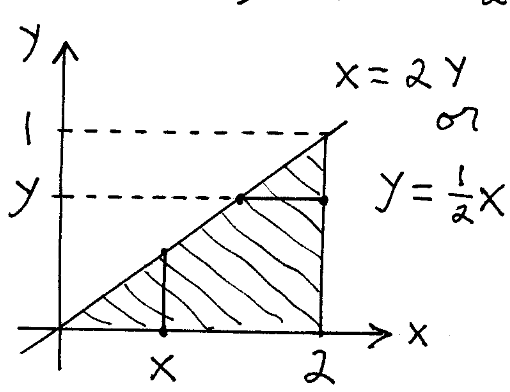
$$AVE = \frac{1}{\text{vol}\, R} \iiint_R f(P)\, dV$$

$$= \frac{1}{2} \int_0^1 \int_0^1 \int_0^2 (x+y-z)\, dz\, dy\, dx$$

can't integrate!

41.) $\displaystyle \int_0^4 \int_0^1 \int_{2y}^{2} \frac{4\cos(x^2)}{2\sqrt{z}}\, dx\, dy\, dz$

$$\left.\begin{array}{l} 0 \le y \le 1 \\ 2y \le x \le 2 \end{array}\right\} \quad \left.\begin{array}{l} 0 \le x \le 2 \\ 0 \le y \le \frac{1}{2}x \end{array}\right\} \quad = \int_0^4 \int_0^2 \int_0^{\frac{1}{2}x} \frac{2\cos(x^2)}{\sqrt{z}}\, dy\, dx\, dz$$



$$= \int_0^4 \int_0^2 \left(\frac{2\cos(x^2)}{\sqrt{z}} \cdot y \Big|_{y=0}^{y=\frac{1}{2}x} \right) dx\, dz$$

$$= \int_0^4 \int_0^2 \frac{1}{\sqrt{z}} \cdot x \cos(x^2)\, dx\, dz$$

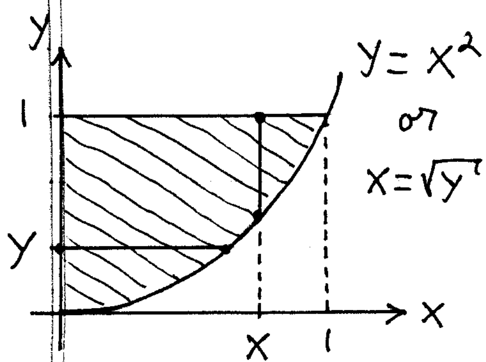
$$= \int_0^4 \left(\frac{1}{\sqrt{z}} \cdot \frac{1}{2} \sin(x^2) \Big|_{x=0}^{x=2} \right) dz$$

$$= \int_0^4 \frac{1}{2\sqrt{z}} (\sin 4 - \cancel{\sin 0}^{\,0})\, dz = \sin 4 \cdot \sqrt{z}\, \Big|_0^4$$

$$= 2 \sin 4$$

can't integrate

42.) $\int_0^1 \int_0^1 \int_{x^2}^1 12xz\, e^{zy^2}\, dy\, dx\, dz$



$= \int_0^1 \int_0^1 \int_0^{\sqrt{y}} 12xz\, e^{zy^2}\, dx\, dy\, dz$

$= \int_0^1 \int_0^1 \left(6x^2 \cdot z\, e^{zy^2} \Big|_{x=0}^{x=\sqrt{y}} \right) dy\, dz$

$= \int_0^1 \int_0^1 6yz\, e^{zy^2}\, dy\, dz$

$= \int_0^1 \left(3e^{zy^2} \Big|_{y=0}^{y=1} \right) dz = \int_0^1 (3e^z - 3e^0)\, dz$

$= \int_0^1 (3e^z - 3)\, dz = (3e^z - 3z)\Big|_0^1$

$= (3e - 3) - (3e^0 - 3(0)) = 3e - 6$