Section 15.6

21.) $\int_0^{\pi}\int_0^{\pi}\int_0^{2\sin\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

$= \int_0^{\pi}\int_0^{\pi} \left(\frac{1}{3}\rho^3 \sin\phi \Big|_{\rho=0}^{\rho=2\sin\phi}\right) d\phi \, d\theta$

$= \int_0^{\pi}\int_0^{\pi} \frac{1}{3}(8\sin^3\phi)\sin\phi \, d\phi \, d\theta$

$= \int_0^{\pi}\int_0^{\pi} \frac{8}{3}\sin^4\phi \, d\phi \, d\theta$

$= \int_0^{\pi}\int_0^{\pi} \frac{8}{3}(\sin^2\phi)^2 \, d\phi \, d\theta$

$= \int_0^{\pi}\int_0^{\pi} \frac{8}{3}\left(\frac{1}{2}(1-\cos 2\phi)\right)^2 d\phi \, d\theta$

$= \int_0^{\pi}\int_0^{\pi} \frac{8}{3}\cdot\frac{1}{4}(1-2\cos 2\phi + \cos^2 2\phi) \, d\phi \, d\theta$

$= \int_0^{\pi}\int_0^{\pi} \frac{2}{3}\left(1-2\cos 2\phi + \frac{1}{2}(1+\cos 4\phi)\right) d\phi \, d\theta$

$= \int_0^{\pi}\int_0^{\pi} \frac{2}{3}\left(\frac{3}{2} - 2\cos 2\phi + \frac{1}{2}\cos 4\phi\right) d\phi \, d\theta$

$= \int_0^{\pi} \frac{2}{3}\left(\frac{3}{2}\phi - 2\cdot\frac{1}{2}\sin 2\phi + \frac{1}{2}\cdot\frac{1}{4}\sin 4\phi\right)\Big|_{\phi=0}^{\phi=\pi} d\theta$

$= \int_0^{\pi} \frac{2}{3}\left(\frac{3}{2}\pi - \cancelto{0}{\sin 2\pi} + \frac{1}{8}\cancelto{0}{\sin 4\pi}\right) - \frac{1}{3}(0) \quad d\theta$

$= \int_0^{\pi} \pi \, d\theta = \pi\theta\Big|_0^{\pi} = \pi(\pi-0) = \pi^2$

24.) $\int_0^{\frac{3}{2}\pi}\int_0^{\pi}\int_0^{1} 5\rho^3 \sin^3\phi \, d\rho \, d\phi \, d\theta$

$$= \int_0^{\frac{3}{2}\pi} \int_0^{\pi} \left(\frac{5}{4} \rho^4 \sin^3\phi \, \Big|_{\rho=0}^{\rho=1} \right) d\phi \, d\theta$$

$$= \int_0^{\frac{3}{2}\pi} \int_0^{\pi} \frac{5}{4} \sin^3\phi \, d\phi \, d\theta$$

$$= \int_0^{\frac{3}{2}\pi} \int_0^{\pi} \frac{5}{4} \sin\phi \cdot \sin^2\phi \, d\phi \, d\theta$$

$$= \int_0^{\frac{3}{2}\pi} \int_0^{\pi} \frac{5}{4} \sin\phi \left(1 - \cos^2\phi\right) d\phi \, d\theta$$

$$= \int_0^{\frac{3}{2}\pi} \int_0^{\pi} \frac{5}{4} \left(\sin\phi - \sin\phi \cos^2\phi\right) d\phi \, d\theta$$

$$= \int_0^{\frac{3}{2}\pi} \frac{5}{4} \left(-\cos\phi + \frac{1}{3}\cos^3\phi\right) \Big|_{\phi=0}^{\phi=\pi} d\theta$$

$$= \int_0^{\frac{3}{2}\pi} \left[\frac{5}{4}\left(-\cos\pi + \frac{1}{3}\cos^3\pi\right) - \frac{5}{4}\left(-\cos 0 + \frac{1}{3}\cos^3 0\right) \right] d\theta$$

$$= \int_0^{\frac{3}{2}\pi} \left[\frac{5}{4}\left(-(-1) + \frac{1}{3}(-1)\right) - \frac{5}{4}\left(-1 + \frac{1}{3}(1)\right) \right] d\theta$$

$$= \int_0^{\frac{3}{2}\pi} \left(\frac{5}{\cancel{4}_2} \cdot \frac{\cancel{2}}{3} - \frac{5}{\cancel{4}_2} \cdot \frac{-\cancel{2}}{3} \right) d\theta$$

$$= \int_0^{\frac{3}{2}\pi} \frac{10}{6} \, d\theta = \int_0^{3/2\pi} \frac{5}{3} \, d\theta$$

$$= \frac{5}{3} \theta \Big|_0^{\frac{3}{2}\pi} = \frac{5}{\cancel{3}} \cdot \frac{\cancel{3}}{2} \pi = \frac{5}{2}\pi$$

26.) $$\int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^{\sec\phi} \rho^3 \cos\phi \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \left(\frac{1}{4} \rho^4 \cdot \cos\phi \sin\phi \, \Big|_{\rho=0}^{\rho=\sec\phi} \right) d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{4}} \frac{1}{4}\sec^4\phi \cos\phi \sin\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{4}} \frac{1}{4}\sec^2\phi \cdot \cancel{\frac{1}{\cos\phi}}\cancel{\cos\phi} \cdot \frac{1}{\cos\phi}\sin\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{4}} \frac{1}{4}\sec^2\phi \tan\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{4}\cdot\frac{1}{2}\tan^2\phi \Big|_{\phi=0}^{\phi=\frac{\pi}{4}}\right) d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{8}\tan^2\frac{\pi}{4} - \frac{1}{8}\cancelto{0}{\tan^2 0}\right) d\theta$$

$$= \int_0^{2\pi} \frac{1}{8}(1)^2 \, d\theta = \frac{1}{8}\theta\Big|_0^{2\pi} = \frac{\pi}{4}$$

27.) $\int_0^2\int_{-\pi}^0\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \rho^3 \sin 2\phi \, d\phi \, d\theta \, d\rho$

$$= \int_0^2\int_{-\pi}^0 \left(\rho^3 \cdot \frac{-1}{2}\cos 2\phi \Big|_{\phi=\frac{\pi}{4}}^{\phi=\frac{\pi}{2}}\right) d\theta \, d\rho$$

$$= \int_0^2\int_{-\pi}^0 \left(\frac{-1}{2}\rho^3 \cancelto{-1}{\cos\pi} - \frac{-1}{2}\rho^3 \cancelto{0}{\cos\frac{\pi}{2}}\right) d\theta \, d\rho$$

$$= \int_0^2\int_{-\pi}^0 \frac{1}{2}\rho^3 \, d\theta \, d\rho = \int_0^2 \left(\frac{1}{2}\rho^3\cdot\theta\Big|_{\theta=-\pi}^{\theta=0}\right) d\rho$$

$$= \int_0^2 \left(\frac{1}{2}\rho^3\cdot(0) - \frac{1}{2}\rho^3(-\pi)\right) d\rho$$

$$= \int_0^2 \frac{\pi}{2}\rho^3 \, d\rho = \frac{\pi}{2}\cdot\frac{1}{4}\rho^4\Big|_0^2$$

$$= \frac{\pi}{2}\cdot\frac{1}{4}\cdot 16 = 2\pi$$

30.) $$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{\csc\phi}^{2} 5\rho^4 \sin^3\phi \, d\rho \, d\theta \, d\phi$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\rho^5 \cdot \sin^3\phi \Big|_{\rho=\csc\phi}^{\rho=2} \right) d\theta \, d\phi$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(32\sin^3\phi - \csc^5\phi \cdot \sin^3\phi \right) d\theta \, d\phi$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(32\sin^3\phi - \csc^2\phi \cdot \frac{1}{\cancel{\sin^3\phi}} \cdot \cancel{\sin^3\phi} \right) d\theta \, d\phi$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(32\sin^3\phi - \csc^2\phi \right) \cdot \theta \Big|_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} \, d\phi$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(32\sin^3\phi - \csc^2\phi \right) \left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right) \right) d\phi$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \pi \left(32 \cdot \sin\phi \cdot \sin^2\phi - \csc^2\phi \right) d\phi$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \pi \left(32 \cdot \sin\phi \left(1 - \cos^2\phi\right) - \csc^2\phi \right) d\phi$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \pi \left(32\sin\phi - 32\sin\phi\cos^2\phi - \csc^2\phi \right) d\phi$$

$$= \pi \left(-32\cos\phi + 32 \cdot \frac{1}{3}\cos^3\phi + \cot\phi \right) \Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= \pi\left(-32\cancelto{0}{\cos\frac{\pi}{2}} + \frac{32}{3}\cancelto{0}{\cos^3\frac{\pi}{2}} + \cancelto{0}{\cot\frac{\pi}{2}}\right)$$

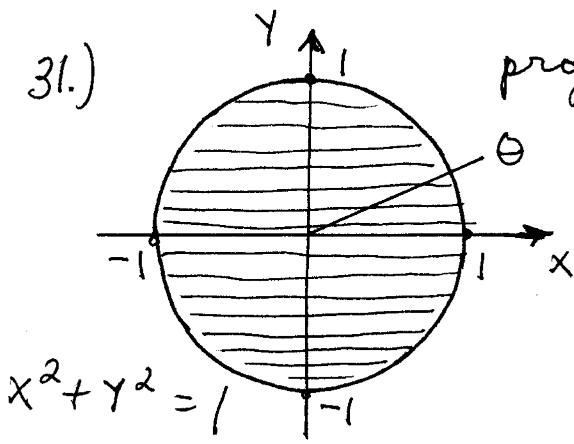
$$-\pi\left(-32\cos\frac{\pi}{6} + \frac{32}{3}\cos^3\frac{\pi}{6} + \cot\frac{\pi}{6}\right)$$

$$= -\pi\left(-32\cdot\frac{\sqrt{3}}{2} + \frac{32}{3}\cdot\left(\frac{\sqrt{3}}{2}\right)^3 + \frac{\sqrt{3}/2}{1/2}\right)$$

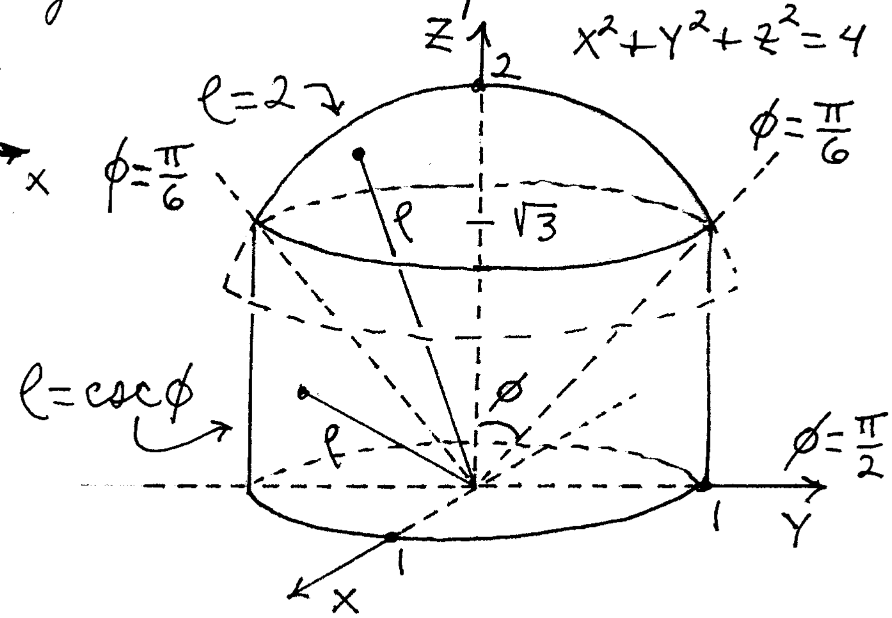
$$= -\pi\left(-16\sqrt{3} + \frac{\overset{4}{\cancel{32}}}{\cancel{3}}\cdot\frac{\cancel{3}\sqrt{3}}{\cancel{8}} + \sqrt{3}\right)$$

$$= -\pi\left(-11\sqrt{3}\right) = 11\sqrt{3}\cdot\pi$$

31.)



projection on XY-plane



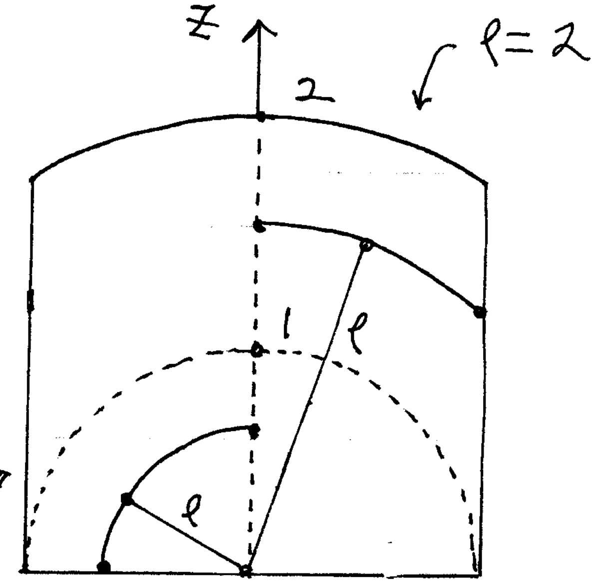
$x^2+y^2+z^2=4$
and $x^2+y^2=1 \rightarrow$
$1+z^2=4 \rightarrow z^2=3$
$\rightarrow z=\sqrt{3}$;

$x^2+y^2=1 \rightarrow r=1 \rightarrow \rho\sin\phi=1 \rightarrow \rho=\csc\phi$

a.) $\text{Vol} = \int_0^{2\pi}\int_0^{\frac{\pi}{6}}\int_0^{2} 1\cdot\rho^2\sin\phi \; d\rho\, d\phi\, d\theta$

$+ \int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\int_0^{\csc\phi} 1\cdot\rho^2\sin\phi \; d\rho\, d\phi\, d\theta$

b.) SIDE VIEW OF SOLID



$r = 1 \rightarrow$

$\rho \sin\phi = 1 \rightarrow$

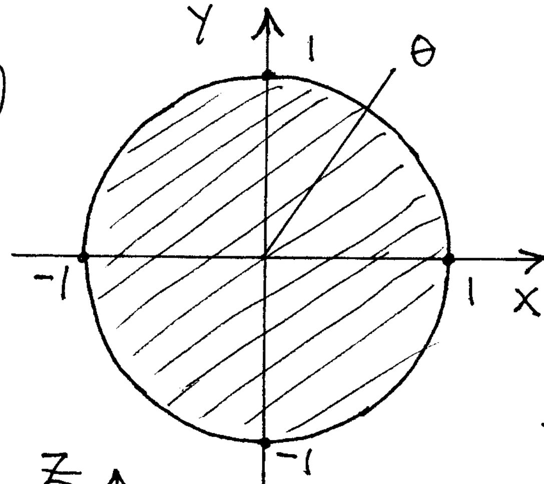
$\sin\phi = \frac{1}{\rho} \rightarrow$

$\phi = \arcsin\left(\frac{1}{\rho}\right)$

$$\text{Vol} = \int_0^{2\pi}\int_0^1\int_0^{\frac{\pi}{2}} 1\cdot\rho^2\sin\phi \, d\phi \, d\rho \, d\theta$$

$$+ \int_0^{2\pi}\int_1^2\int_0^{\arcsin\left(\frac{1}{\rho}\right)} 1\cdot\rho^2\sin\phi \, d\phi \, d\rho \, d\theta$$

32.)

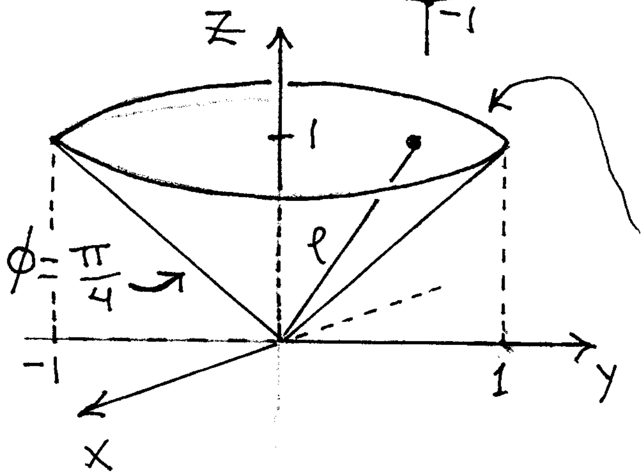


$z = \sqrt{x^2+y^2}$ and $z = 1$

$\rightarrow 1 = \sqrt{x^2+y^2} \rightarrow$

$x^2 + y^2 = 1$;

projection on xy-plane ;



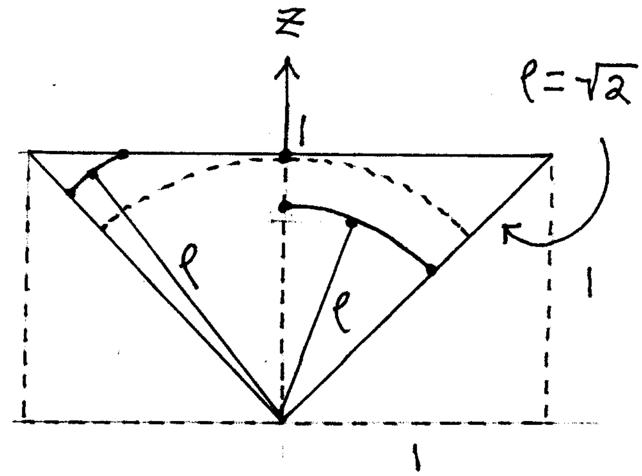
$z = 1 \rightarrow \rho\cos\phi = 1 \rightarrow$

$\rho = 1/\cos\phi \rightarrow$

$\rho = \sec\phi$

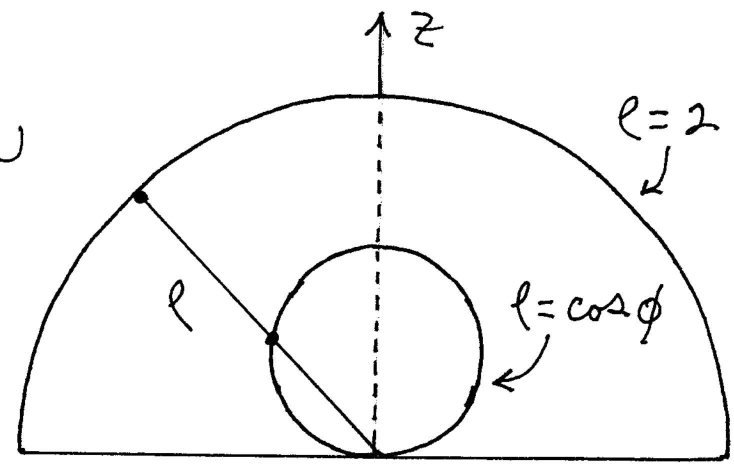
a.) $\text{Vol} = \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^{\sec\phi} 1 \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

b.) SIDE VIEW OF SOLID



$$\text{Vol} = \int_0^{2\pi} \int_0^{1} \int_0^{\frac{\pi}{4}} 1 \cdot \rho^2 \sin\phi \, d\phi \, d\rho \, d\theta + \int_0^{2\pi} \int_1^{\sqrt{2}} \int_{\sec\phi}^{\frac{\pi}{4}} 1 \cdot \rho^2 \sin\phi \, d\phi \, d\rho \, d\theta$$

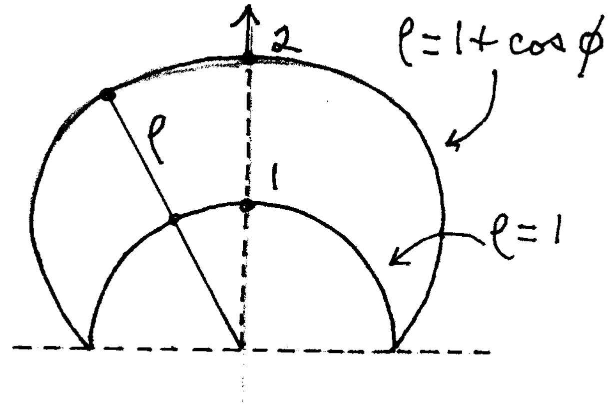
33.) SIDE VIEW OF SOLID



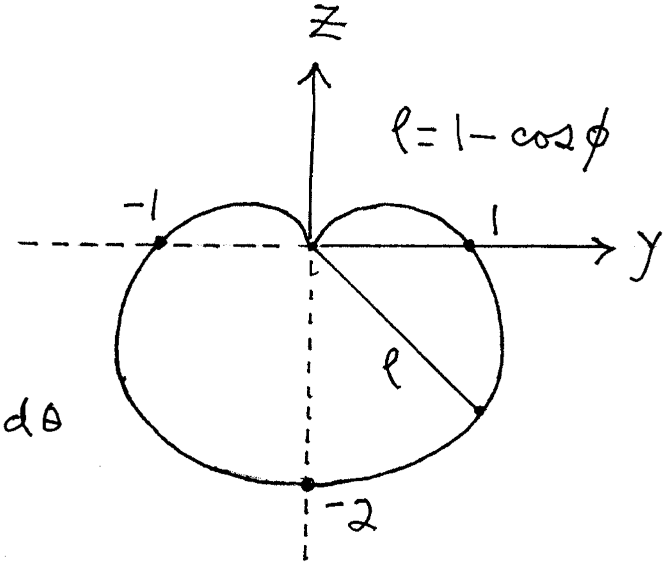
$$\text{Vol} = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_{\cos\phi}^{2} 1 \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

34.)

$$\text{Vol} = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_1^{1+\cos\phi} 1 \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$



35.) SIDE VIEW

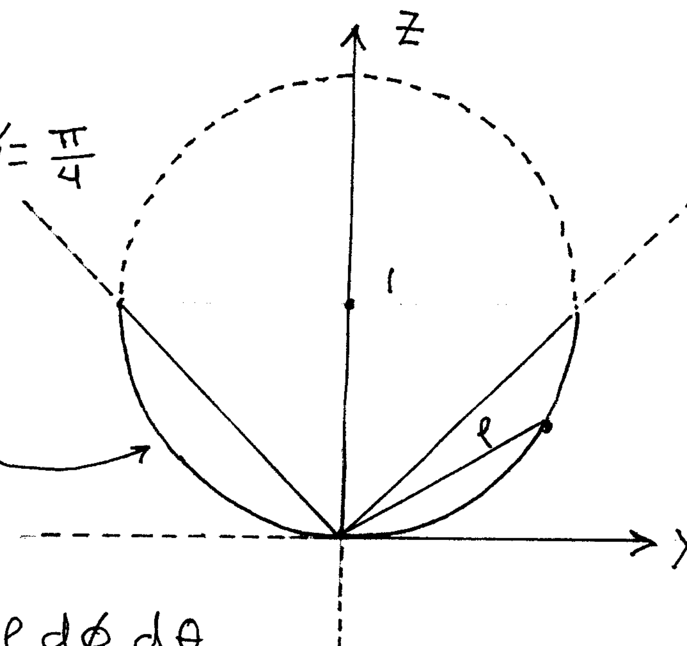


$$\text{Vol} = \int_0^{2\pi}\int_0^{\pi}\int_0^{1-\cos\phi} 1\cdot\rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

37.) $z = \sqrt{x^2+y^2}$ (cone)

$\rightarrow \phi = \frac{\pi}{4}$

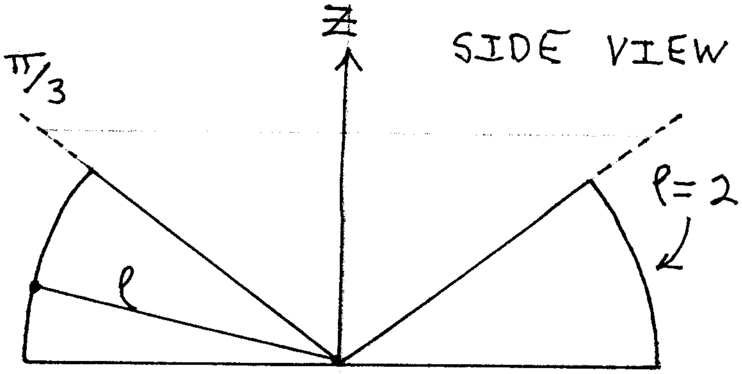
SIDE VIEW



$$\text{Vol} = \int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{2\cos\phi} 1\cdot\rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

38.)

SIDE VIEW

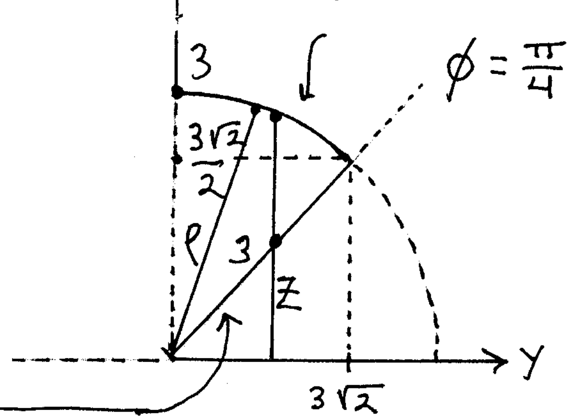


$$\text{Vol} = \int_0^{2\pi}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\int_0^{2} 1\cdot\rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

40.)

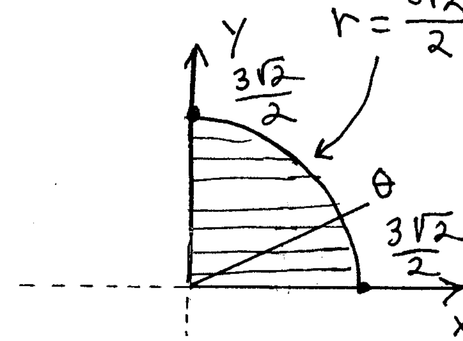
$z = \sqrt{9-r^2}$
$\rho = 3$ or

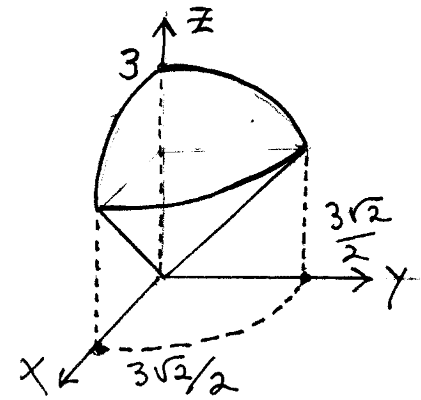
$z = \sqrt{x^2+y^2}$
or
$z = r$



SIDE VIEW

proj. on
xy-plane

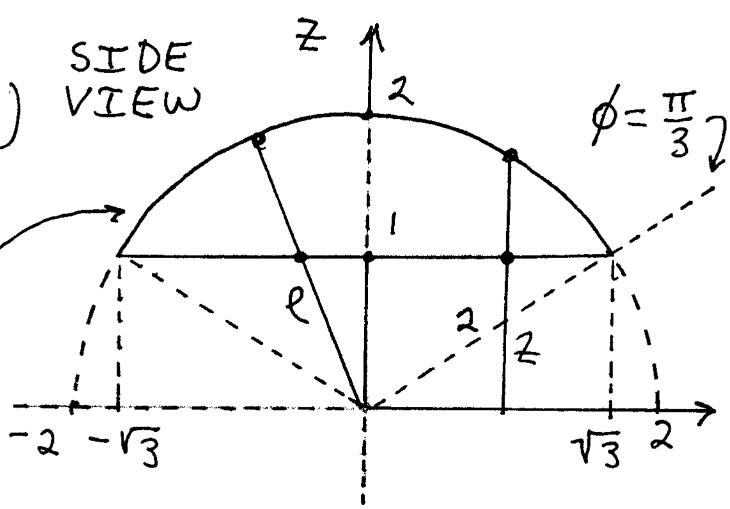




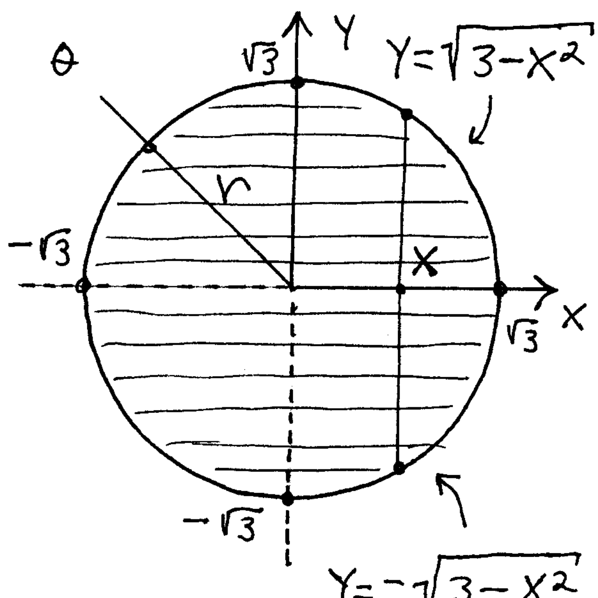
a.) $\text{Vol} = \int_0^{\frac{\pi}{2}} \int_0^{\frac{3\sqrt{2}}{2}} \int_r^{\sqrt{9-r^2}} 1 \cdot r \, dz \, dr \, d\theta$

b.) $\text{Vol} = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{4}} \int_0^{3} 1 \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

41.) SIDE VIEW

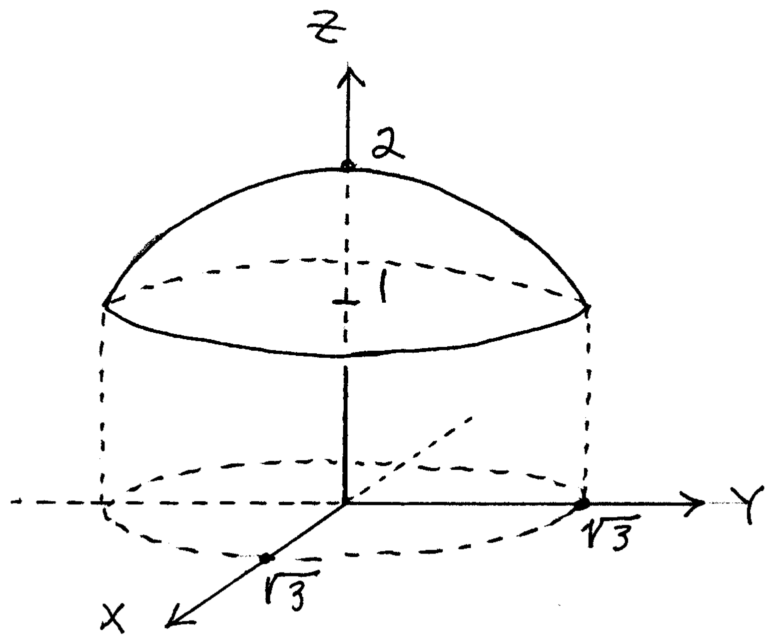


proj. on xy-plane



$z = \sqrt{4-x^2-y^2}$ or
$z = \sqrt{4-r^2}$ or
$\rho = 2$;

$z = 1$ or $\rho \cos\phi = 1$ or $\rho = \sec\phi$

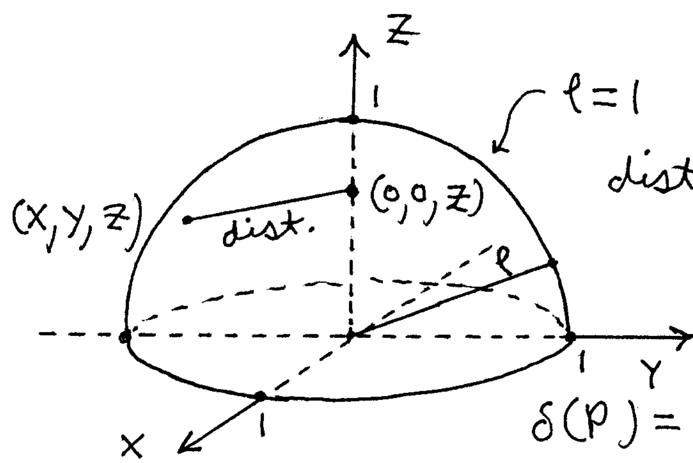


a.) $Vol = \int_0^{2\pi} \int_0^{\frac{\pi}{3}} \int_{\sec\phi}^{\rho} 1 \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

b.) $Vol = \int_0^{2\pi} \int_0^{\sqrt{3}} \int_1^{\sqrt{4-r^2}} 1 \cdot r \, dz \, dr \, d\theta$

c.) $Vol = \int_{-\sqrt{3}}^{\sqrt{3}} \int_{-\sqrt{3-x^2}}^{\sqrt{3-x^2}} \int_1^{\sqrt{4-x^2-y^2}} 1 \, dz \, dY \, dx$

42.) b.)



$dist = \sqrt{(x-0)^2 + (Y-0)^2 + (z-z)^2}$

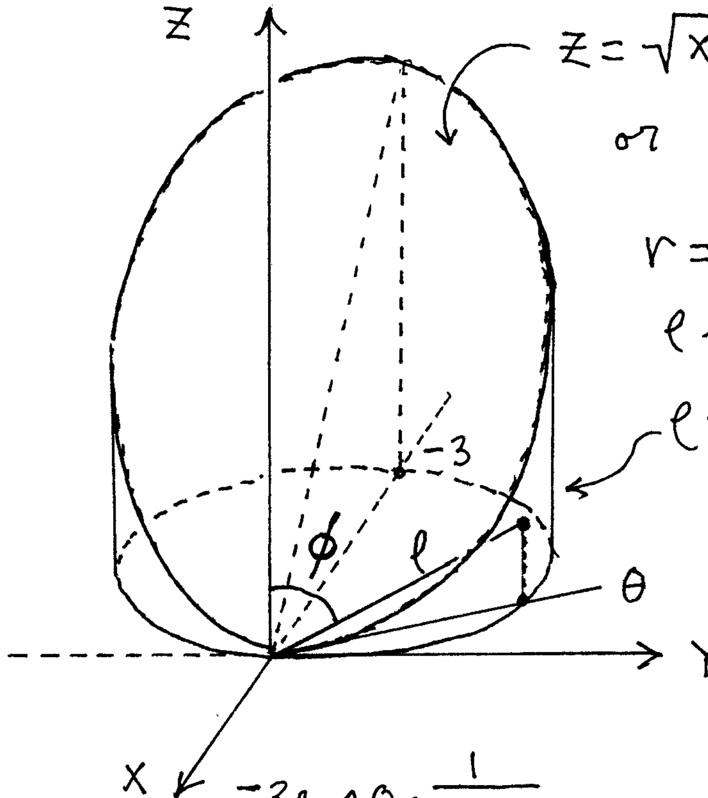
$= \sqrt{x^2+y^2} = \sqrt{r^2}$

$= r = \rho \sin\phi$,

$\delta(P) = z = \rho \cos\phi$;

$I_z = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^1 (\rho \sin\phi)^2 \cdot (\rho\cos\phi) \cdot \rho^2 \sin\phi \cdot d\rho \, d\phi \, d\theta$

46.)



$z = \sqrt{x^2+y^2}$ (cone)

or $\phi = \frac{\pi}{4}$

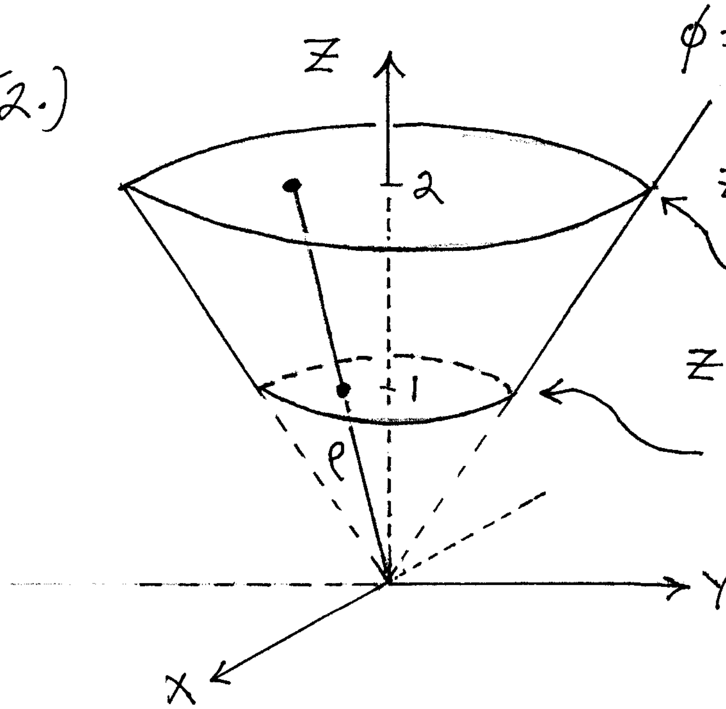
$r = -3\cos\theta \rightarrow$

$\rho \sin\phi = -3\cos\theta \rightarrow$

$\rho = -3\cos\theta \cdot \frac{1}{\sin\phi}$

$$\text{Vol} = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{-3\cos\theta \cdot \frac{1}{\sin\phi}} 1 \cdot \rho^2 \sin\phi \cdot d\rho\, d\phi\, d\theta$$

52.)



$\phi = \frac{\pi}{4}$

$z = 2 \rightarrow \rho\cos\phi = 2 \rightarrow$

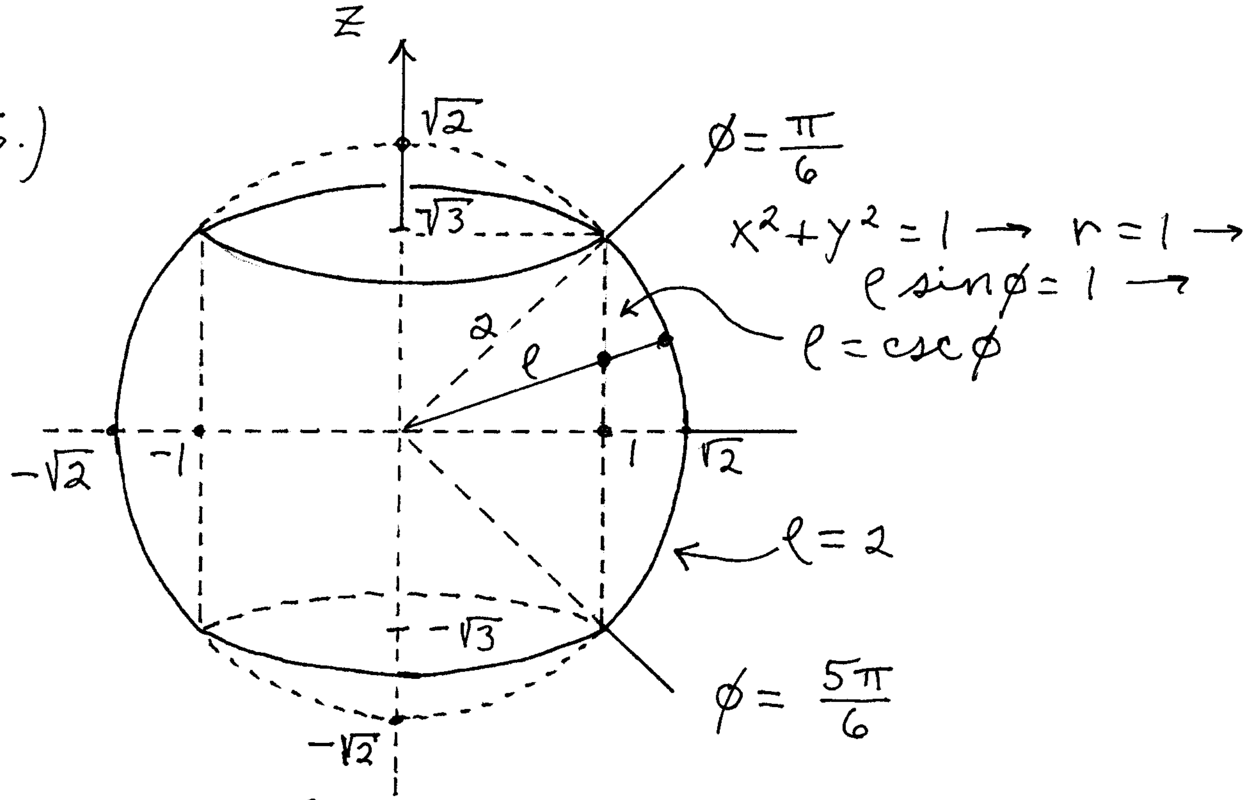
$\rho = 2\sec\phi$

$z = 1 \rightarrow \rho\cos\phi = 1 \rightarrow$

$\rho = \sec\phi$

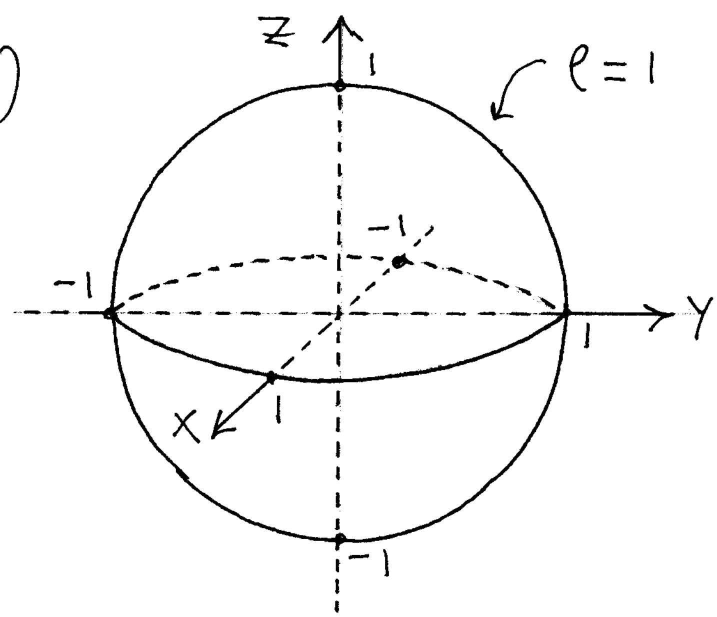
$$\text{Vol} = \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_{\sec\phi}^{2\sec\phi} 1 \cdot \rho^2 \sin\phi \; d\rho\, d\phi\, d\theta$$

56.)



$$\text{Vol} = \int_0^{2\pi} \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \int_{\csc\phi}^{2} 1 \cdot \rho^2 \sin\phi \cdot d\rho\, d\phi\, d\theta$$

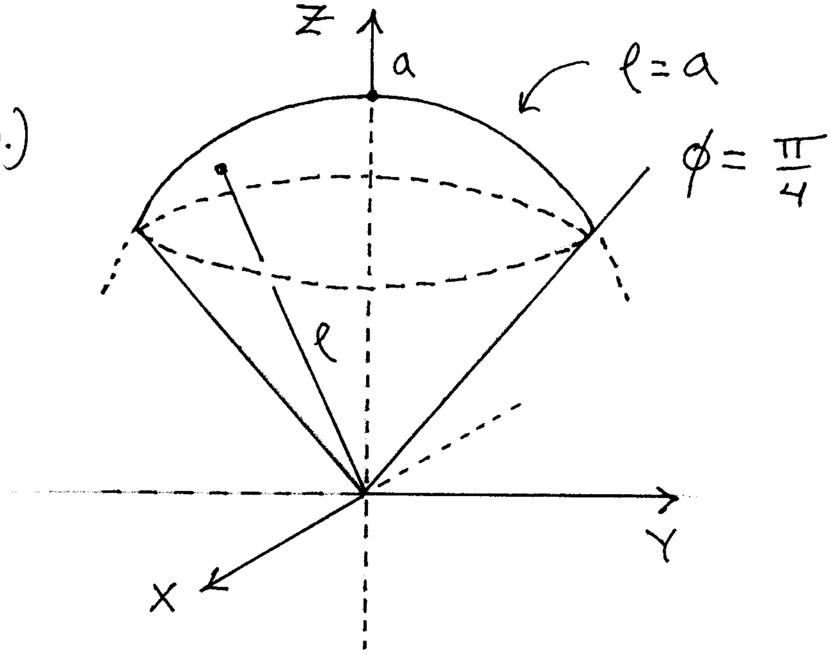
65)



$$\text{AVE} = \frac{1}{\text{vol}\, R} \iiint_R f(P)\, dV$$

$$= \frac{1}{\int_0^{2\pi}\int_0^{\pi}\int_0^1 1 \cdot \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta} \cdot \int_0^{2\pi}\int_0^{\pi}\int_0^1 \rho \cdot \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

70.)



$$\bar{X} = \iiint_R x \, dV \Big/ \iiint_R 1 \, dV$$

$$= \frac{\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{a} (\rho \cdot \sin\phi \cos\theta)\cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta}{\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{a} 1\cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta}$$