Section 15.7

1.) a.) $\begin{cases} u = x - y \\ v = 2x + y \end{cases} \longrightarrow \begin{cases} y = x - u \\ v = 2x + y \end{cases}$ (SUB)

$\rightarrow v = 2x + (x - u) \rightarrow 3x = u + v \rightarrow$

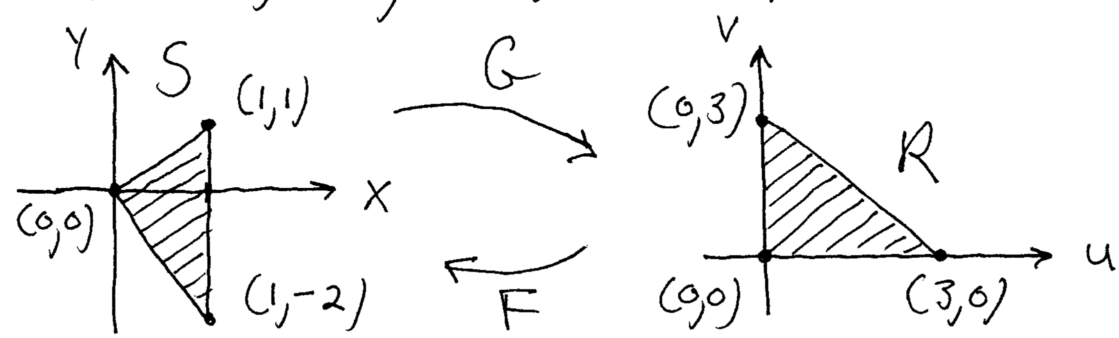
$x = \frac{1}{3}u + \frac{1}{3}v \rightarrow y = \left(\frac{1}{3}u + \frac{1}{3}v\right) - u \rightarrow$

$y = \frac{1}{3}v - \frac{2}{3}u$, so

$$F(u,v) = \left(\frac{1}{3}u + \frac{1}{3}v, \frac{1}{3}v - \frac{2}{3}u\right) = (x,y);$$

$$J(P) = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 1/3 & 1/3 \\ -2/3 & 1/3 \end{vmatrix} = \frac{1}{9} - \left(\frac{-2}{9}\right) = \frac{3}{9} = 1/3$$

b.) Let $G(x,y) = (x-y, 2x+y) = (u,v)$, a linear mapping (which preserves lines); then $G(0,0) = (0,0)$, $G(1,1) = (0,3)$, $G(1,-2) = (3,0) \rightarrow$



4.) a.) $\begin{cases} u = 2x - 3y \\ v = -x + y \end{cases} \longrightarrow \begin{cases} u = 2x - 3y \\ y = x + v \end{cases}$ (SUB)

$\rightarrow u = 2x - 3(x + v) = 2x - 3x - 3v \rightarrow$

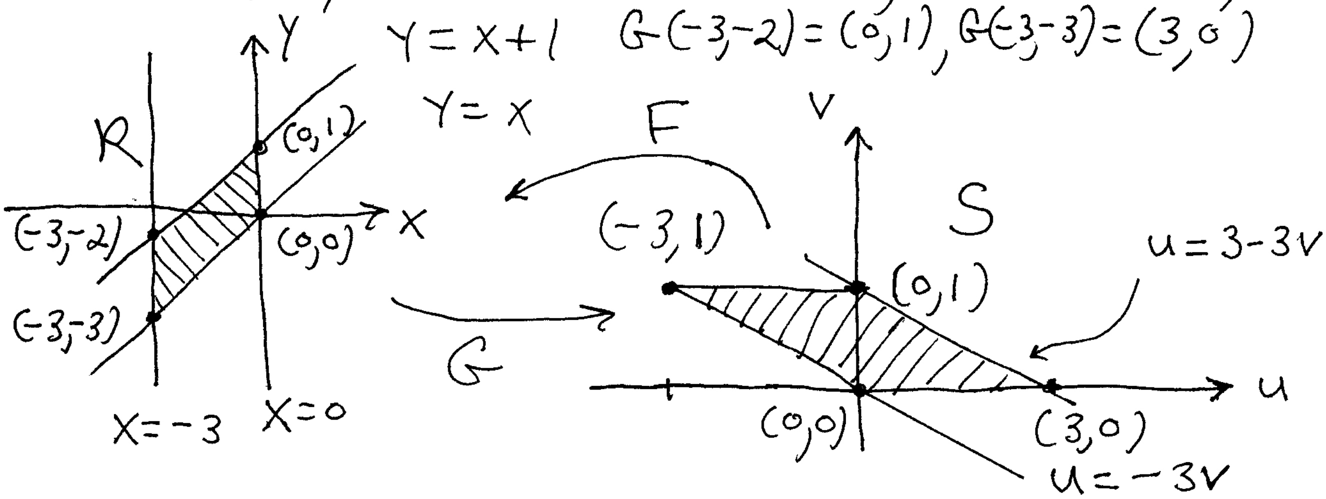
$u + 3v = -x \rightarrow x = -u - 3v$ so

$y = (-u-3v) + v = -u-2v$, so

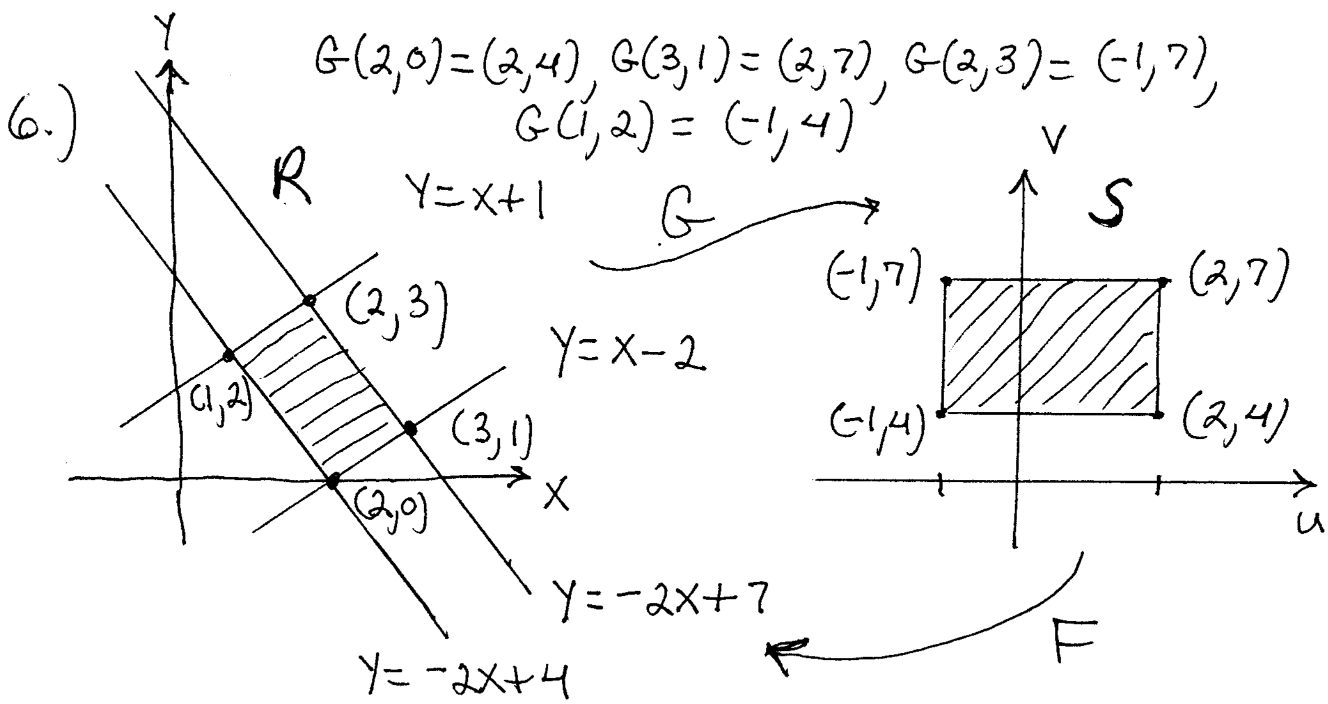
$$F(u,v) = (-u-3v, -u-2v) = (x,y);$$

$$J(P) = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} -1 & -3 \\ -1 & -2 \end{vmatrix} = (-1)(-2) - (-3)(-1) = -1$$

b.) Let $G(x,y) = (2x-3y, -x+y) = (u,v)$, a linear mapping (which preserves lines); then $G(0,0)=(0,0)$, $G(0,1)=(-3,1)$, $G(-3,-2)=(0,1)$, $G(-3,-3)=(3,0)$



6.) $G(2,0)=(2,4)$, $G(3,1)=(2,7)$, $G(2,3)=(-1,7)$, $G(1,2)=(-1,4)$



Then $\iint_R f(x,y)\,dx\,dy = \iint_S f(F(u,v))\cdot|J(P)|\,dv\,du$

$$\to \iint_R (2x^2 - xy - y^2)\,dx\,dy$$

$$= \iint_S \left[2\left(\tfrac{1}{3}u + \tfrac{1}{3}v\right)^2 - \left(\tfrac{1}{3}u + \tfrac{1}{3}v\right)\left(\tfrac{1}{3}v - \tfrac{2}{3}u\right) - \left(\tfrac{1}{3}v - \tfrac{2}{3}u\right)^2\right]\cdot\left|\tfrac{1}{3}\right|\,dv\,du$$

$$= \int_{-1}^{2}\int_{4}^{7} \left[2\left(\tfrac{1}{9}u^2 + \tfrac{2}{9}uv + \tfrac{1}{9}v^2\right) - \left(\tfrac{1}{9}uv - \tfrac{2}{9}u^2 + \tfrac{1}{9}v^2 - \tfrac{2}{9}uv\right) - \left(\tfrac{1}{9}v^2 - \tfrac{4}{9}uv + \tfrac{4}{9}u^2\right)\right]\left(\tfrac{1}{3}\right)dv\,du$$

$$= \cdots = \frac{1}{3}\int_{-1}^{2}\int_{4}^{7} uv\,dv\,du$$

$$= \frac{1}{3}\int_{-1}^{2}\left(u\cdot\tfrac{1}{2}v^2\Big|_{v=4}^{v=7}\right)du$$

$$= \frac{1}{3}\int_{-1}^{2}\left(\frac{49}{2}u - \frac{16}{2}u\right)du = \frac{1}{3}\int_{-1}^{2}\frac{33}{2}u\,du$$

$$= \frac{11}{2}\cdot\frac{1}{2}u^2\Big|_{-1}^{2} = \frac{11}{4}(4-1) = \frac{33}{4}$$

8.) $\iint_R f(x,y)\,dx\,dy = \iint_S f(F(u,v))\,|J(P)|\,du\,dv \to$

$$\iint_R 2(x-y)\,dx\,dy = \iint_S 2((-u-3v)-(-u-2v))\cdot|-1|\,du\,dv$$

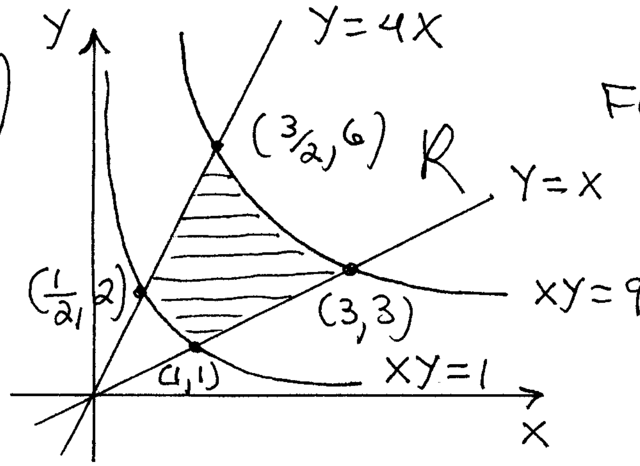
$$= \int_0^1 \int_{-3v}^{3-3v} 2(-v)\, du\, dv$$

$$= \int_0^1 \left(-2uv \Big|_{u=-3v}^{u=3-3v} \right) dv$$

$$= \int_0^1 [-2(3-3v)v - -2(-3v)v]\, dv$$

$$= \int_0^1 (-6v + \cancel{6v^2} - \cancel{6v^2})\, dv$$

$$= -3v^2 \Big|_0^1 = -3$$

9.) 

Given

$F(u,v) = (u/v, uv) = (x,y) \rightarrow$

$\begin{cases} x = u/v \rightarrow u = xv \\ y = uv \end{cases}$ (SUB)

$\rightarrow y = (xv)v = xv^2 \rightarrow$

$v^2 = y/x \rightarrow v = \sqrt{y/x}$

and $u = x\sqrt{y/x} = \sqrt{xy}$, so

$G(x,y) = (\sqrt{xy}, \sqrt{y/x}) = (u,v)$;

$G(1,1) = (1,1)$, $G(3,3) = (3,1)$, $G(\frac{1}{2}, 2) = (1,2)$, and

$G(3/2, 6) = (3,2)$; image of line $y = 4x$ under G is

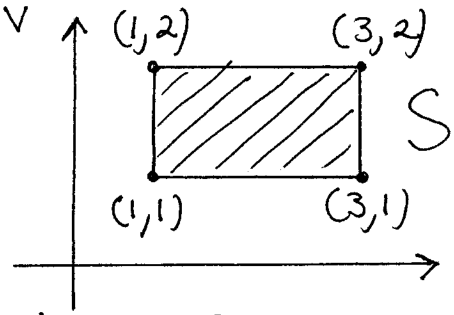
$G(x, 4x) = (\sqrt{4x^2}, \sqrt{4}) = (2x, 2) = (u,v) \rightarrow$

$v = 2$, $u = 2x$ for $\frac{1}{2} \le x \le 3/2$; image of line $y = x$ under G is

$G(x,x) = (\sqrt{x^2}, \sqrt{1}) = (x, 1) = (u,v) \rightarrow$

$v = 1$, $u = x$ for $1 \le x \le 3$; image of $y = 1/x$

under G is $G(x, \frac{1}{x}) = (\sqrt{1}, \sqrt{\frac{1}{x^2}}) = (1, \frac{1}{x}) = (u, v) \to$
$u = 1$, $v = 1/x$ for $\frac{1}{2} \le x \le 1$; image of $y = 9/x$
under G is $G(x, 9/x) = (\sqrt{9}, \sqrt{9/x^2}) = (3, \frac{3}{x}) = (u, v) \to$
$u = 3$, $v = 3/x$ for $3/2 \le x \le 3$:



$$J(P) = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1}{v} & \frac{-u}{v^2} \\ v & u \end{vmatrix} = \frac{u}{v} - \frac{-u}{v} = \frac{2u}{v} \; ;$$

then $\iint_R f(x,y)\, dx\, dy = \iint_S f(F(u,v)) |J(P)|\, dv\, du$

$$= \int_1^3 \int_1^2 (v + u) \cdot \frac{2u}{v}\, dv\, du$$

$$= \int_1^3 \int_1^2 \left(2u + \frac{2u^2}{v}\right) dv\, du$$

$$= \int_1^3 \left(2uv + 2u^2 \cdot \ln|v|\right)\Big|_{v=1}^{v=2} du$$

$$= \int_1^3 \left[(4u + 2u^2 \ln 2) - (2u + 2u^2 \overset{0}{\cancel{\ln 1}})\right] du$$

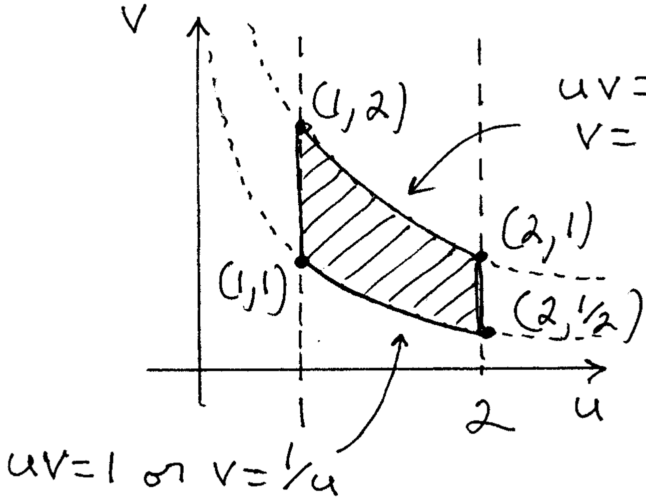
$$= \int_1^3 (2u + 2\ln 2 \cdot u^2)\, du$$

$$= \left(u^2 + 2\ln 2 \cdot \frac{1}{3} u^3\right)\Big|_1^3$$

$$= (9 + 18 \ln 2) - (1 + \frac{2}{3} \ln 2)$$

$$= 8 + \frac{52}{3} \ln 2$$

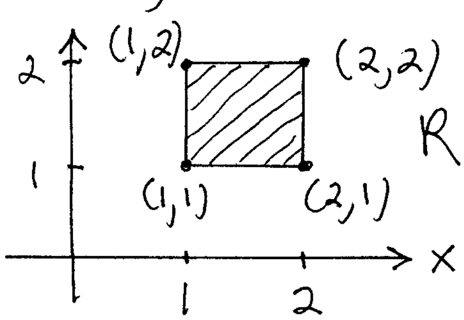
10.) a.) $F(u,v) = (u, uv) = (x,y)$



$F(1,1) = (1,1)$,
$F(1,2) = (1,2)$,
$F(2,\frac{1}{2}) = (2,1)$,
$F(2,1) = (2,2)$;

$$J(P) = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ v & u \end{vmatrix} = u - 0 = u \ ;$$

b.) the image of $u=1$ under F is $F(u,v) = F(1,v) = (1,v) = (x,y) \to x=1,\ y=v$ for $1 \le v \le 2$; the image of $u=2$ under F is $F(u,v) = F(2,v) = (2,2v) = (x,y) \to x=2,\ y=2v$ for $\frac{1}{2} \le v \le 1$; the image of $v = 2/u$ is $F(u,v) = F(u, 2/u) = (u,2) = (x,y) \to x=u,\ y=2$ for $1 \le u \le 2$; the image of $v = 1/u$ is $F(u,v) = F(u,\frac{1}{u}) = (u,1) = (x,y) \to x=u,\ y=1$ for $1 \le u \le 2$;



$$\iint_R f(x,y)\,dy\,dx = \iint_S f(F(u,v)) \cdot |J(P)| \cdot dv\,du \ ;$$

$$\iint_R f(x,y)\,dy\,dx = \int_1^2\int_1^2 \frac{y}{x}\,dy\,dx = \int_1^2 \left(\frac{1}{x}\cdot\frac{1}{2}y^2\Big|_1^2\right)dx$$

$$= \int_1^2 \frac{3}{2}\cdot\frac{1}{x}\,dx = \frac{3}{2}\ln|x|\Big|_1^2 = \frac{3}{2}\ln 2 - \frac{3}{2}\cancel{\ln 1}^{\,0}$$

$$= \frac{3}{2}\ln 2$$ and

$$\iint_S f(F(u,v))\cdot|J(P)|\cdot dv\,du$$

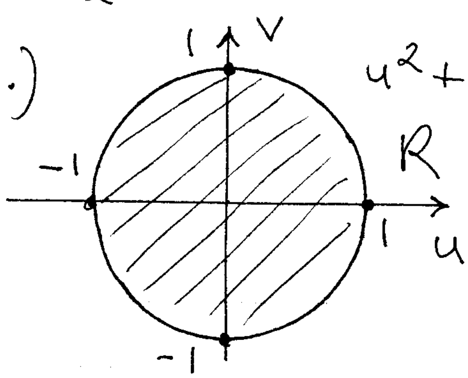
$$= \int_1^2\int_{1/u}^{2/u} \frac{\cancel{u}v}{\cancel{u}}\cdot|u|\,dv\,du = \int_1^2\int_{\frac{1}{u}}^{\frac{2}{u}} uv\,dv\,du$$

$$= \int_1^2 \left(u\cdot\frac{1}{2}v^2\Big|_{v=1/u}^{v=2/u}\right)du$$

$$= \int_1^2 \left(\frac{u}{2}\left(\frac{2}{u}\right)^2 - \frac{u}{2}\left(\frac{1}{u}\right)^2\right)du = \int_1^2 \frac{3}{2}\cdot\frac{1}{u}\,du$$

$$= \frac{3}{2}\ln|u|\Big|_1^2 = \frac{3}{2}\ln 2 - \frac{3}{2}\cancel{\ln 1}^{\,0} = \frac{3}{2}\ln 2$$

12.)

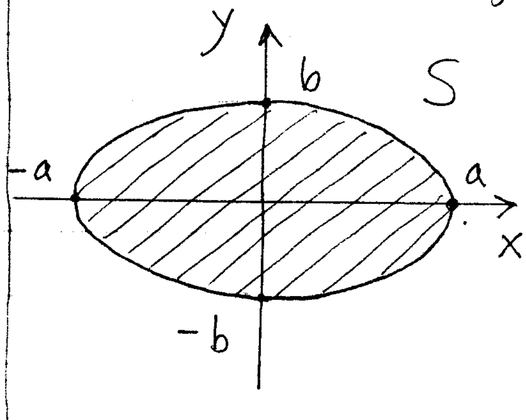


$u^2+v^2=1$

$F(u,v)=(au,bv)=(x,y) \to$

$$\begin{cases} x=au \\ y=bv \end{cases} \to \begin{cases} u = x/a \\ v = y/b \end{cases}$$ and

$$u^2+v^2=1 \to \left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1 \quad \text{(ellipse)};$$



$$J(P) = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

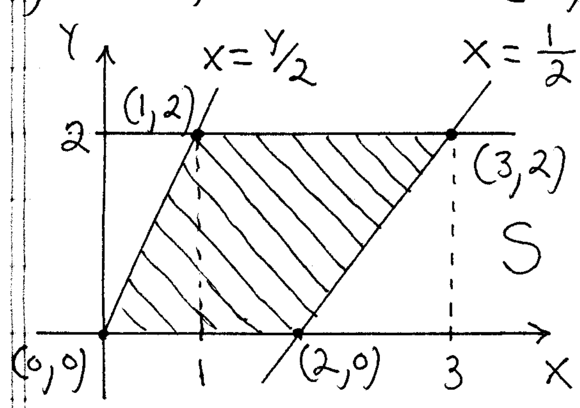
$$= \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab - 0 = ab,$$

$$\iint_S 1 \, dy\, dx = \iint_R 1 \cdot |J(P)| \, dv\, du$$

$$= \iint_R ab \, dv\, du = ab \cdot \iint_R 1 \, dv\, du$$

$$= ab \cdot \pi (1)^2 = ab\pi .$$

14.) $F(u,v) = (u + \frac{1}{2}v, v) = (x,y)$

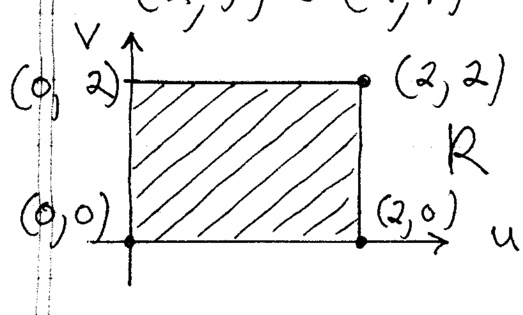


$\begin{cases} x = u + \frac{1}{2}v \\ y = v \end{cases}$ (SUB) $\rightarrow$

$x = u + \frac{1}{2}y \rightarrow u = x - \frac{1}{2}y$ and $v = y$ $\rightarrow$

$G(x,y) = (x - \frac{1}{2}y, y) = (u,v)$,

$G(0,0) = (0,0)$, $G(2,0) = (2,0)$, $G(3,2) = (2,2)$, $G(1,2) = (0,2)$; the image of $y=0$ under G is $G(x,y) = G(x,0) = (x,0) = (u,v) \rightarrow u = x,\ v = 0$ for $0 \le x \le 2$; the image of $y=2$ under G is $G(x,y) = G(x,2) = (x-1, 2) = (u,v) \rightarrow$ $u = x-1$, $v = 2$ for $1 \le x \le 3$; the image of $x = y/2$ under G is $G(x,y) = G(y/2, y) = (0,y) = (u,v)$ $\rightarrow u = 0,\ v = y$ for $0 \le y \le 2$; the image of $x = \frac{1}{2}y + 2$ under G is $G(x,y) = G(\frac{1}{2}y + 2, y)$ $= (2,y) = (u,v) \rightarrow u = 2,\ v = y$ for $0 \le y \le 2$;



$$J(P) = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 1 & 1/2 \\ 0 & 1 \end{vmatrix}$$

$$= (1)(1) - (0)(1/2) = 1 \ ;$$

$$\int_0^2 \int_{\frac{y}{2}}^{\frac{1}{2}y+2} y^3(2x-y)\, e^{(2x-y)^2}\, dx\, dy$$

$$= \iint_R v^3\,(2u)\,e^{(2u)^2} \cdot |J(P)|\, dv\, du$$

$$= \int_0^2 \int_0^2 2v^3\, u\, e^{4u^2} \cdot (1)\, dv\, du$$

$$= \int_0^2 \left(\frac{2}{4} v^4 \cdot u e^{4u^2} \Big|_{v=0}^{v=2} \right) du$$

$$= \int_0^2 8 \cdot u e^{4u^2} du = 8 \cdot \frac{1}{8} e^{4u^2} \Big|_0^2$$

$$= e^{16} - 1 \; .$$

15.) a.) $x = u\cos v$, $y = u\sin v$ $\rightarrow$

$$J(P) = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} \cos v & -u\sin v \\ \sin v & u\cos v \end{vmatrix}$$

$$= u\cos^2 v - {}^-u\sin^2 v$$

$$= u(\cos^2 v + \sin^2 v) = u\cdot(1) = u$$

16.) a.) $x = u\cos v$, $y = u\sin v$, $z = w$

$$J(P) = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix} = \begin{vmatrix} \cos v & -u\sin v & 0 \\ \sin v & u\cos v & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \cos v \cdot \begin{vmatrix} u\cos v & 0 \\ 0 & 1 \end{vmatrix} - {}^-u\sin v \begin{vmatrix} \sin v & 0 \\ 0 & 1 \end{vmatrix}$$

$$+ (0) \cdot \begin{vmatrix} \sin v & u\cos v \\ 0 & 0 \end{vmatrix}$$

$= \cos v\,(u\cos v - 0) + u\sin v\,(\sin v - 0) + (0)$
$= u\cos^2 v + u\sin^2 v = u(\cos^2 v + \sin^2 v)$
$= u\cdot(1) = u$

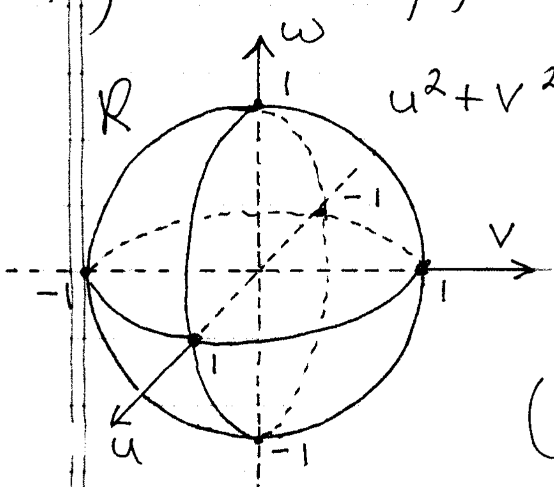
b.) $x = 2u - 1,\ y = 3v - 4,\ z = \frac{1}{2}w - 2$

$$J(P) = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix} = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1/2 \end{vmatrix}$$

$$= 2\cdot\begin{vmatrix} 3 & 0 \\ 0 & 1/2 \end{vmatrix} - 0\cdot\begin{vmatrix} 0 & 0 \\ 0 & 1/2 \end{vmatrix} + 0\cdot\begin{vmatrix} 0 & 3 \\ 0 & 0 \end{vmatrix}$$

$$= 2(3)(1/2) - 0 + 0 = 3$$

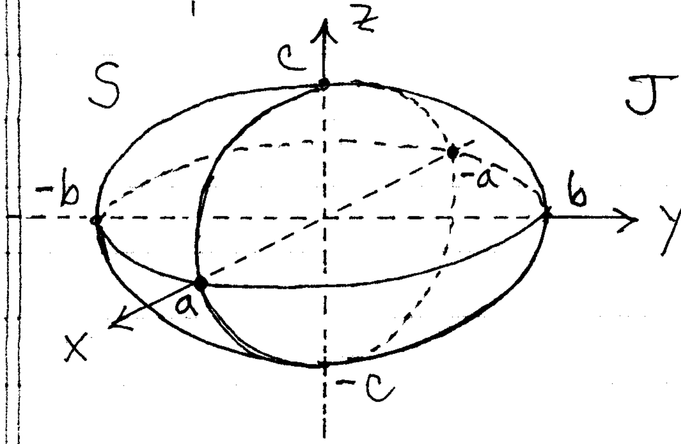
20.) Let $F(u,v,w) = (au, bv, cw) = (x,y,z)$



$u^2 + v^2 + z^2 = 1$ (sphere)

$$\begin{cases} x = au \\ y = bv \\ z = cw \end{cases} \rightarrow \begin{cases} u = x/a \\ v = y/b \\ w = z/c \end{cases} \rightarrow$$

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1 \text{ (ellipse);}$$



$$J(P) = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$

$$= \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$$

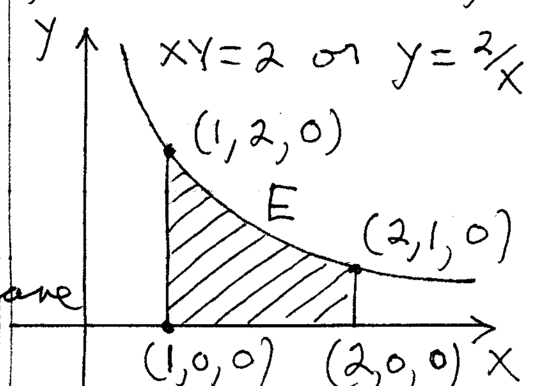
$$\iiint_S 1\, dz\, dy\, dx = \iiint_R 1 \cdot |J(P)| \cdot dw\, dv\, du$$

$$= \iiint_R abc\, dw\, dv\, du = abc \iiint_R 1\, dw\, dv\, du$$

$$= abc \cdot \frac{4}{3}\pi^{3}$$

22.) $D:\ 1 \le x \le 2,\ 0 \le xy \le 2,\ 0 \le z \le 1$

proj. of D on xy-plane



$G(x,y,z) = (x, xy, 3z) = (u,v,w)$

$$\begin{cases} u = x & \rightarrow x = u \\ v = xy & \\ w = 3z & \end{cases} \quad \text{(SUB)}$$

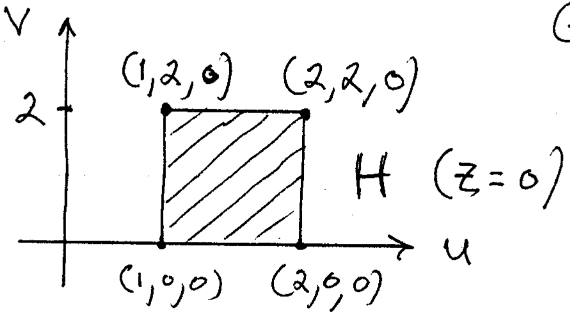
$\rightarrow v = uy \rightarrow y = v/u$ and

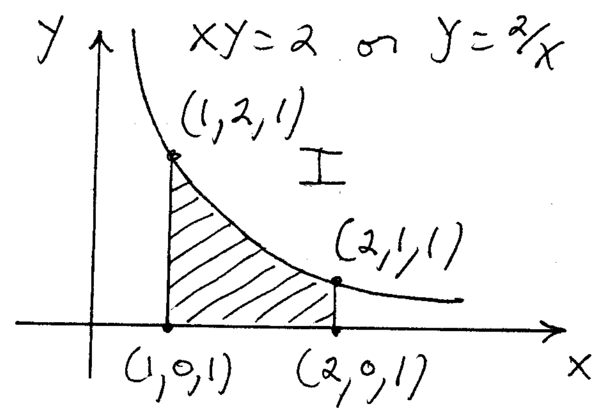
$z = \frac{1}{3}w$, so $F(u,v,w) = \left(u, \frac{v}{u}, \frac{1}{3}w\right) = (x,y,z)$;

$$J(P) = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_y & z_w \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ -\frac{v}{u^2} & \frac{1}{u} & 0 \\ 0 & 0 & 1/3 \end{vmatrix} = \frac{1}{3u} \quad ;$$

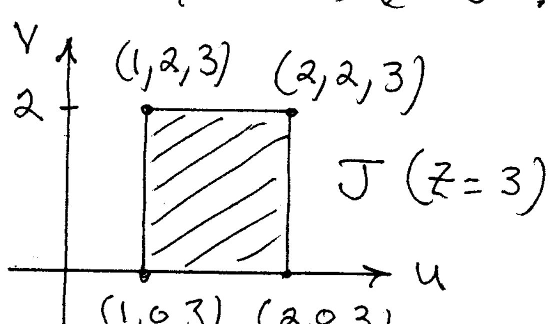
the image of $z=0$ under G is $G(x,y,z) = G(x,y,0) = (x, xy, 0) \rightarrow u = x,\ v = xy$, and $w = 0$ for x, y in region E;
$G(1,0,0) = (1,0,0)$, $G(2,0,0) = (2,0,0)$, $G(2,1,0) = (2,2,0)$, $G(1,2,0) = (1,2,0)$; the image of $x=1$ under G is
$G(x,y,z) = G(1,y,0) = (1,y,0) = (u,v,w) \rightarrow$ $u = 1,\ v = y$, and $w = 0$ for $0 \le y \le 2$;

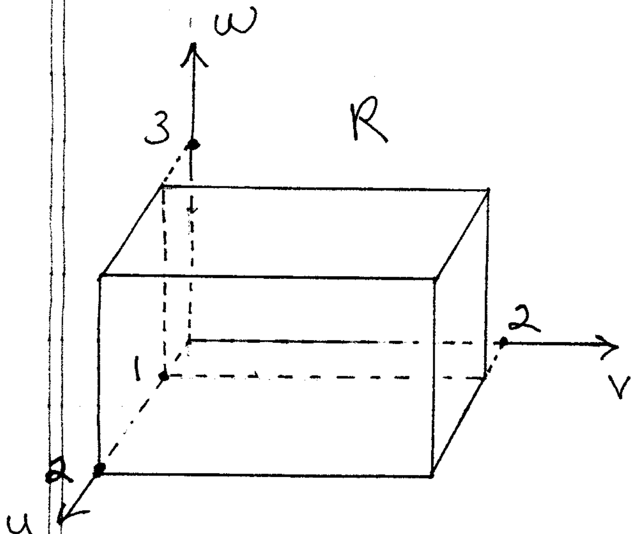
the image of $x=2$ under G is
$G(x,y,z) = G(2,y,0) = (2,2y,0) = (u,v,w) \rightarrow$
$u=2$, $v=2y$, and $w=0$ for $0 \le y \le 1$;
the image of $y=0$ under G is
$G(x,y,z) = G(x,0,0) = (x,0,0) = (u,v,w) \rightarrow$
$u=x$, $v=0$, and $w=0$ for $1 \le x \le 2$; the image of $y = 2/x$ under G is $G(x,y,z) = G(x, 2/x, 0)$
$= (x,2,0) = (u,v,w) \rightarrow u=x$, $v=2$, and $w=0$
for $1 \le x \le 2$; thus image of E under G is **H** ; consider region I in plane $z=1$:

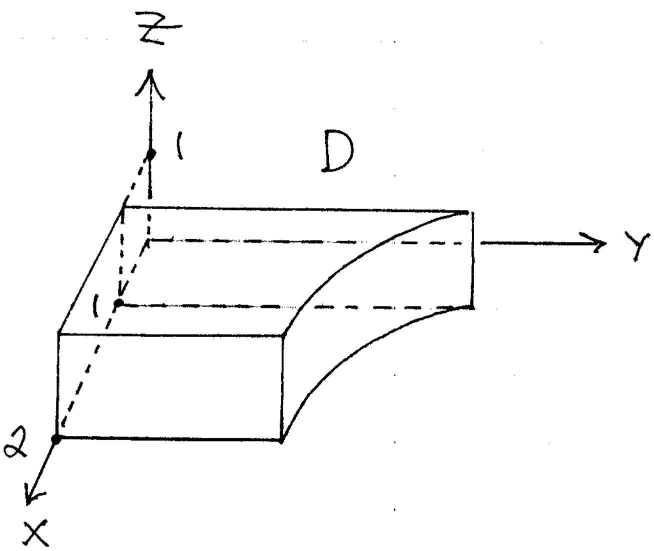




the image of I under G is J:







Then

$$\iiint_D (x^2y + 3xyz)\, dx\, dy\, dz$$

$$= \iiint_R \left[(u)^2\left(\frac{v}{u}\right) + 3(u)\left(\frac{v}{u}\right)\left(\frac{1}{3}w\right)\right] \cdot |J(P)| \cdot dw\, dv\, du$$

$$= \int_1^2\int_0^2\int_0^3 (uv + vw) \cdot \frac{1}{3u}\, dw\, dv\, du$$

$$= \int_1^2\int_0^2\int_0^3 \left(\frac{1}{3}v + \frac{1}{3}\cdot\frac{vw}{u}\right) dw\, dv\, du$$

$$= \frac{1}{3}\int_1^2\int_0^2\int_0^3 \left(v + \frac{vw}{u}\right) dw\, dv\, du$$

$$= \frac{1}{3}\int_1^2\int_0^2 \left(vw + \frac{v}{u}\cdot\frac{1}{2}w^2\right)\Big|_{w=0}^{w=3} dv\, du$$

$$= \frac{1}{3}\int_1^2\int_0^2 \left(3v + \frac{9}{2}\cdot\frac{v}{u}\right) dv\, du$$

$$= \frac{1}{3}\int_1^2 \left(3\cdot\frac{1}{2}v^2 + \frac{9}{2}\cdot\frac{1}{u}\cdot\frac{1}{2}v^2\right)\Big|_{v=0}^{v=2} du$$

$$= \frac{1}{3}\int_1^2 \left(6 + 9\cdot\frac{1}{u}\right) du$$

$$= \frac{1}{3}\left(6u + 9\ln|u|\right)\Big|_1^2$$

$$= \frac{1}{3}\left(12 + 9\ln|2|\right) - \frac{1}{3}\left(6 + 9\cancel{\ln 1}^{\,0}\right)$$

$$= 4 + 3\ln 2 - 2$$

$$= 2 + \ln 8$$