Section 16.2

1.) $f(x,y,z) = (x^2+y^2+z^2)^{-1/2} \longrightarrow$

$$\vec{\nabla} f(x,y,z) = \frac{-1}{\cancel{2}}(x^2+y^2+z^2)^{-3/2}\cdot \cancel{2}x\,\vec{i} + \frac{-1}{\cancel{2}}(x^2+y^2+z^2)^{-3/2}\cdot \cancel{2}y\,\vec{j} + \frac{-1}{\cancel{2}}(x^2+y^2+z^2)^{-3/2}\cdot \cancel{2}z\,\vec{k}$$

$$= \frac{-x}{(x^2+y^2+z^2)^{3/2}}\vec{i} + \frac{-y}{(x^2+y^2+z^2)^{3/2}}\vec{j} + \frac{-z}{(x^2+y^2+z^2)^{3/2}}\vec{k}$$

3.) $g(x,y,z) = e^z - \ln(x^2+y^2) \longrightarrow$

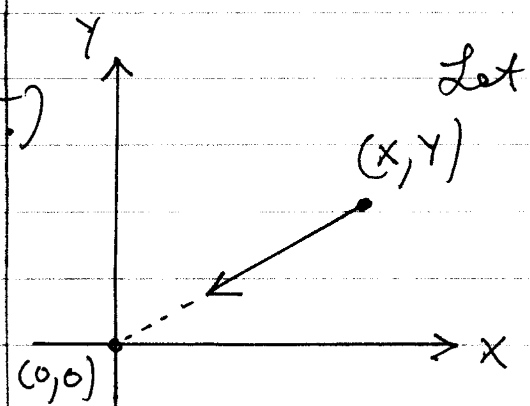
$$\vec{\nabla} g(x,y,z) = \frac{-1}{x^2+y^2}\cdot 2x\,\vec{i} + \frac{-1}{x^2+y^2}2y\,\vec{j} + e^z\,\vec{k}$$

$$= \frac{-2x}{x^2+y^2}\vec{i} + \frac{-2y}{x^2+y^2}\vec{j} + e^z\,\vec{k}$$

4.) $g(x,y,z) = xy + yz + xz \longrightarrow$

$$\vec{\nabla} g(x,y,z) = (y+z)\vec{i} + (x+z)\vec{j} + (x+y)\vec{k}$$

5.)



Let $\vec{v} = (0-x)\,\vec{i} + (0-y)\,\vec{j}$

$= (-x)\,\vec{i} + (-y)\,\vec{j}$ and

$|\vec{v}| = \sqrt{(-x)^2 + (-y)^2}$

$= \sqrt{x^2+y^2}$;

so unit vector is $\vec{u} = \dfrac{-x}{\sqrt{x^2+y^2}}\vec{i} + \dfrac{-y}{\sqrt{x^2+y^2}}\vec{j}$;

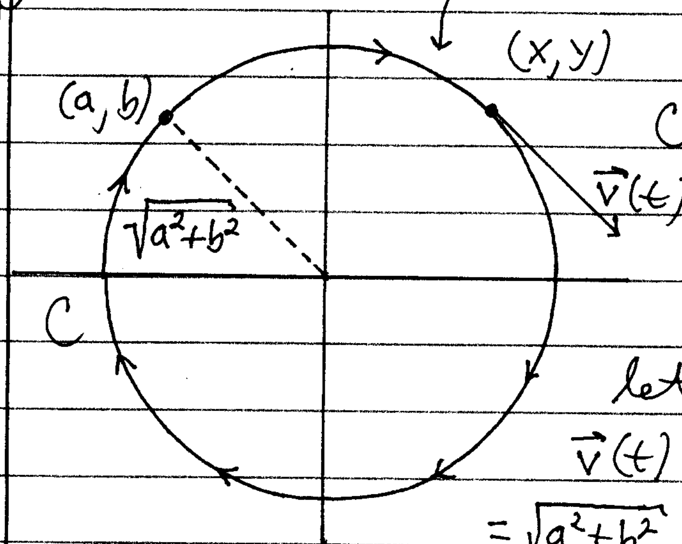
magnitude of $\vec{F}$ is

$$|\vec{F}(x,y)| = k \cdot \frac{1}{x^2+y^2} \quad \text{so vector field is}$$

$$\vec{F}(x,y) = k \cdot \frac{1}{x^2+y^2} \cdot \vec{u} \qquad (k \text{ is a constant.})$$

$$= \frac{-kx}{(x^2+y^2)^{3/2}} \vec{i} + \frac{-ky}{(x^2+y^2)^{3/2}} \vec{j}$$

6.) circle: $x^2+y^2 = (\sqrt{a^2+b^2})^2$



$$C : \begin{cases} x = \sqrt{a^2+b^2} \sin t \\ y = \sqrt{a^2+b^2} \cos t \end{cases} \quad \text{for } 0 \le t \le 2\pi \ ;$$

let $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} \xrightarrow{D}$

$$\vec{v}(t) = x'(t)\vec{i} + y'(t)\vec{j}$$

$$= \sqrt{a^2+b^2} \cos t \cdot \vec{i} - \sqrt{a^2+b^2} \sin t \cdot \vec{j}$$

$$= y\vec{i} - x\vec{j} \ ;$$

$$|\vec{v}(t)| = \sqrt{(\sqrt{a^2+b^2} \cdot \cos t)^2 + (-\sqrt{a^2+b^2} \cdot \sin t)^2}$$

$$= \sqrt{(a^2+b^2)\underbrace{(\cos^2 t + \sin^2 t)}_{1}} = \sqrt{a^2+b^2} \ ;$$

let

$$\vec{F}(x,y) = \vec{v}(t) = y\vec{i} - x\vec{j} .$$

7.) $\vec{F}(x,y,z) = (3y)\vec{i} + (2x)\vec{j} + (4z)\vec{k}$

a.) $C_1 : \vec{r}(t) = (t)\vec{i} + (t)\vec{j} + (t)\vec{k}$ for $0 \le t \le 1$

D, $\vec{r}'(t) = (1)\vec{i} + (1)\vec{j} + (1)\vec{k}$ and

$\vec{F}(x,y,z) = \vec{F}(t,t,t) = (3t)\vec{i} + (2t)\vec{j} + (4t)\vec{k}$,

so

$$\text{Work} = \int_{C_1} \vec{F}\cdot\vec{T}\,ds = \int_{C_1} \vec{F}\cdot\vec{r}'(t)\,dt$$

$$= \int_0^1 [(3t)(1) + (2t)(1) + (4t)(1)]\,dt$$

$$= \int_0^1 9t\,dt = 9\cdot\frac{1}{2}t^2\Big|_0^1 = \frac{9}{2}(1)^2 = \frac{9}{2}$$

b.) C_2: $\vec{r}(t) = (t)\vec{i} + (t^2)\vec{j} + (t^4)\vec{k}$ for $0 \le t \le 1$

D, $\vec{r}'(t) = (1)\vec{i} + (2t)\vec{j} + (4t^3)\vec{k}$ and

$$\vec{F}(x,y,z) = \vec{F}(t,t^2,t^4)$$

$$= (3t^2)\vec{i} + (2t)\vec{j} + (4t^4)\vec{k}$$

so

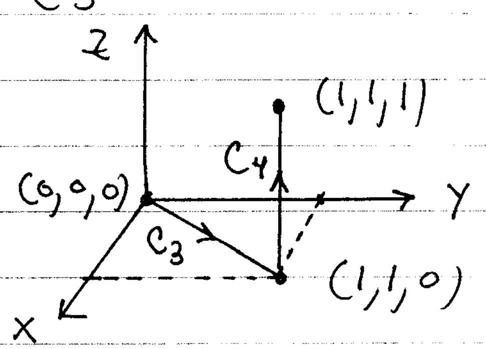
$$\text{Work} = \int_{C_2} \vec{F}\cdot\vec{T}\,ds = \int_{C_2} \vec{F}\cdot\vec{r}'(t)\,dt$$

$$= \int_0^1 [(3t^2)(1) + (2t)(2t) + (4t^4)(4t^3)]\,dt$$

$$= \int_0^1 (3t^2 + 4t^2 + 16t^7)\,dt = \int_0^1 (7t^2 + 16t^7)\,dt$$

$$= \left(\frac{7}{3}t^3 + 2t^8\right)\Big|_0^1 = \frac{7}{3} + \frac{6}{3} = \frac{13}{3}$$

c.)



C_3: $\begin{cases} x = t \\ y = t \\ z = 0 \end{cases}$ for $0 \le t \le 1$,

C_4: $\begin{cases} x = 1 \\ y = 1 \\ z = t \end{cases}$ for $0 \le t \le 1$.

$\vec{r}_3(t) = (t)\vec{i} + (t)\vec{j} \rightarrow \vec{r}_3\,'(t) = (1)\vec{i} + (1)\vec{j}$ for $0 \le t \le 1$ and

$\vec{F}(x,y,z) = \vec{F}(t,t,0) = (3t)\vec{i} + (2t)\vec{j} + (0)\vec{k}$;

$\vec{r}_4(t) = (1)\vec{i} + (1)\vec{j} + (t)\vec{k} \Rightarrow \vec{r}_4\,'(t) = (1)\vec{k}$ for $0 \le t \le 1$ and

$\vec{F}(x,y,z) = \vec{F}(1,1,t) = (3)\vec{i} + (2)\vec{j} + (4t)\vec{k}$; ~~then~~

$$\text{work} = \int_{C_3} \vec{F} \cdot \vec{T}\, ds + \int_{C_4} \vec{F} \cdot \vec{T}\, ds$$

$$= \int_{C_3} \vec{F} \cdot \vec{r}_3\,'(t)\, dt + \int_{C_4} \vec{F} \cdot \vec{r}_4\,'(t)\, dt$$

$$= \int_0^1 [(3t)(1) + (2t)(1)]\, dt + \int_0^1 [(4t)(1)]\, dt$$

$$= \int_0^1 5t\, dt + \int_0^1 4t\, dt = \frac{5}{2}t^2 \Big|_0^1 + 2t^2 \Big|_0^1$$

$$= \frac{5}{2} + \frac{4}{2} = \frac{9}{2}$$

12.) $\vec{F}(x,y,z) = (y+z)\vec{i} + (x+z)\vec{j} + (x+y)\vec{k}$

a.) $C_1 : \vec{r}(t) = (t)\vec{i} + (t)\vec{j} + (t)\vec{k}$ for $0 \le t \le 1$

$\Rightarrow \vec{r}\,'(t) = (1)\vec{i} + (1)\vec{j} + (1)\vec{k}$ and

$\vec{F}(x,y,z) = \vec{F}(t,t,t) = (t+t)\vec{i} + (t+t)\vec{j} + (t+t)\vec{k}$

$= (2t)\vec{i} + (2t)\vec{j} + (2t)\vec{k}$

so

$$\text{work} = \int_{C_1} \vec{F} \cdot \vec{T}\, ds = \int_{C_1} \vec{F} \cdot \vec{r}\,'(t)\, dt$$

$$= \int_0^1 [(2t)(1)+(2t)(1)+(2t)(1)]\,dt = \int_0^1 6t\,dt$$

$$= 3t^2 \Big|_0^1 = 3$$

b.) $C_2: \vec{r}(t) = (t)\vec{i} + (t^2)\vec{j} + (t^4)\vec{k}$ for $0 \le t \le 1$

$\xrightarrow{D} \vec{r}\,'(t) = (1)\vec{i} + (2t)\vec{j} + (4t^3)\vec{k}$ and

$$\vec{F}(x,y,z) = \vec{F}(t,t^2,t^4)$$
$$= (t^2+t^4)\vec{i} + (t+t^4)\vec{j} + (t+t^2)\vec{k};$$

then

$$\text{work} = \int_{C_2} \vec{F}\cdot\vec{T}\,ds = \int_{C_2} \vec{F}\cdot\vec{r}\,'(t)\,dt$$

$$= \cancel{\int_0^1 [(t^2+t^4)(1)+(t+t^4)(2t)+(t+t^2)(4t^3)]\,dt}$$

$$= \int_0^1 [t^2+t^4+2t^2+2t^5+4t^4+4t^5]\,dt$$

$$= \int_0^1 [3t^2+5t^4+6t^5]\,dt$$

$$= (t^3+t^5+t^5)\Big|_0^1 = 1+1+1 = 3$$

c.) SEE solution 7.) c.)

$\vec{r}_3(t) = (t)\vec{i} + (t)\vec{j} \xrightarrow{D} \vec{r}_3\,'(t) = (1)\vec{i} + (1)\vec{j}$ for $0 \le t \le 1$

and $\vec{F}(x,y,z) = \vec{F}(t,t,0) = (t+0)\vec{i} + (t+0)\vec{j} + (t+t)\vec{k}$
$= (t)\vec{i} + (t)\vec{j} + (2t)\vec{k}$;

for $0 \le t \le 1$ → $\vec{r}_4(t) = (1)\vec{i} + (1)\vec{j} + (t)\vec{k} \xrightarrow{D} \vec{r}_4\,'(t) = (1)\vec{k}$ and

$\vec{F}(x,y,z) = \vec{F}(1,1,t) = (1+t)\vec{i} + (1+t)\vec{j} + (1+1)\vec{k}$
$= (1+t)\vec{i} + (1+t)\vec{j} + (2)\vec{k}$, ~~then~~

$$\text{Work} = \int_{C_3} \vec{F}\cdot\vec{T}\,ds + \int_{C_4} \vec{F}\cdot\vec{T}\,ds$$

$$= \int_{C_3} \vec{F}\cdot\vec{r}_3\,'(t)\,dt + \int_{C_4} \vec{F}\cdot\vec{r}_4\,'(t)\,dt$$

$$= \int_0^1 [(t)(1)+(t)(1)+(2t)(0)]\,dt$$

$$+ \int_0^1 [(1+t)(0)+(1+t)(0)+(2)(1)]\,dt$$

$$= \int_0^1 2t\,dt + \int_0^1 2\,dt = t^2\Big|_0^1 + 2t\Big|_0^1$$

$$= 1+2 = 3 .$$

13.) $\vec{F}(x,y,z) = (xy)\,\vec{i} + (y)\,\vec{j} + (-yz)\,\vec{k}$

C: $\vec{r}(t) = (t)\,\vec{i} + (t^2)\,\vec{j} + (t)\,\vec{k}$ for $0 \le t \le 1 \xrightarrow{D}$

$\vec{r}\,'(t) = (1)\,\vec{i} + (2t)\,\vec{j} + (1)\,\vec{k}$, and

$\vec{F}(x,y,z) = \vec{F}(t, t^2, t)$

$= (t\cdot t^2)\,\vec{i} + (t^2)\,\vec{j} + (-t^2\cdot t)\,\vec{k}$

$= (t^3)\,\vec{i} + (t^2)\,\vec{j} + (-t^3)\,\vec{k}$; then

$$\text{Work} = \int_C \vec{F}\cdot\vec{T}\,ds = \int_C \vec{F}\cdot\vec{r}\,'(t)\,dt$$

$$= \int_0^1 [(t^3)(1) + (t^2)(2t) + (-t^3)(1)]\,dt$$

$$= \int_0^1 [\cancel{t^3} + 2t^3 - \cancel{t^3}]\,dt = \int_0^1 2t^3\,dt$$

$$= \frac{1}{2}t^4\Big|_0^1 = \frac{1}{2}$$

15.) $\vec{F}(x,y,z) = (z)\vec{i} + (x)\vec{j} + (y)\vec{k}$,

$C: \vec{r}(t) = (\sin t)\vec{i} + (\cos t)\vec{j} + (t)\vec{k}$

for $0 \le t \le 2\pi$ $\xrightarrow{D}$

$\vec{r}\,'(t) = (\cos t)\vec{i} + (-\sin t)\vec{j} + (1)\vec{k}$, and

$\vec{F}(x,y,z) = \vec{F}(\sin t, \cos t, t)$

$= (t)\vec{i} + (\sin t)\vec{j} + (\cos t)\vec{k}$; ~~then~~

$$\text{Work} = \int_C \vec{F}\cdot\vec{T}\,ds = \int_C \vec{F}\cdot\vec{r}\,'(t)\,dt$$

$$= \int_0^{2\pi} [t\cos t - \sin^2 t + \cos t]\,dt$$

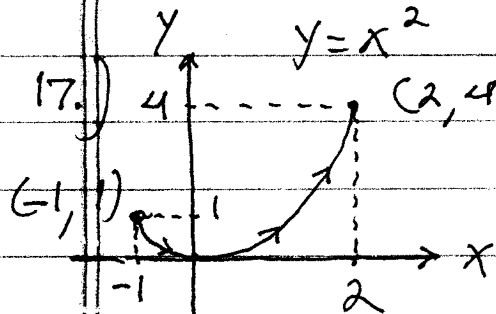
$$= \int_0^{2\pi} t\cos t\,dt - \int_0^{2\pi} \frac{1}{2}(1-\cos 2t)\,dt + \sin t\Big|_0^{2\pi}$$

(Let $u = t$, $dv = \cos t\,dt$
$\rightarrow du = dt$, $v = \sin t$)

$$= t\sin t\Big|_0^{2\pi} - \int_0^{2\pi} \sin t\,dt + \frac{-1}{2}\left(t - \frac{1}{2}\sin 2t\right)\Big|_0^{2\pi} + \sin t\Big|_0^{2\pi}$$

$$= 2\pi \underbrace{\sin 2\pi}_{0} - (0)\underbrace{\sin 0}_{0} + \cos t\Big|_0^{2\pi} - \frac{1}{2}\left[\left(2\pi - \frac{1}{2}\underbrace{\sin 4\pi}_{0}\right) - \left(0 - \frac{1}{2}\underbrace{\sin 0}_{0}\right)\right] + \underbrace{\cos 2\pi}_{1} - \underbrace{\cos 0}_{1}$$

$$= \underbrace{\cos 2\pi - \cos 0}_{0} - \pi = -\pi$$

17.) 

$$C: \begin{cases} x = t \\ y = t^2 \end{cases} \text{ for } -1 \le t \le 2$$

$$\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = 2t \longrightarrow$$

$$\int_C xy\,dx + (x+y)\,dy$$

$$= \int_{-1}^{2} \left[(t)(t^2)\frac{dx}{dt} + (t+t^2)\frac{dy}{dt} \right] dt$$

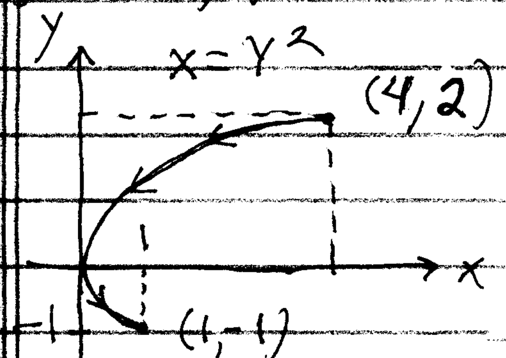
$$= \int_{-1}^{2} \left[t^3 \cdot (1) + (t+t^2)(2t) \right] dt$$

$$= \int_{-1}^{2} (3t^3 + 2t^2)\,dt = \left(\frac{3}{4}t^4 + \frac{2}{3}t^3 \right) \Big|_{-1}^{2}$$

$$= \left(12 + \frac{16}{3}\right) - \left(\frac{3}{4} - \frac{2}{3}\right)$$

$$= \frac{144}{12} + \frac{64}{12} - \frac{9}{12} + \frac{8}{12} = \frac{207}{12} = \frac{69}{4}$$

19.) $\vec{F}(x,y) = (x^2)\,\vec{i} + (-y)\,\vec{j}$ and



$$C: \begin{cases} x = t^2 \\ y = t \end{cases} \text{ for } t = 2 \text{ to } t = -1$$

$$\int_C \vec{F} \cdot \vec{T}\,ds = \int_C M\,dx + N\,dy$$

$$= \int_C \left[x^2 \cdot \frac{dx}{dt} + (-y)\frac{dy}{dt} \right] dt$$

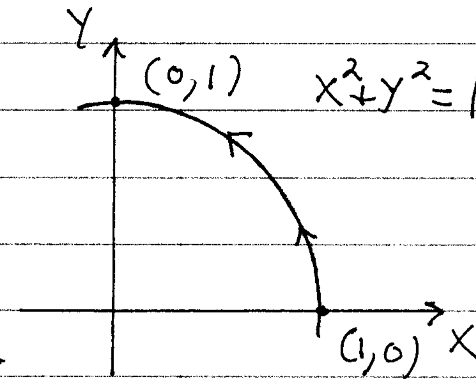
$$= \int_2^{-1} \left[(t^2)^2 (2t) + (-t)(1) \right] dt$$

$$= \int_2^{-1} \left[2t^5 - t \right] dt = \left(\frac{1}{3}t^6 - \frac{1}{2}t^2 \right) \Big|_2^{-1}$$

$$= \left(\frac{1}{3} - \frac{1}{2} \right) - \left(\frac{64}{3} - 2 \right) = -\frac{63}{3} + \frac{3}{2}$$

$$= -21 + \frac{3}{2} = -\frac{42}{2} + \frac{3}{2} = -\frac{39}{2}$$

20.) $\vec{F}(x,y) = (y)\,\vec{i} + (-x)\,\vec{j}$



$$C: \begin{cases} x = \cos t \\ y = \sin t \end{cases} \text{ for } 0 \le t \le \frac{\pi}{2}$$

$$\frac{dx}{dt} = -\sin t, \quad \frac{dy}{dt} = \cos t \,;$$

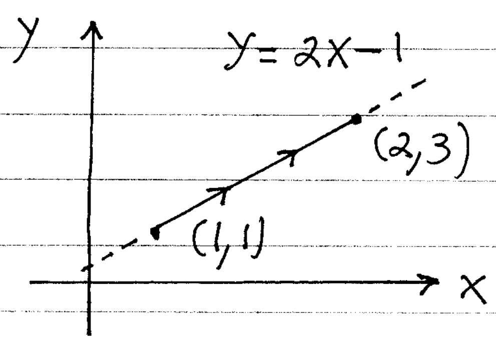
$$\int_C \vec{F} \cdot d\vec{r} = \int_C M\,dx + N\,dy$$

$$= \int_C \left[y \cdot \frac{dx}{dt} + (-x)\frac{dy}{dt} \right] dt$$

$$= \int_0^{\frac{\pi}{2}} \left[\sin t \cdot {}^-\sin t + {}^-\cos t \cdot \cos t \right] dt$$

$$= \int_0^{\frac{\pi}{2}} -(\underbrace{\sin^2 t + \cos^2 t}_{1})\,dt = -t \Big|_0^{\frac{\pi}{2}} = -\frac{\pi}{2}$$

21.) $\vec{F}(x,y) = (xy)\,\vec{i} + (y-x)\,\vec{j}$



$C: \begin{cases} x = t \\ y = 2t-1 \end{cases}$ for $1 \le t \le 2$;

$$\text{work} = \int_C \vec{F} \cdot \vec{T}\, ds = \int_C M\,dx + N\,dy$$

$$= \int_C \left[xy \cdot \frac{dx}{dt} + (y-x)\frac{dy}{dt} \right] dt$$

$$= \int_C \left[(t)(2t-1)(1) + ((2t-1)-t)(2) \right] dt$$

$$= \int_1^2 \left[2t^2 - t + 2t - 2 \right] dt = \int_1^2 \left[2t^2 + t - 2 \right] dt$$

$$= \left(\frac{2}{3}t^3 + \frac{1}{2}t^2 - 2t \right) \Big|_1^2$$

$$= \left(\frac{16}{3} + 2 - 4 \right) - \left(\frac{2}{3} + \frac{1}{2} - 2 \right)$$

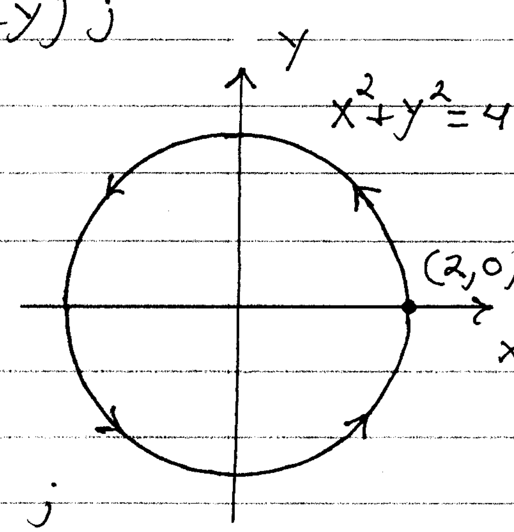
$$= \frac{14}{3} - \frac{1}{2} = \frac{25}{6}$$

22.) $f(x,y) = (x+y)^2 \longrightarrow \vec{\nabla} f = f_x \vec{i} + f_y \vec{j} \longrightarrow$

$\vec{\nabla} f(x,y) = 2(x+y)\vec{i} + 2(x+y)\vec{j}$

~~so let~~

$\vec{F}(x,y) = 2(x+y)\vec{i} + 2(x+y)\vec{j}$;



$C: \begin{cases} x = 2\cos t \\ y = 2\sin t \end{cases}$ for $0 \le t \le 2\pi$

$\frac{dx}{dt} = -2\sin t, \ \frac{dy}{dt} = 2\cos t$;

$$\text{Work} = \int_C \vec{F}\cdot\vec{T}\,ds = \int_C M\,dx + N\,dy$$

$$= \int_C \left[2(x+y)\frac{dx}{dt} + 2(x+y)\frac{dy}{dt}\right] dt$$

$$= \int_0^{2\pi} \left[2(2\cos t + 2\sin t)\cdot {}^-2\sin t + 2(2\cos t + 2\sin t)\cdot 2\cos t\right] dt$$

$$= \int_0^{2\pi} \left(-8\cancel{\sin t\cos t} - 8\sin^2 t + 8\cos^2 t + 8\cancel{\sin t\cos t}\right) dt$$

$$= \int_0^{2\pi} 8\cdot(\cos^2 t - \sin^2 t)\,dt$$

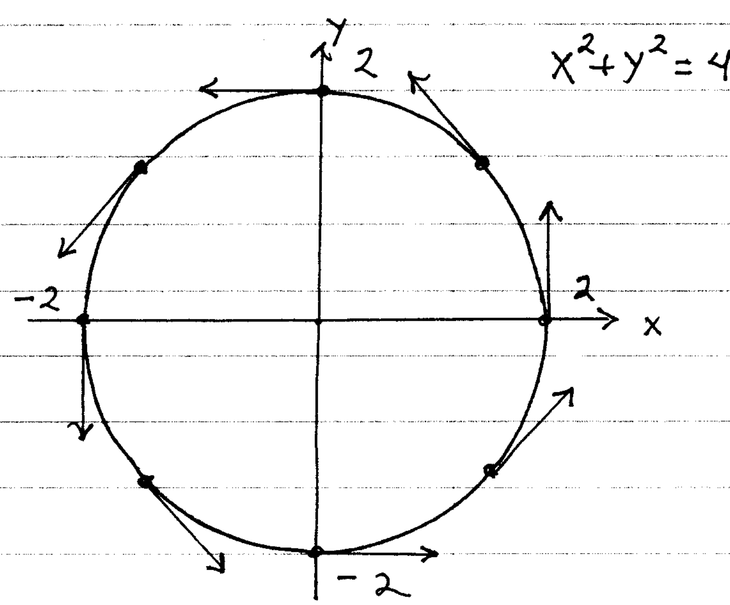
$$= \int_0^{2\pi} 8\cdot\cos 2t\,dt = 8\cdot\frac{1}{2}\sin 2t \Big|_0^{2\pi}$$

$$= 4\underbrace{\sin 4\pi}_{0} - 4\underbrace{\sin 0}_{0} = 0$$

3(.)

(x,y)	$\vec{F}(x,y)$
$(2,0)$	$\vec{j}$
$(\sqrt{2},\sqrt{2})$	$-\frac{\sqrt{2}}{2}\vec{i} + \frac{\sqrt{2}}{2}\vec{j}$
$(0,2)$	$-\vec{i}$
$(-\sqrt{2},\sqrt{2})$	$-\frac{\sqrt{2}}{2}\vec{i} - \frac{\sqrt{2}}{2}\vec{j}$
$(-2,0)$	$-\vec{j}$
$(-\sqrt{2},-\sqrt{2})$	$\frac{\sqrt{2}}{2}\vec{i} - \frac{\sqrt{2}}{2}\vec{j}$
$(0,-2)$	$\vec{i}$
$(\sqrt{2},-\sqrt{2})$	$\frac{\sqrt{2}}{2}\vec{i} + \frac{\sqrt{2}}{2}\vec{j}$

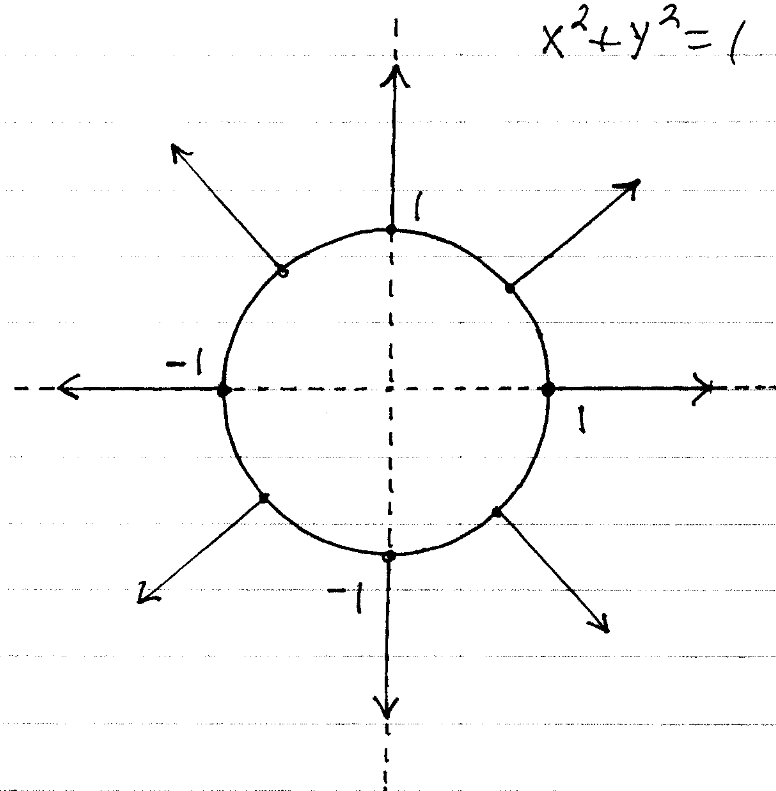
$$\vec{F}(x,y) = \frac{-y}{\sqrt{x^2+y^2}}\vec{i} + \frac{x}{\sqrt{x^2+y^2}}\vec{j}$$



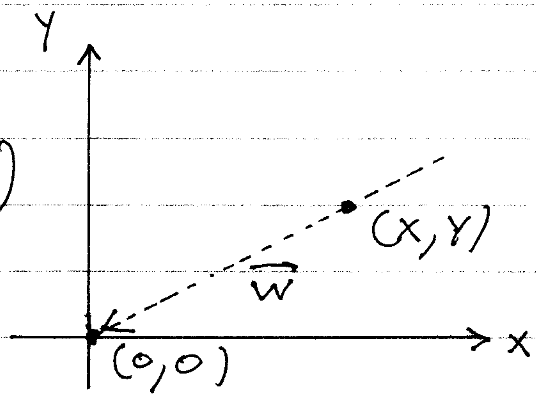
(x,y)	$\vec{F}(x,y)$
$(1,0)$	$\vec{i}$
$(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2})$	$\frac{\sqrt{2}}{2}\vec{i}+\frac{\sqrt{2}}{2}\vec{j}$
$(0,1)$	$\vec{j}$
$(-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2})$	$-\frac{\sqrt{2}}{2}\vec{i}+\frac{\sqrt{2}}{2}\vec{j}$
$(-1,0)$	$-\vec{i}$
$(-\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2})$	$-\frac{\sqrt{2}}{2}\vec{i}-\frac{\sqrt{2}}{2}\vec{j}$
$(0,-1)$	$-\vec{j}$
$(\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2})$	$\frac{\sqrt{2}}{2}\vec{i}-\frac{\sqrt{2}}{2}\vec{j}$

$$\vec{F}(x,y)=(x)\vec{i}+(y)\vec{j}$$

$x^2+y^2=1$



35.)



The vector at point (x,y) pointing at origin is

$$\vec{w}=(0-x)\vec{i}+(0-y)\vec{j}=(-x)\vec{i}+(-y)\vec{j},$$

it magnitude is

$$|\vec{w}|=\sqrt{(-x)^2+(-y)^2}=\sqrt{x^2+y^2};$$

so unit vector is

$$\vec{F}(x,y)=\frac{\vec{w}}{|\vec{w}|}=\frac{-x}{\sqrt{x^2+y^2}}\vec{i}+\frac{-y}{\sqrt{x^2+y^2}}\vec{j}$$