Section 16.3

1.) $\vec{F}(x,y,z) = \underset{M}{(yz)}\vec{i} + \underset{N}{(xz)}\vec{j} + \underset{P}{(xy)}\vec{k} \longrightarrow$

$M_y = z = N_x$, $N_z = x = P_y$, $P_x = y = M_z$, so $\vec{F}$ is conservative.

2.) $\vec{F}(x,y,z) = \underset{M}{(y\sin z)}\vec{i} + \underset{N}{(x\sin z)}\vec{j} + \underset{P}{(xy\cos z)}\vec{k} \longrightarrow$

$M_y = \sin z = N_x$, $N_z = x\cos z = P_y$,
$P_x = y\cos z = M_z$, so $\vec{F}$ is conservative.

3.) $\vec{F}(x,y,z) = \underset{M}{(y)}\vec{i} + \underset{N}{(x+z)}\vec{j} + \underset{P}{(-y)}\vec{k} \longrightarrow$

$M_y = 1 = N_x$, $N_z = 1 \neq P_y = -1$, so $\vec{F}$ is not conservative.

4.) $\vec{F}(x,y) = \underset{M}{(-y)}\vec{i} + \underset{N}{(x)}\vec{j} \longrightarrow$

$M_y = -1 \neq N_x = 1$, so $\vec{F}$ is not conservative

5.) $\vec{F}(x,y,z) = \underset{M}{(z+y)}\vec{i} + \underset{N}{(z)}\vec{j} + \underset{P}{(y+x)}\vec{k} \longrightarrow$

$M_y = 1 \neq N_x = 0$, so $\vec{F}$ is not conservative

6.) $\vec{F}(x,y,z) = \underset{M}{(e^x\cos y)}\vec{i} + \underset{N}{(-e^x\sin y)}\vec{j} + \underset{P}{(z)}\vec{k} \longrightarrow$

$M_y = -e^x\sin y = N_x$, $N_z = 0 = P_y$, $M_z = 0 = P_x$

7) $\vec{F}(x,y,z) = (2x)\vec{i} + (3y)\vec{j} + (4z)\vec{k} \rightarrow$

$$\qquad\quad f_x \qquad\quad f_y \qquad\quad f_z$$

$f_x = 2x \xrightarrow{\int_x} f = x^2 + g(y,z) \xrightarrow{D_y}$

$f_y = 0 + g_y(y,z) = 3y \xrightarrow{\int_y} g(y,z) = \frac{3}{2}y^2 + h(z)$

$\rightarrow f = x^2 + \frac{3}{2}y^2 + h(z) \xrightarrow{D_z} f_z = 0 + 0 + h'(z) = 4z$

$\rightarrow h(z) = 2z^2$ so

$f(x,y,z) = x^2 + \frac{3}{2}y^2 + 2z^2$

8) $\vec{F}(x,y,z) = (y+z)\vec{i} + (x+z)\vec{j} + (x+y)\vec{k} \longrightarrow$

$$\qquad\quad f_x \qquad\quad f_y \qquad\quad f_z$$

$f_x = y + z \xrightarrow{\int_x} f = xy + xz + g(y,z) \xrightarrow{D_y}$

$f_y = x + g_y(y,z) = x + z \rightarrow g_y(y,z) = z \xrightarrow{\int_y}$

$g(y,z) = yz + h(z) \rightarrow f = xy + xz + yz + h(z) \xrightarrow{D_z}$

$f_z = x + y + h'(z) = x + y \longrightarrow h'(z) = 0 \rightarrow$

$h(z) = C = 0$, so $f(x,y,z) = xy + xz + yz$

9) $\vec{F}(x,y,z) = e^{y+2z} \cdot \vec{i} + xe^{y+2z} \cdot \vec{j} + 2xe^{y+2z} \cdot \vec{k} \longrightarrow$

$$\qquad\quad f_x \qquad\quad f_y \qquad\quad f_z$$

$f_x = e^{y+2z} \xrightarrow{\int_x} f = xe^{y+2z} + g(y,z) \xrightarrow{D_y}$

$f_y = xe^{y+2z} + g_y(y,z) = xe^{y+2z} \rightarrow g_y(y,z) = 0$

$\xrightarrow{\int_y} g(y,z) = h(z) \rightarrow f = xe^{y+2z} + h(z) \xrightarrow{D_z}$

$f_z = 2xe^{y+2z} + h'(z) = 2xe^{y+2z} \longrightarrow$

$h'(z) = 0 \longrightarrow h(z) = C = 0$, so

$f(x,y,z) = xe^{y+2z}$

10.) $\vec{F}(x,y,z) = \underbrace{(y \sin z)}_{f_x} \vec{i} + \underbrace{(x \sin z)}_{f_y} \vec{j} + \underbrace{(xy \cos z)}_{f_z} \vec{k} \rightarrow$

$f_x = y \sin z \xrightarrow{\int_x} f = xy \sin z + g(y,z) \xrightarrow{D_y}$

$f_y = x \sin z + g_y(y,z) = x \sin z \rightarrow$

$g_y(y,z) = 0 \xrightarrow{\int_y} g(y,z) = h(z) \rightarrow$

$f = xy \sin z + h(z) \xrightarrow{D_z}$

$f_z = xy \cos z + h'(z) = xy \cos z \rightarrow$

$h'(z) = 0 \rightarrow h(z) = C = 0$, so

$f(x,y,z) = xy \sin z$

13.) $d(x^2+y^2+z^2) = (2x)dx + (2y)dy + (2z)dz$,

so $\displaystyle\int_{(0,0,0)}^{(2,3,-6)} (2x)dx + (2y)dy + (2z)dz$

$= \displaystyle\int_{(0,0,0)}^{(2,3,-6)} 1 \cdot d(x^2+y^2+z^2) = (x^2+y^2+z^2)\Big|_{(0,0,0)}^{(2,3,-6)}$

$= 4 + 9 + 36 = 49$

14.) $d(xyz) = (yz)dx + (xz)dy + (xy)dz$,

so $\displaystyle\int_{(1,1,2)}^{(3,5,0)} (yz)dx + (xz)dy + (xy)dz$

$= \displaystyle\int_{(1,1,2)}^{(3,5,0)} 1 \cdot d(xyz) = (xyz)\Big|_{(1,1,2)}^{(3,5,0)}$

$= (3)(5)(0) - (1)(1)(2) = -2$

16.) $d\left(x^2-\frac{1}{3}y^3-4\arctan z\right)$

$= (2x)\,dx + (-y^2)\,dy + \left(\frac{-4}{1+z^2}\right)dz$, so

$\int_{(0,0,0)}^{(3,3,1)} (2x)\,dx + (-y^2)\,dy + \left(\frac{-4}{1+z^2}\right)dz$

$= \int_{(0,0,0)}^{(3,3,1)} 1\cdot d\left(x^2-\frac{1}{3}y^3-4\arctan z\right)$

$= \left(x^2-\frac{1}{3}y^3-4\arctan z\right)\Big|_{(0,0,0)}^{(3,3,1)}$

$= \left[(3)^2-\frac{1}{3}(3)^3-4\cancelto{\frac{\pi}{4}}{\arctan 1}\right]$

$- \left[(0)^2-\frac{1}{3}(0)^3-4\cancelto{0}{\arctan 0}\right] = -\pi$

17.) $d(\sin x \sin y + z)$

$= (\cos x \sin y)\,dx + (\sin x \cos y)\,dy + (1)\,dz$, so

$\int_{(1,0,0)}^{(0,1,1)} (\cos x \sin y)\,dx + (\sin x \cos y)\,dy + (1)\,dz$

$= \int_{(1,0,0)}^{(0,1,1)} 1\cdot d(\sin x \sin y + z)$

$= (\sin x \sin y + z)\Big|_{(1,0,0)}^{(0,1,1)}$

$= (\cancelto{0}{\sin 0}\cdot \sin 1 + 1) - (\sin 1\cdot \cancelto{0}{\sin 0} + 0) = 1$

19.) If $f(x,y,z) = x^3 + z^2 \ln y$, then

$\vec{\nabla} f(x,y,z) = (3x^2)\,\vec{i} + \left(\frac{z^2}{y}\right)\vec{j} + (2z\ln y)\,\vec{k}$,

so that

$$\int_{(1,1,1)}^{(1,2,3)} (3x^2)\,dx + \left(\frac{z^2}{y}\right)dy + (2z\ln y)\,dz$$
$$= \int_{(1,1,1)}^{(1,2,3)} \vec{\nabla} f(x,y,z)\cdot\vec{T}\,ds$$
$$= f(x,y,z)\Big|_{(1,1,1)}^{(1,2,3)} = (x^3 + z^2\ln y)\Big|_{(1,1,1)}^{(1,2,3)}$$
$$= \left((1)^3 + (3)^2\ln 2\right) - \left((1)^3 + (1)^2\cancel{\ln 1}^{\,0}\right)$$
$$= \cancel{1} + 9\ln 2 \cancel{-1} = 9\ln 2$$

20.) If $f(x,y,z) = x^2\ln y - xyz$, then
$\vec{\nabla} f(x,y,z) = (2x\ln y - yz)\vec{i} + \left(\frac{x^2}{y} - xz\right)\vec{j} + (-xy)\vec{k}$,
so that
$$\int_{(1,2,1)}^{(2,1,1)} (2x\ln y - yz)\,dx + \left(\frac{x^2}{y} - xz\right)dy + (-xy)\,dz$$
$$= \int_{(1,2,1)}^{(2,1,1)} \vec{\nabla} f(x,y,z)\cdot\vec{T}\,ds$$
$$= f(x,y,z)\Big|_{(1,2,1)}^{(2,1,1)} = (x^2\ln y - xyz)\Big|_{(1,2,1)}^{(2,1,1)}$$
$$= \left((2)^2\cancel{\ln 1}^{\,0} - (2)(1)(1)\right) - \left((1)^2\ln 2 - (1)(2)(1)\right)$$
$$= -2 - \ln 2 + 2 = -\ln 2$$

24.) If $f(x,y,z) = \frac{1}{3}x^3 + \frac{1}{2}y^2 z$, then
$\vec{\nabla} f(x,y,z) = (x^2)\vec{i} + (yz)\vec{j} + \left(\frac{1}{2}y^2\right)\vec{k}$,
so that

$$\int_C (x^2)\,dx + (yz)\,dy + \left(\tfrac{1}{2}y^2\right)dz$$

$$= \int_{(0,0,0)}^{(0,3,4)} 1\cdot d\left(\tfrac{1}{3}x^3 + \tfrac{1}{2}y^2 z\right)$$

$$= \left(\tfrac{1}{3}x^3 + \tfrac{1}{2}y^2 z\right)\Big|_{(0,0,0)}^{(0,3,4)}$$

$$= \left(\tfrac{1}{3}(0)^3 + \tfrac{1}{2}(3)^2(4)\right) - (0+0) = 18$$

25.) If $f(x,y,z) = xz^2 + y^2$, then $\vec{\nabla} f(x,y,z) = (2xz)\,\vec{i} + (2y)\,\vec{j} + (2xz)\,\vec{k}$, so that

$$\int_A^B (2xz)\,dx + (2y)\,dy + (2xz)\,dz$$

$$= \int_A^B \vec{\nabla} f(x,y,z)\cdot\vec{T}\,dS,$$ which is a path independent work line integral since the vector field is a gradient field.

37.) If $\vec{F}(x,y,z) = \underbrace{(a)}_{M}\vec{i} + \underbrace{(b)}_{N}\vec{j} + \underbrace{(c)}_{P}\vec{k}$, then

$M_y = 0 = N_x$, $N_z = 0 = P_y$, $P_x = 0 = M_z$, so $\vec{F}$ is a gradient $\vec{\nabla} f$, where $f(x,y,z) = ax + by + cz$; then if $A = (x_1, y_1, z_1)$ and $B = (x_2, y_2, z_2)$ we have $\overrightarrow{AB} = (x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j} + (z_2 - z_1)\vec{k}$;

$$\text{Work} = \int_A^B \vec{F} \cdot \vec{T}\, ds = \int_{(x_1,y_1,z_1)}^{(x_2,y_2,z_2)} \vec{\nabla} f(x,y,z) \cdot \vec{T}\, ds$$

$$= f(x,y,z) \Big|_{(x_1,y_1,z_1)}^{(x_2,y_2,z_2)} = (ax+by+cz) \Big|_{(x_1,y_1,z_1)}^{(x_2,y_2,z_2)}$$

$$= (ax_2+by_2+cz_2) - (ax_1+by_1+cz_1$$

$$= a(x_2-x_1) + b(y_2-y_1) + c(z_2-z_1)$$

$$= (a\vec{i}+b\vec{j}+c\vec{k})((x_2-x_1)\vec{i}+(y_2-y_1)\vec{j}+(z_2-z_1)\vec{k})$$

$$= \vec{F} \cdot \overrightarrow{AB}\ .$$