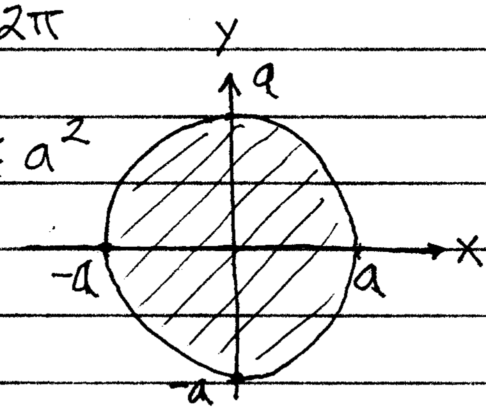
Section 16.4

$C: \begin{cases} x = a\cos t \\ y = a\sin t \end{cases}$ for $0 \le t \le 2\pi$



$\frac{dx}{dt} = -a\sin t$,

$\frac{dy}{dt} = a\cos t$;

1.) $\vec{F}(x,y) = (-y)\vec{i} + (x)\vec{j}$

a.) $\oint_C \vec{F}\cdot\vec{n}\,ds = \int_C M\,dy - N\,dx$

$= \int_0^{2\pi} \left[(-y)\frac{dy}{dt} - (x)\frac{dx}{dt}\right] dt$

$= \int_0^{2\pi} \left[(-a\sin t)(a\cos t) - (a\cos t)(-a\sin t)\right] dt$

$= \int_0^{2\pi} 0\,dt = 0$;

$\iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA = \iint_R (0+0)\,dA = 0$

b.) $\oint_C \vec{F}\cdot\vec{T}\,ds = \int_C M\,dx + N\,dy$

$= \int_0^{2\pi} \left[(-y)\frac{dx}{dt} + (x)\frac{dy}{dt}\right] dt$

$= \int_0^{2\pi} \left[(-a\sin t)(-a\sin t) + (a\cos t)(a\cos t)\right] dt$

$= \int_0^{2\pi} \left[a^2 \underbrace{(\sin^2 t + \cos^2 t)}_{1}\right] dt$

$= a^2 t \Big|_0^{2\pi} = 2a^2\pi$;

$$= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \iint_R (1-(-1))\, dA$$

$$= \iint_R 2\, dA = 2 \cdot \iint_R 1\, dA = 2(\text{area}\, R)$$

$$= 2(\pi a^2) = 2a^2\pi$$

2.) $\vec{F}(x,y) = (y)\,\vec{i} + (0)\,\vec{j}$

a.) $\oint_C \vec{F}\cdot\vec{n}\, ds = \int_C M\,dy - N\,dx$

$$= \int_0^{2\pi} \left[(y)\frac{dy}{dt} - (0)\frac{dx}{dt}\right] dt$$

$$= \int_0^{2\pi} [(a\sin t)(a\cos t)]\, dt$$

$$= a^2 \cdot \frac{1}{2}\sin^2 t \Big|_0^{2\pi} = \frac{a^2}{2}\cancel{\sin^2 2\pi}^{\,0} - \frac{a^2}{2}\cancel{\sin^2 0}^{\,0} = 0\ ;$$

$$\iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA = \iint_R (0+0)\, dA = 0 .$$

b.) $\oint_C \vec{F}\cdot\vec{T}\, ds = \int_C M\,dx + N\,dy$

$$= \int_0^{2\pi} \left[(y)\frac{dx}{dt} + (0)\frac{dy}{dt}\right] dt$$

$$= \int_0^{2\pi} (a\sin t)(-a\sin t)\, dt$$

$$= -a^2 \int_0^{2\pi} \frac{1}{2}(1-\cos 2t)\, dt$$

$$= \frac{-a^2}{2}\left(t - \frac{1}{2}\sin 2t\right)\Big|_0^{2\pi}$$

$$= \frac{-a^2}{2}\left(2\pi - \frac{1}{2}\cancel{\sin}^{0} 4\pi\right) - \frac{-a^2}{2}\left(0 - \frac{1}{2}\cancel{\sin}^{0} 0\right)$$

$$= -a^2\pi \quad ;$$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \iint_R (0 - (1))\, dA$$

$$= -\iint_R 1\, dA = -(\text{area } R) = -\pi a^2 = -a^2\pi .$$

4.) $\vec{F}(x,y) = (-x^2y)\,\vec{i} + (xy^2)\,\vec{j}$

a.) $\oint_C \vec{F}\cdot\vec{n}\, ds = \int_C M\,dy - N\,dx$

$$= \int_0^{2\pi} \left[(-x^2y)\frac{dy}{dt} - (xy^2)\frac{dx}{dt}\right] dt$$

$$= \int_0^{2\pi} [-(a^2\cos^2 t)(a\sin t)\cdot(a\cos t)$$
$$- (a\cos t)(a^2\sin^2 t)(-a\sin t)]\, dt$$

$$= \int_0^{2\pi} \left[-a^4\cos^3 t\sin t + a^4\sin^3 t\cos t\right] dt$$

$$= \left(-a^4\cdot\frac{-1}{4}\cos^4 t + a^4\cdot\frac{1}{4}\sin^4 t\right)\Big|_0^{2\pi}$$

$$= \left(\frac{1}{4}a^4\cos^4 2\pi + \frac{1}{4}a^4\cancel{\sin^4}^{0} 2\pi\right)$$
$$- \left(\frac{1}{4}a^4\cos^4 0 + \frac{1}{4}a^4\cancel{\sin^4}^{0} 0\right)$$

$$= \frac{1}{4}a^4(1)^4 - \frac{1}{4}a^4(1)^4 = 0 \quad ;$$

$$\iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA = \iint_R (-2xy + 2xy)\, dA$$

$$= \iint_R 0 \, dA = 0$$

b.) $\oint_C \vec{F} \cdot \vec{T} \, ds = \int_C M\,dx + N\,dy$

$$= \int_0^{2\pi} \left[(-x^2 y) \frac{dx}{dt} + (xy^2) \cdot \frac{dy}{dt} \right] dt$$

$$= \int_0^{2\pi} \left[(-a^2\cos^2 t)(a \sin t)(-a \sin t) + (a\cos t)(a^2 \sin^2 t)(a \cos t) \right] dt$$

$$= \int_0^{2\pi} \left[a^4 \cos^2 t \sin^2 t + a^4 \cos^2 t \sin^2 t \right] dt$$

$$= 2a^4 \int_0^{2\pi} [\cos t \sin t]^2 \, dt$$

$$= 2a^4 \int_0^{2\pi} \left[\frac{1}{2} \sin 2t \right]^2 dt$$

$$= 2a^4 \cdot \frac{1}{4} \int_0^{2\pi} \sin^2 2t \, dt$$

$$= \frac{1}{2} a^4 \int_0^{2\pi} \frac{1}{2} (1 - \cos 4t) \, dt$$

$$= \frac{1}{4} a^4 \left(t - \frac{1}{4} \sin 4t \right) \Big|_0^{2\pi}$$

$$= \frac{1}{4} a^4 \left(2\pi - \frac{1}{4} \cancel{\sin 8\pi}^{\,0} \right) - \frac{1}{4} a^4 \left(0 - \frac{1}{4} \cancel{\sin 0}^{\,0} \right)$$

$$= \frac{1}{2} a^4 \pi \quad ;$$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA = \iint_R \left(y^2 - (-x^2) \right) dA$$

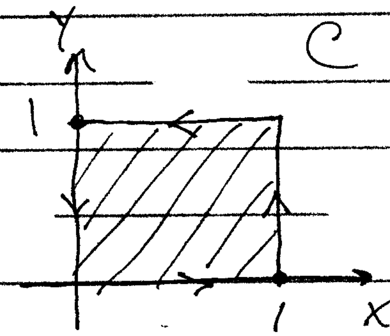
$$= \iint_R \left(a^2 \sin^2 t + a^2 \cos^2 t \right) dA$$

$$= \iint_R a^2 (\underbrace{\sin^2 t + \cos^2 t}_{1})\, dA$$

$$= a^2 \iint_R 1\, dA = a^2 (\text{area } R)$$

$$= a^2 \cdot \pi (a^2) = a^4 \pi .$$

5.) $\vec{F}(x,y) = (x-y)\vec{i} + (y-x)\vec{j}$



a.) Circ $= \oint_C \vec{F} \cdot \vec{T}\, ds$

$$= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

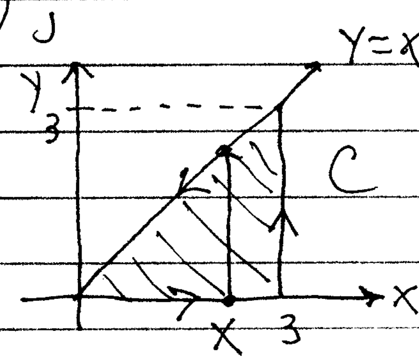
$$= \iint_R (-1-(-1))\, dA = \iint_R 0\, dA = 0$$

b.) Flux $= \oint_C \vec{F} \cdot \vec{n}\, ds = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dA$

$$= \iint_R (1+1)\, dA = 2 \iint_R 1\, dA = 2 (\text{area } R)$$

$$= 2(1) = 2$$

7.) $\vec{F}(x,y) = (y^2 - x^2)\vec{i} + (x^2 + y^2)\vec{j}$



a.) Circ $= \oint_C \vec{F} \cdot \vec{T}\, ds$

$$= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

$$= \iint_R (2x - (2y))\, dA = \int_0^3 \int_0^x (2x - 2y)\, dy\, dx$$

$$= \int_0^3 (2xy - y^2) \Big|_{y=0}^{y=x} dx$$

$$= \int_0^3 (2x^2 - x^2)\, dx = \int_0^3 x^2\, dx$$

$$= \frac{1}{3}x^3 \Big|_0^3 = 9$$

6.) $\text{Flux} = \oint_C \vec{F} \cdot \vec{n}\, ds = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA$

$$= \iint_R (-2x + 2y)\, dA = \int_0^3 \int_0^x (2y - 2x)\, dy\, dx$$

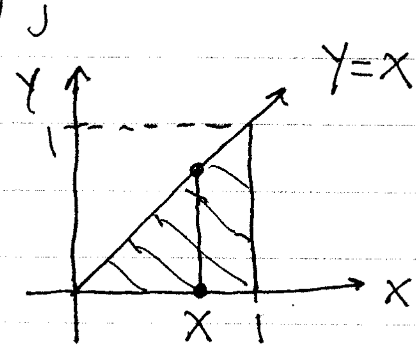
$$= \int_0^3 (y^2 - 2xy) \Big|_{y=0}^{y=x} dx$$

$$= \int_0^3 (x^2 - 2x^2)\, dx = \int_0^3 -x^2\, dx$$

$$= \frac{-1}{3}x^3 \Big|_0^3 = -9$$

8.) $\vec{F}(x,y) = (x+y)\vec{i} + (-x^2 - y^2)\vec{j}$

a.) $\text{Circ} = \oint_C \vec{F} \cdot \vec{T}\, ds$



$$= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

$$= \iint_R (-2x - 1)\, dA = \int_0^1 \int_0^x (-2x - 1)\, dy\, dx$$

$= \int_0^1 (-2x-1)\, y \Big|_{y=0}^{y=x} dx$

$= \int_0^1 (-2x-1)\, x \, dx = \int_0^1 (-2x^2 - x)\, dx$

$= \left(\frac{-2}{3}x^3 - \frac{1}{2}x^2\right)\Big|_0^1 = \frac{-2}{3} - \frac{1}{2} = \frac{-7}{6}$

6.) $\text{Flux} = \oint_C \vec{F} \cdot \vec{n}\, ds = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA$

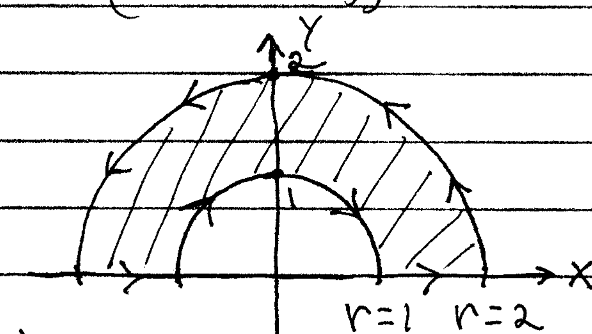
$= \iint_R (1 + (-2y))\, dA = \int_0^1 \int_0^x (1-2y)\, dy\, dx$

$= \int_0^1 (y - y^2)\Big|_{y=0}^{y=x} dx = \int_0^1 (x - x^2)\, dx$

$= \left(\frac{1}{2}x^2 - \frac{1}{3}x^3\right)\Big|_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$

10.) $\vec{F}(x,y) = \left(\arctan\left(\frac{y}{x}\right)\right)\vec{i} + \left(\ln(x^2+y^2)\right)\vec{j}$

a.) $\text{Circ} = \oint_C \vec{F} \cdot \vec{T}\, ds$



$= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$

$= \iint_R \left(\frac{2x}{x^2+y^2} - \frac{\frac{1}{x}}{1+\left(\frac{y}{x}\right)^2}\right) dA$

$= \iint_R \left(\frac{2x}{x^2+y^2} - \frac{x}{x^2+y^2}\right) dA = \iint_R \frac{x}{x^2+y^2}\, dA$

$= \int_0^{\pi} \int_1^2 \frac{\cancel{r}\cos\theta}{\cancel{r^2}} \cdot \cancel{r}\, dr\, d\theta$

$$= \int_0^{\pi} r\cos\theta \Big|_{r=1}^{r=2} d\theta$$

$$= \int_0^{\pi} (2\cos\theta - 1\cos\theta)\, d\theta = \int_0^{\pi} \cos\theta\, d\theta$$

$$= \sin\theta \Big|_0^{\pi} = \cancel{\sin}\overset{0}{\pi} - \cancel{\sin}\overset{0}{0} = 0 .$$

b.) $\text{Flux} = \oint_C \vec{F}\cdot\vec{n}\, ds = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA$

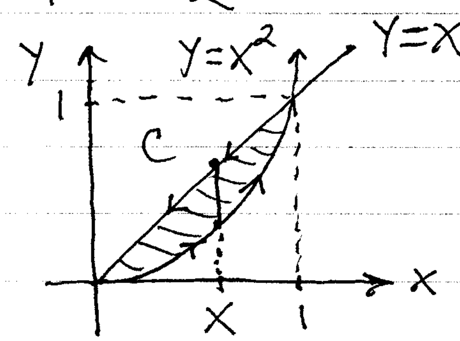
$$= \iint_R \left(\frac{-y/x^2}{1+\left(\frac{y}{x}\right)^2} + \frac{2y}{x^2+y^2} \right) dA$$

$$= \iint_R \left(\frac{-y}{x^2+y^2} + \frac{2y}{x^2+y^2} \right) dA = \iint_R \frac{y}{x^2+y^2}\, dA$$

$$= \int_0^{\pi}\int_1^2 \frac{\cancel{r}\sin\theta}{\cancel{r^2}} \cdot \cancel{r}\, dr\, d\theta = \int_0^{\pi} r\sin\theta \Big|_{r=1}^{r=2} d\theta$$

$$= \int_0^{\pi} \sin\theta\, d\theta = -\cos\theta \Big|_0^{\pi}$$

$$= -\cos\pi - (-\cos 0) = -(-1) + 1 = 2$$



11.) $\vec{F}(x,y) = (xy)\,\vec{i} + (y^2)\,\vec{j}$

a.) $\text{Circ} = \oint_C \vec{F}\cdot\vec{T}\, ds$

$$= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

$$= \iint_R (0 - x)\, dA = \int_0^1 \int_{x^2}^{x} -x\, dy\, dx$$

$$= \int_0^1 (-xy)\Big|_{y=x^2}^{y=x} dx = \int_0^1 \left[-x^2 - (-x^3)\right] dx$$

$$= \int_0^1 (x^3 - x^2)\, dx = \left(\frac{1}{4}x^4 - \frac{1}{3}x^3\right)\Big|_0^1$$

$$= \frac{1}{4} - \frac{1}{3} = \frac{3}{12} - \frac{4}{12} = \frac{-1}{12}$$

6.) Flux $= \oint_C \vec{F} \cdot \vec{n}\, ds$

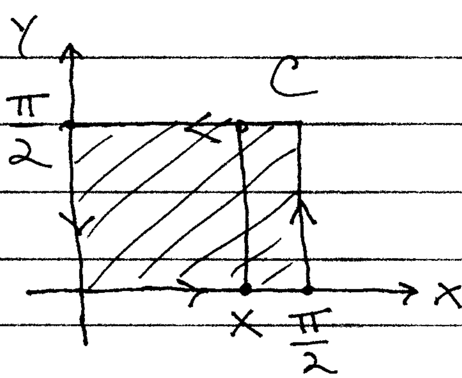
$$= \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA = \iint_R (y + 2y)\, dA$$

$$= \int_0^1 \int_{x^2}^{x} 3y\, dy\, dx = \int_0^1 \left(\frac{3}{2}y^2 \Big|_{y=x^2}^{y=x}\right) dx$$

$$= \int_0^1 \left(\frac{3}{2}x^2 - \frac{3}{2}x^4\right) dx = \left(\frac{1}{2}x^3 - \frac{3}{10}x^5\right)\Big|_0^1$$

$$= \frac{1}{2} - \frac{3}{10} = \frac{5}{10} - \frac{3}{10} = \frac{2}{10} = \frac{1}{5}$$

12.) $\vec{F}(x,y) = (-\sin y)\,\vec{i} + (x\cos y)\,\vec{j}$



a.) Circ $= \oint_C \vec{F} \cdot \vec{T}\, ds$

$$= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \iint_R \left(\cos y - (-\cos y)\right) dA$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} 2\cos y\, dy\, dx = \int_0^{\frac{\pi}{2}} \left(2\sin y \Big|_{y=0}^{y=\frac{\pi}{2}}\right) dx$$

$$= \int_0^{\frac{\pi}{2}} \left(2\overset{1}{\cancel{\sin\frac{\pi}{2}}} - 2\overset{0}{\cancel{\sin 0}}\right) dx = 2x \Big|_0^{\pi/2} = \pi$$

b.) $\text{Flux} = \oint_C \vec{F} \cdot \vec{n}\, ds = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA$

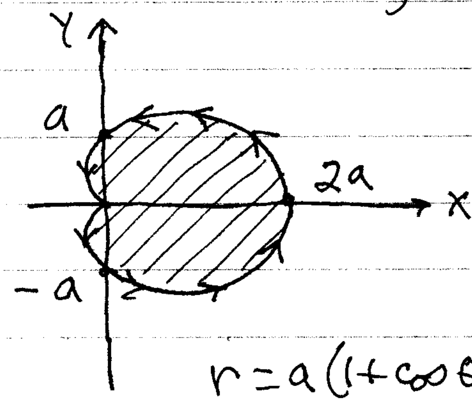
$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \left(0 + (-x \sin y)\right) dy\, dx$

$= \int_0^{\frac{\pi}{2}} \left(x \cos y \Big|_{y=0}^{y=\frac{\pi}{2}}\right) dx$

$= \int_0^{\frac{\pi}{2}} \left(x \overset{0}{\cancel{\cos}} \frac{\pi}{2} - x \overset{1}{\cancel{\cos}}\, 0\right) dx$

$= \int_0^{\frac{\pi}{2}} -x\, dx = \frac{-1}{2} x^2 \Big|_0^{\frac{\pi}{2}} = \frac{-1}{8}\pi^2$

13.) $\vec{F}(x,y) = \left(3xy - \frac{x}{1+y^2}\right)\vec{i} + \left(e^x + \arctan y\right)\vec{j}$,



$r = a(1+\cos\theta)$

$\text{Flux} = \oint_C \vec{F} \cdot \vec{n}\, ds$

$= \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA$

$= \iint_R \left(\left(3y - \cancel{\frac{1}{1+y^2}}\right) + \cancel{\frac{1}{1+y^2}}\right) dA$

$= \int_0^{2\pi} \int_0^{a(1+\cos\theta)} 3 \cdot r \sin\theta \cdot r\, dr\, d\theta$

$= \int_0^{2\pi} \int_0^{a(1+\cos\theta)} 3r^2 \cdot \sin\theta\, dr\, d\theta$

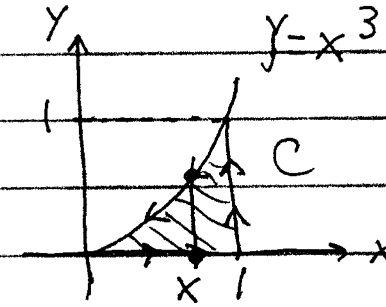
$= \int_0^{2\pi} \left(r^3 \sin\theta \Big|_{r=0}^{r=a(1+\cos\theta)}\right) d\theta$

$= \int_0^{2\pi} a^3 (1+\cos\theta)^3 \sin\theta\, d\theta$

$$= \frac{-1}{4} a^3 (1+\cos\theta)^4 \Big|_0^{2\pi}$$

$$= \frac{-1}{4} a^3 (1+\cancel{\cos 2\pi}^{\,1})^4 - \frac{-1}{4} a^3 (1+\cancel{\cos 0}^{\,1})^4 = 0$$

15.) $\text{Work} = \oint_C \vec{F}\cdot\vec{T}\, ds$



$$= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

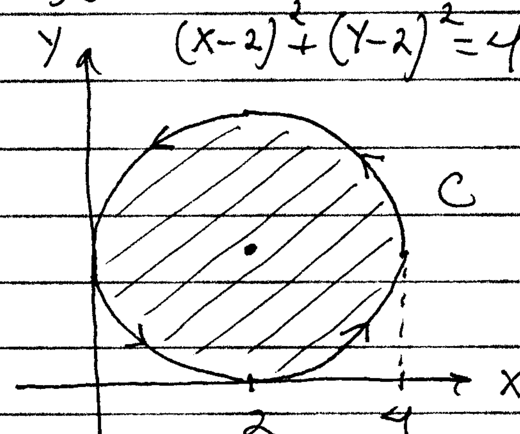
$$= \iint_R (8xy^2 - 6xy^2)\, dA \qquad \vec{F}(x,y) = (2xy^3)\vec{i} + (4x^2y^2)\vec{j}$$

$$= \int_0^1 \int_0^{x^3} 2xy^2\, dy\, dx = \int_0^1 \left(2x\cdot\frac{1}{3}y^3 \Big|_{y=0}^{y=x^3}\right) dx$$

$$= \int_0^1 \frac{2}{3}x\cdot x^9\, dx = \int_0^1 \frac{2}{3}x^{10}\, dx = \frac{2}{3}\cdot\frac{1}{11}x^{11}\Big|_0^1$$

$$= 2/33$$

16.) $\vec{F}(x,y) = (4x-2y)\vec{i} + (2x-4y)\vec{j}$



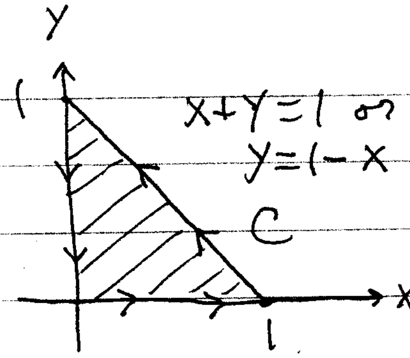
$\text{Work} = \oint_C \vec{F}\cdot\vec{T}\, ds$

$$= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

$$= \iint_R (2-(-2))\, dA$$

$$= 4\iint_R 1\, dA = 4(\text{area } R) = 4\cdot\pi(2)^2 = 16\pi$$

17.) $\oint_C y^2\,dx + x^2\,dy$



$(\vec{F}(x,y) = (y^2)\vec{i} + (x^2)\vec{j}\ )$

$= \oint_C \vec{F}\cdot\vec{T}\,ds = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$

$= \iint_R (2x-2y)\,dA = \int_0^1\int_0^{1-x} (2x-2y)\,dy\,dx$

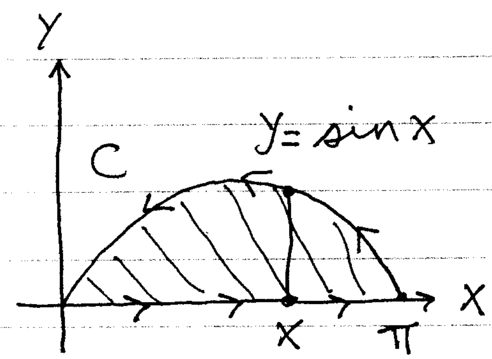
$= \int_0^1 \left(2xy - y^2\right)\Big|_{y=0}^{y=1-x} dx$

$= \int_0^1 \left[2x(1-x) - (1-x)^2\right] dx = \int_0^1 \left[2x - 2x^2 - (x^2 - 2x + 1)\right] dx$

$= \int_0^1 (-3x^2 + 4x - 1)\,dx = (-x^3 + 2x^2 - x)\Big|_0^1$

$= -1 + 2 - 1 = 0$;

18.) $\oint_C 3y\,dx + 2x\,dy$



$(\vec{F}(x,y) = (3y)\vec{i} + (2x)\vec{j}\ )$

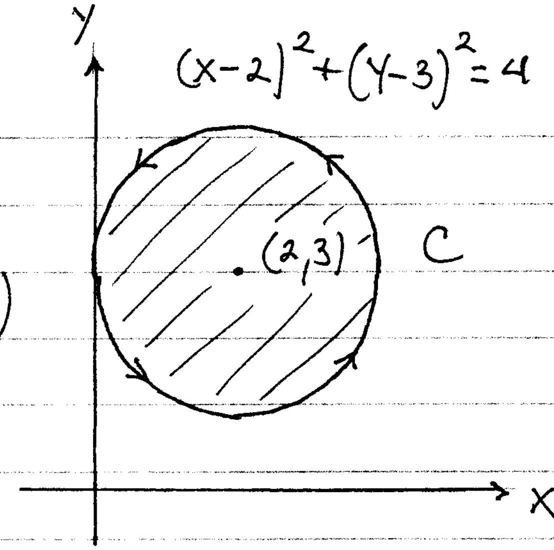
$= \oint_C \vec{F}\cdot\vec{T}\,ds = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$

$= \iint_R (2-3)\,dA = \int_0^{\pi}\int_0^{\sin x} -1\,dy\,dx$

$= \int_0^{\pi} \left(-y\Big|_{y=0}^{y=\sin x}\right) dx = \int_0^{\pi} -\sin x\,dx$

$= \cos x\Big|_0^{\pi} = \cos\pi - \cos 0 = -1 - 1 = -2$

19.) $\oint_C (6y+x)\,dx + (y+2x)\,dy$



$(\vec{F}(x,y) = (6y+x)\vec{i} + (y+2x)\vec{j}\ )$

$= \oint_C \vec{F}\cdot\vec{T}\,ds$

$= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$

$= \iint_R (2-6)\,dA = -4 \iint_R 1\,dA$

$= -4\,(\text{area } R) = -4\cdot\pi(2)^2 = -16\pi$

20.) $\oint_C (2x+y^2)\,dx + (2xy+3y)\,dy$

$(\vec{F}(x,y) = (2x+y^2)\vec{i} + (2xy+3y)\vec{j}\ )$

$= \oint_C \vec{F}\cdot\vec{T}\,ds$ (and $M_y = 2y = N_x$, so $\vec{F}$ is conservative)

$= 0$ (since C is a closed curve)

22.) $\vec{r}(t) = (a\cos t)\vec{i} + (b\sin t)j,\ 0 \le t \le 2\pi$

$C: \begin{cases} x = a\cos t \\ y = b\sin t \end{cases}$ for $0 \le t \le 2\pi$

$\frac{dx}{dt} = -a\sin t,\ \frac{dy}{dt} = b\cos t$

Area $= \frac{1}{2}\oint_C x\,dy - y\,dx$

$= \frac{1}{2}\int_0^{2\pi}\left[(a\cos t)\frac{dy}{dt} - (b\sin t)\frac{dx}{dt}\right]dt$

$$= \frac{1}{2}\int_0^{2\pi} [(a\cos t)(b\cos t) - (b\sin t)(-a\sin t)]\, dt$$

$$= \cancel{\frac{1}{2}\int_0^{2\pi} ab(\underbrace{\cos^2 t + \sin^2 t}_{1})\, dt}$$

$$= \frac{1}{2} ab\, t \Big|_0^{2\pi} = ab\pi$$

24.) $C: \begin{cases} x = t^2 \\ y = \frac{1}{3}t^3 - t \end{cases}$ for $-\sqrt{3} \le t \le \sqrt{3}$

$\frac{dx}{dt} = 2t$, $\frac{dy}{dt} = t^2 - 1$, then

$$\text{Area} = \frac{1}{2}\oint_C x\,dy - y\,dx$$

$$= \frac{1}{2}\oint_C [t^2\cdot(t^2-1) - (\tfrac{1}{3}t^3 - t)(2t)]\, dt$$

$$= \frac{1}{2}\int_{-\sqrt{3}}^{\sqrt{3}} [t^4 - t^2 - (\tfrac{2}{3}t^4 - 2t^2)]\, dt$$

$$= \frac{1}{2}\int_{-\sqrt{3}}^{\sqrt{3}} [\tfrac{1}{3}t^4 + t^2]\, dt$$

$$= \frac{1}{2}\left(\frac{1}{15}t^5 + \frac{1}{3}t^3\right)\Big|_{-\sqrt{3}}^{\sqrt{3}}$$

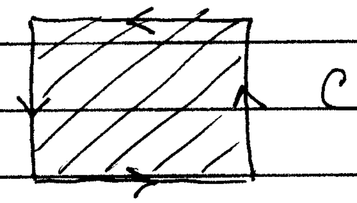
$$= \frac{1}{2}\left(\frac{1}{15}\cdot 9\sqrt{3} - \frac{1}{3}3\sqrt{3}\right)$$

$$= \frac{1}{2}\cdot\frac{4}{15}\sqrt{3} = \frac{2}{15}\sqrt{3}$$

26.) $\oint_C (xy^2)\,dx + (x^2y + 2x)\,dy$

$(\vec{F}(x,y) = (xy^2)\vec{i} + (x^2y + 2x)\vec{j}$

Square of area A



$$= \oint_C \vec{F} \cdot \vec{T}\, ds$$

$$= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

$$\cancel{= \iint_R \left((2xy+2) - (2xy) \right) dA}$$

$$= 2 \cdot \iint_R 1\, dA = 2 \cdot (\text{area of square})$$

$$= 2A \quad \text{(independent of location of square)}$$

29.) Since

$$\oint_C \vec{F} \cdot \vec{n}\, ds = \oint_C M\,dy - N\,dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dA,$$

a.) let $\vec{F}(x,y) = (x)\vec{i} + (0)\vec{j}$ then

$$\oint_C x\,dy - (0)\,dx = \iint_R 1\, dA \rightarrow$$

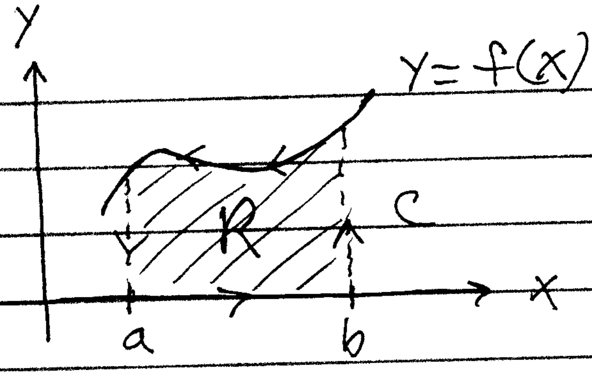
$$\text{Area } R = \oint_C x\,dy .$$

b.) let $\vec{F}(x,y) = (0)\vec{i} + (y)\vec{j}$ then

$$\oint_C (0)\,dy - y\,dx = \iint_R 1\, dA \longrightarrow$$

$$\text{Area } R = -\oint_C y\,dx .$$

30.)



Let C be loop around region, then by problem 29

$$\int_a^b f(x)\,dx = \text{Area } R = -\oint_C y\,dx \, .$$

33.) assume that $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = 0$.

Show that $\displaystyle\oint_C \frac{\partial f}{\partial y}\,dx - \frac{\partial f}{\partial x}\,dy = 0$:

$$\oint_C \frac{\partial f}{\partial y}\,dx + \left(-\frac{\partial f}{\partial x}\right)dy \longrightarrow$$

$$\vec{F} = \underbrace{\left(\frac{\partial f}{\partial y}\right)}_{M}\vec{i} + \underbrace{\left(-\frac{\partial f}{\partial x}\right)}_{N}\vec{j} \quad \text{so that}$$

$$M_y = \frac{\partial^2 f}{\partial y^2}, \quad N_x = \frac{-\partial^2 f}{\partial x^2} \; ; \text{ then}$$

$$\oint_C \frac{\partial f}{\partial y}\,dx + \left(\frac{-\partial f}{\partial x}\right)dy = \oint_C \vec{F}\cdot\vec{T}\,ds$$

$$= \iint_R (N_x - M_y)\,dA = \iint_R \left(-\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2}\right)dA$$

$$= -\iint_R \underbrace{\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right)}_{0}dA = 0$$