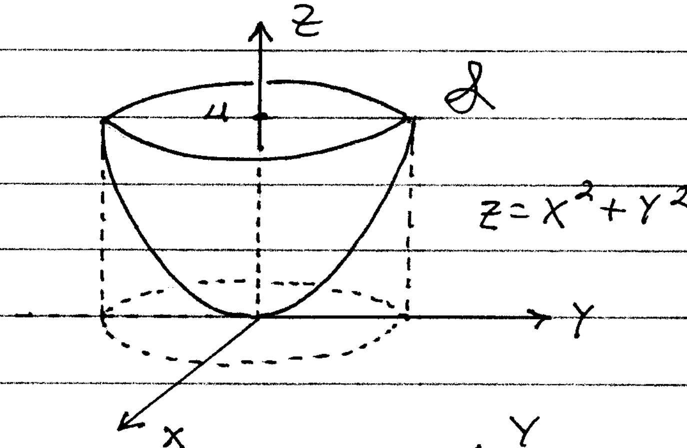
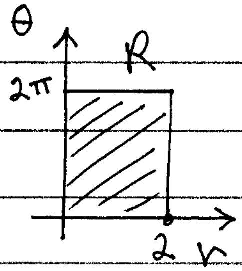
Section 16.6

1.) $z = x^2 + y^2$ and $z \le 4 \rightarrow 4 = x^2 + y^2$

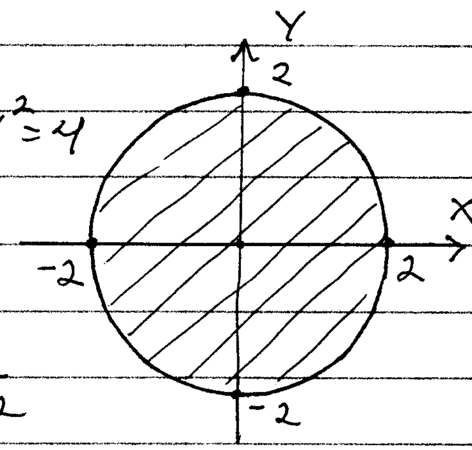


$$S: \begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = x^2+y^2 = r^2 \end{cases}$$

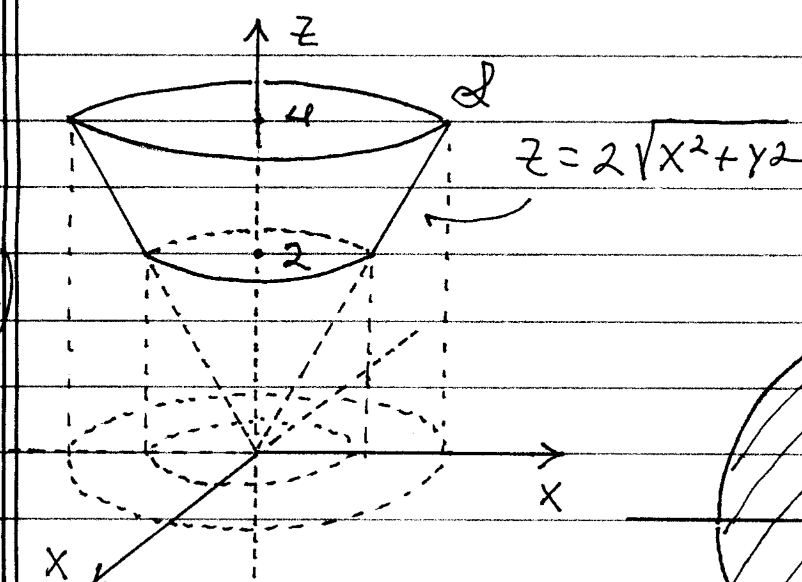
for $0 \le \theta \le 2\pi$, $0 \le r \le 2$.

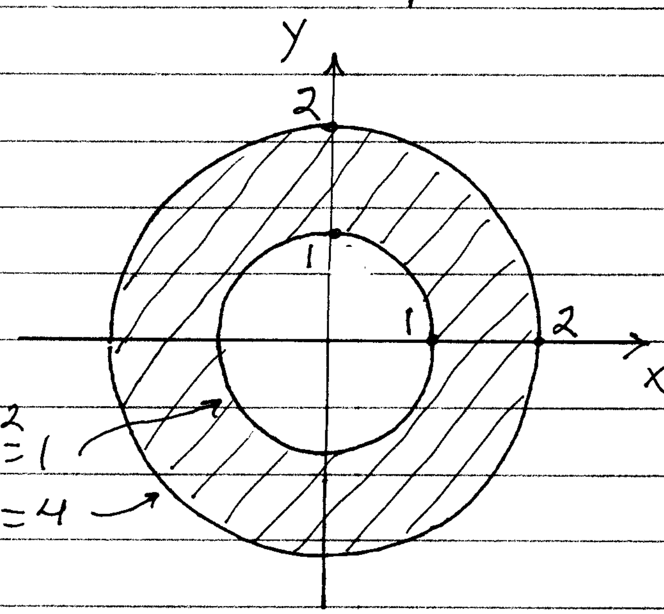


$x^2+y^2=4$



4.)



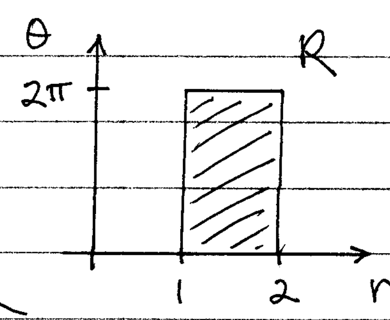


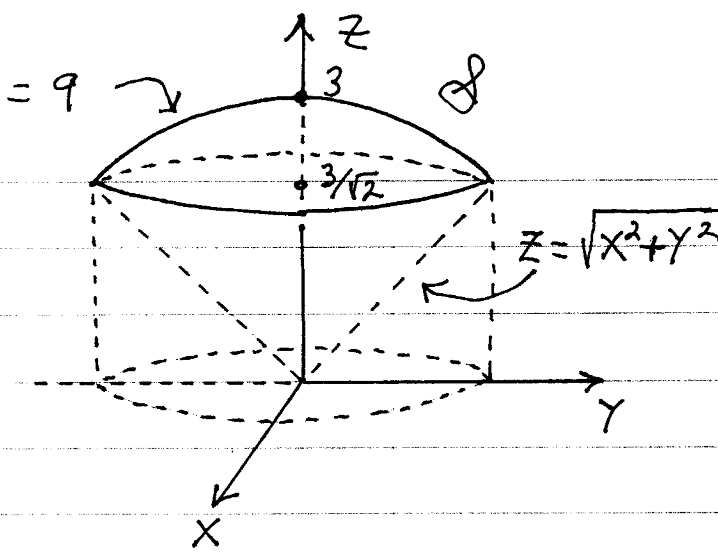
$2 = 2\sqrt{x^2+y^2} \rightarrow x^2+y^2 = 1$

$4 = 2\sqrt{x^2+y^2} \rightarrow x^2+y^2 = 4$

$$S: \begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = 2\sqrt{x^2+y^2} = 2\sqrt{r^2} = 2r \end{cases}$$

for $0 \le \theta \le 2\pi$, $1 \le r \le 2$

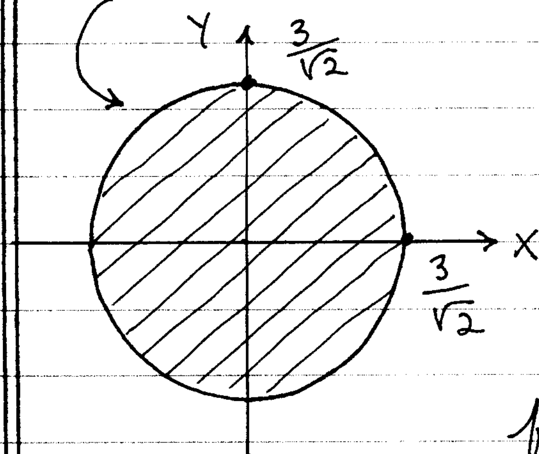


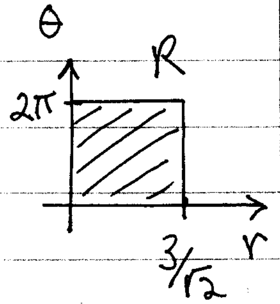


5.) $x^2+y^2+\left(\sqrt{x^2+y^2}\right)^2=9$

$\rightarrow 2x^2+2y^2=9$

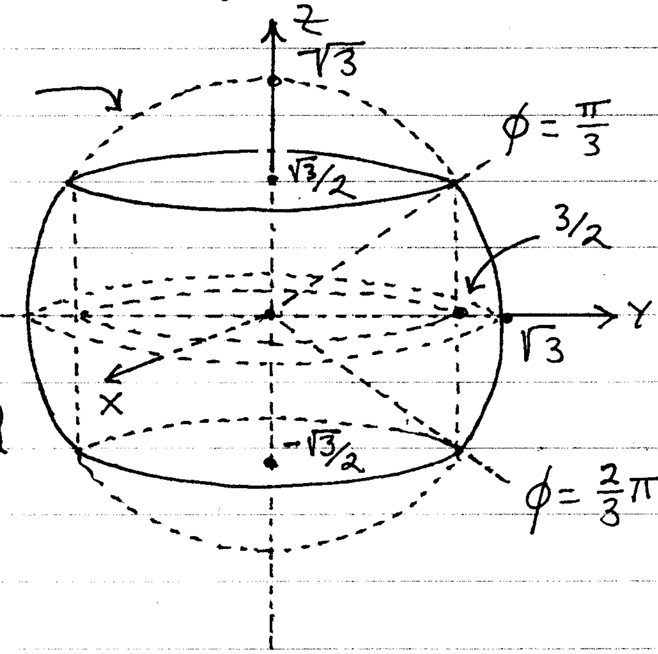
$\rightarrow x^2+y^2=9/2$





$$S:\begin{cases} x=r\cos\theta \\ y=r\sin\theta \\ z=\sqrt{9-(x^2+y^2)}=\sqrt{9-r^2} \end{cases}$$

for $0\le\theta\le 2\pi,\ 0\le r\le 3/\sqrt{2}$



7.) $z=\pm\sqrt{3}/2 \rightarrow$

$x^2+y^2+\frac{3}{4}=3 \rightarrow$

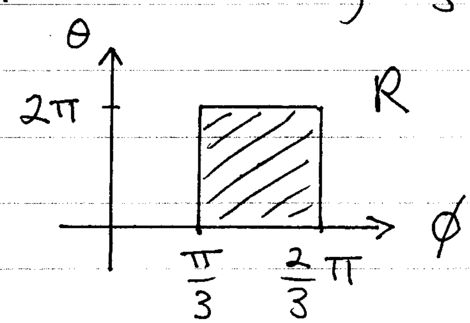
$x^2+y^2=9/4=\left(3/2\right)^2$,

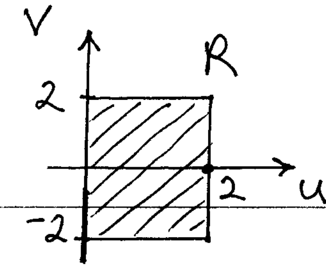
use spherical coordinates: S

S

$$S:\begin{cases} x=\sqrt{3}\sin\phi\cos\theta \\ y=\sqrt{3}\sin\phi\sin\theta \\ z=\sqrt{3}\cos\phi \end{cases}$$

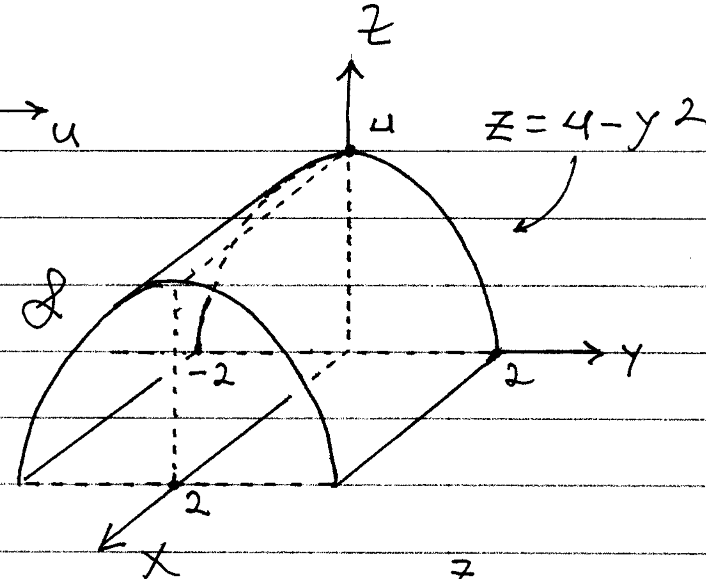
for $0\le\theta\le 2\pi,\ \frac{\pi}{3}\le\phi\le\frac{2}{3}\pi$





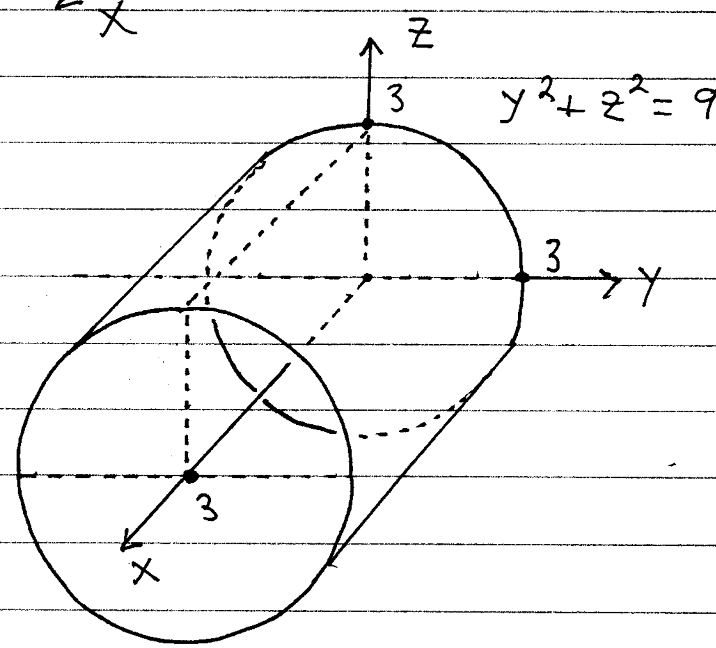
9.) $\mathcal{S}: \begin{cases} x = u \\ y = v \\ z = 4 - v^2 \end{cases}$

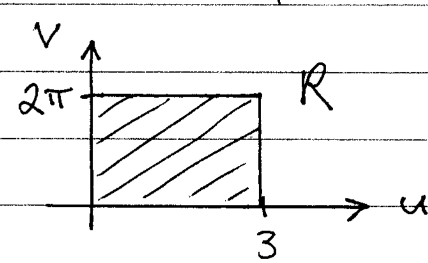
for $0 \le u \le 2$,
$-2 \le v \le 2$



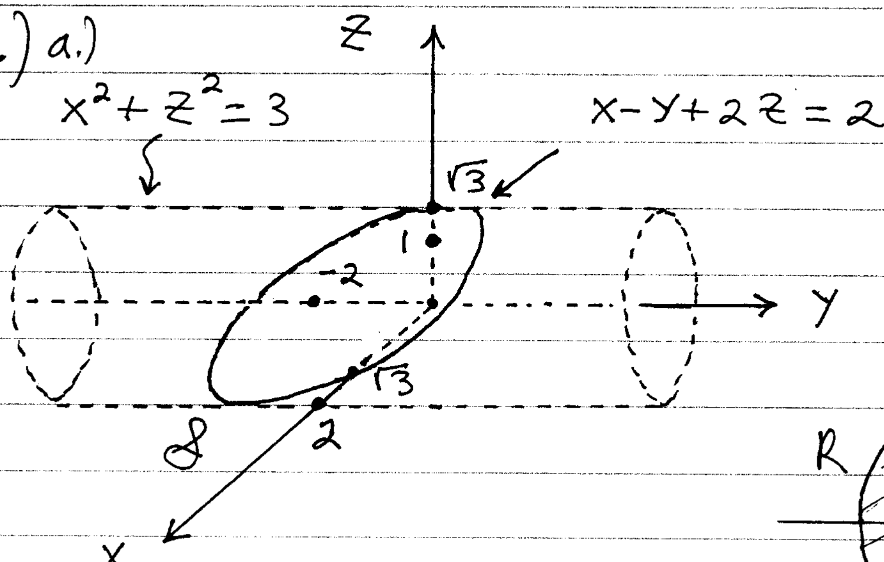
11.) $\mathcal{S}: \begin{cases} x = u \\ y = 3\cos v \\ z = 3\sin v \end{cases}$

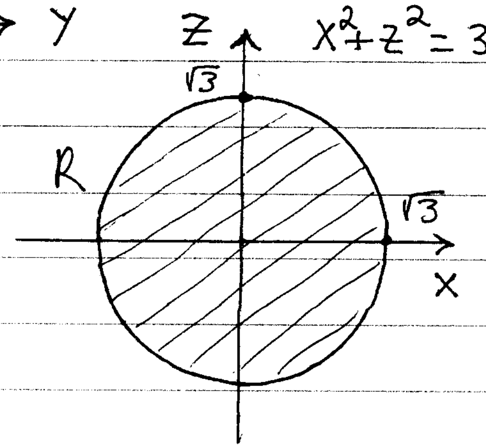
for $0 \le u \le 3$,
$0 \le v \le 2\pi$

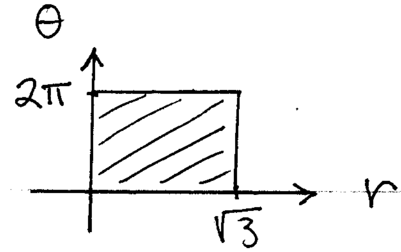




14.) a.)



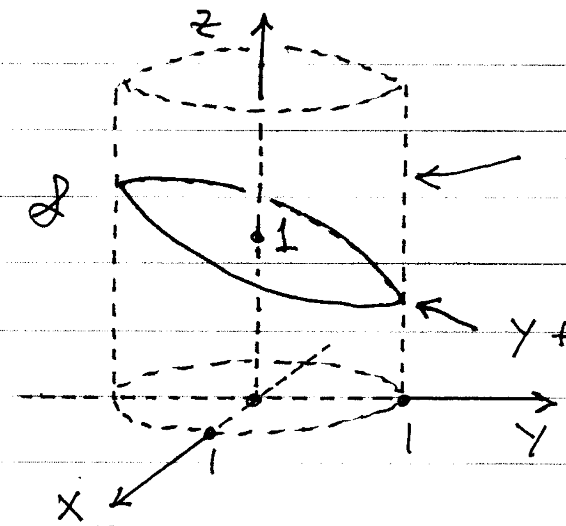


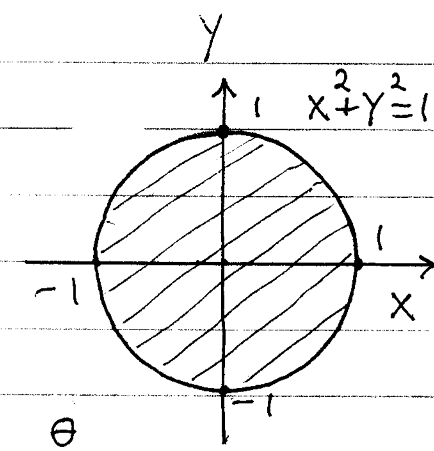


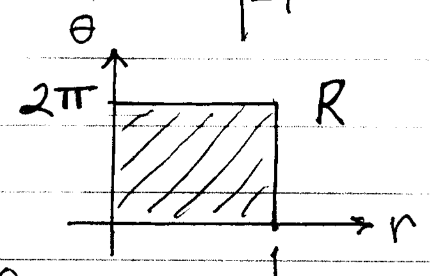
$$\mathcal{S}: \begin{cases} x = r\cos\theta \\ z = r\sin\theta \\ y = x + 2z - 2 = r\cos\theta + 2r\sin\theta - 2 \end{cases}$$

for $0 \le r \le \sqrt{3}$, $0 \le \theta \le 2\pi$

17.







$$\mathcal{S}: \begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = 1 - \frac{1}{2}y = 1 - \frac{1}{2}r\sin\theta \end{cases}$$

for $0 \le r \le 1$, $0 \le \theta \le 2\pi$; then

$$\vec{r}_\theta = (-r\sin\theta)\vec{i} + (r\cos\theta)\vec{j} + \left(\frac{-1}{2}r\cos\theta\right)\vec{k},$$

$$\vec{r}_r = (\cos\theta)\vec{i} + (\sin\theta)\vec{j} + \left(\frac{-1}{2}\sin\theta\right)\vec{k};$$

$$\vec{r}_\theta \times \vec{r}_r = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -r\sin\theta & r\cos\theta & \frac{-1}{2}r\cos\theta \\ \cos\theta & \sin\theta & -\frac{1}{2}\sin\theta \end{vmatrix}$$

$$= \left(\frac{-1}{2}r\sin\theta\cos\theta - \frac{-1}{2}r\sin\theta\cos\theta\right)\vec{i}$$

$$- \left(\frac{1}{2}r\sin^2\theta - \frac{-1}{2}r\cos^2\theta\right)\vec{j}$$

$+ (-r\sin^2\theta - r\cos^2\theta)\vec{k}$

$= (0)\vec{i} - \frac{1}{2}r(\underbrace{\sin^2\theta + \cos^2\theta}_{1})\vec{j} - r(\underbrace{\sin^2\theta+\cos^2\theta}_{1})\vec{k}$

$= (0)\vec{i} + (-\frac{1}{2}r)\vec{j} + (-r)\vec{k}$, and

$|\vec{r}_\theta \times \vec{r}_r| = \sqrt{(-\frac{1}{2}r)^2 + (-r)^2} = \sqrt{\frac{1}{4}r^2 + r^2}$

$= \frac{\sqrt{5}}{2} r$; then

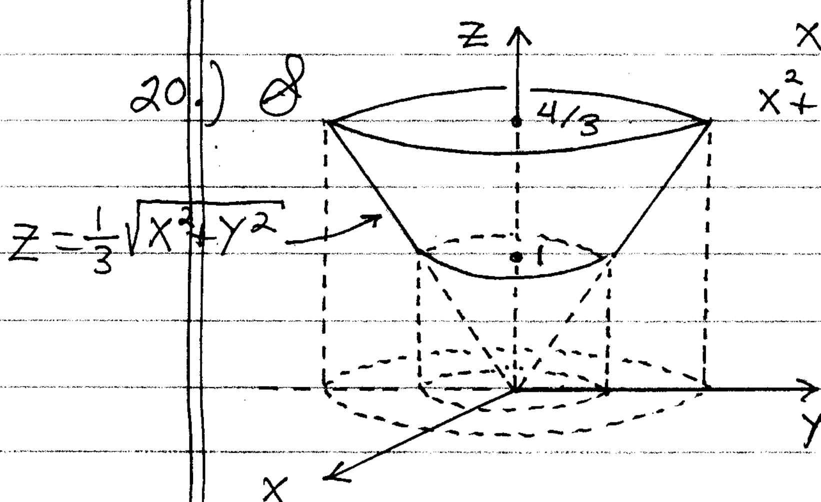
Area $S = \iint_S 1\, dS = \iint_R |\vec{r}_\theta \times \vec{r}_r|\, dA$

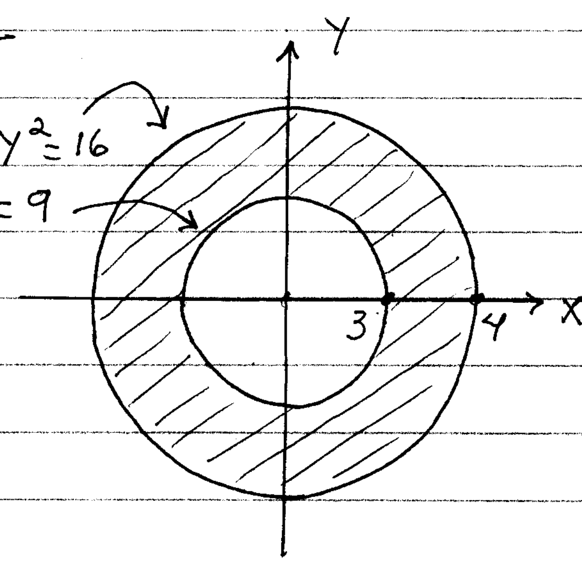
$= \int_0^{2\pi}\int_0^1 \frac{\sqrt{5}}{2} r\, dr\, d\theta$

$= \int_0^{2\pi} \left(\frac{\sqrt{5}}{2}\cdot \frac{1}{2}r^2 \Big|_{r=0}^{r=1}\right) d\theta = \int_0^{2\pi} \frac{1}{4}\sqrt{5}\, d\theta$

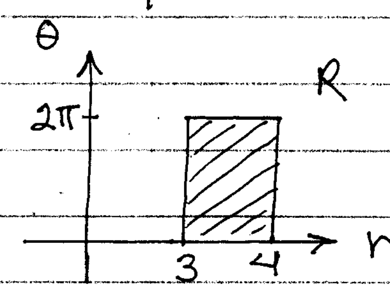
$= \frac{1}{4}\sqrt{5}\cdot 2\pi = \frac{1}{2}\sqrt{5}\,\pi$

20.) S





$S: \begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = \frac{1}{3}\sqrt{r^2} = \frac{1}{3}r \end{cases}$ for $3 \le r \le 4$, $0 \le \theta \le 2\pi$



$$\vec{r}_\theta = (-r\sin\theta)\,\vec{i} + (r\cos\theta)\,\vec{j} + (0)\,\vec{k},$$

$$\vec{r}_r = (\cos\theta)\,\vec{i} + (\sin\theta)\,\vec{j} + (\tfrac{1}{3})\,\vec{k}, \quad \text{then}$$

$$\vec{r}_\theta \times \vec{r}_r = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -r\sin\theta & r\cos\theta & 0 \\ \cos\theta & \sin\theta & 1/3 \end{vmatrix}$$

$$= (\tfrac{1}{3}r\cos\theta - 0)\,\vec{i} - (-\tfrac{1}{3}r\sin\theta - 0)\,\vec{j} + (-r\sin^2\theta - r\cos^2\theta)\,\vec{k}$$

$$= (\tfrac{1}{3}r\cos\theta)\,\vec{i} + (\tfrac{1}{3}r\sin\theta)\,\vec{j} - r(\underbrace{\sin^2\theta + \cos^2\theta}_{1})\,\vec{k}$$

$$= (\tfrac{1}{3}r\cos\theta)\,\vec{i} + (\tfrac{1}{3}r\sin\theta)\,\vec{j} + (-r)\,\vec{k};$$

$$|\vec{r}_\theta \times \vec{r}_r| = \sqrt{(\tfrac{1}{3}r\cos\theta)^2 + (\tfrac{1}{3}r\sin\theta)^2 + (-r)^2}$$

$$= \sqrt{\tfrac{1}{9}r^2\cos^2\theta + \tfrac{1}{9}r^2\sin^2\theta + r^2}$$

$$= \sqrt{\tfrac{1}{9}r^2(\underbrace{\cos^2\theta + \sin^2\theta}_{1}) + r^2}$$

$$= \sqrt{\tfrac{10}{9}r^2} = \frac{\sqrt{10}}{3}r \; ; \quad \text{then}$$

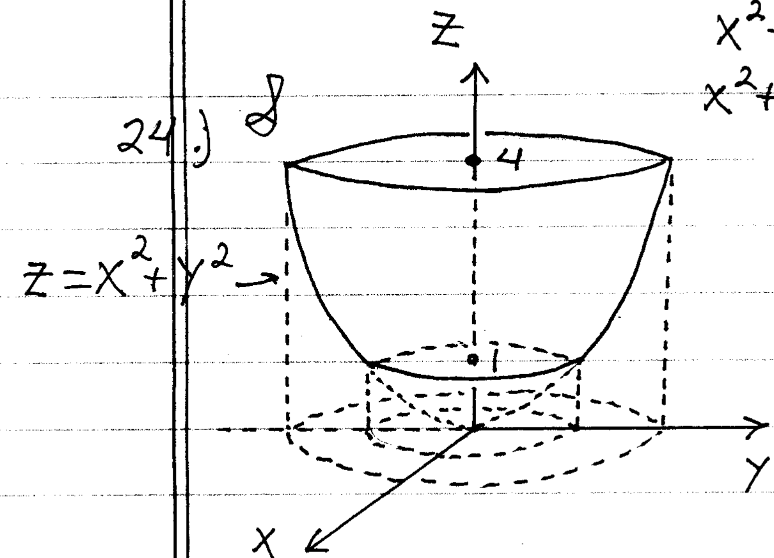
$$\text{Area } \mathcal{S} = \iint_{\mathcal{S}} 1\,dS = \iint_R |\vec{r}_\theta \times \vec{r}_r|\,dA$$

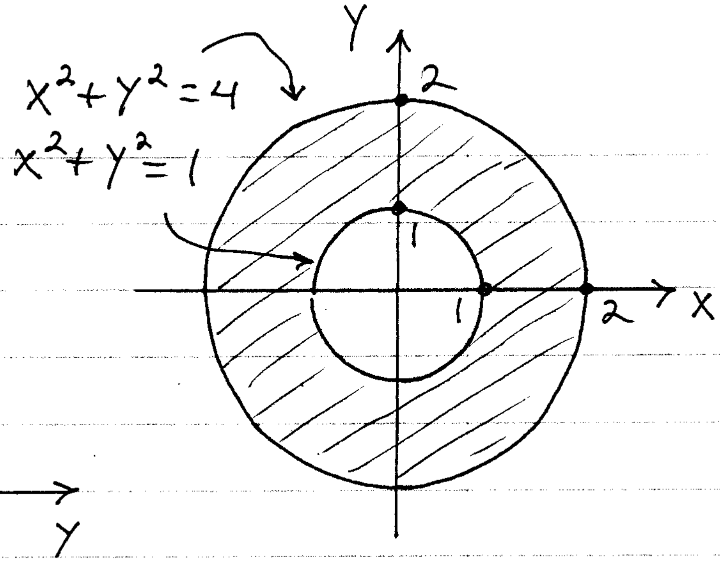
$$= \int_0^{2\pi}\int_3^4 \frac{\sqrt{10}}{3}r\,dr\,d\theta = \int_0^{2\pi}\left(\frac{\sqrt{10}}{3}\cdot\frac{1}{2}r^2\Big|_{r=3}^{r=4}\right)d\theta$$

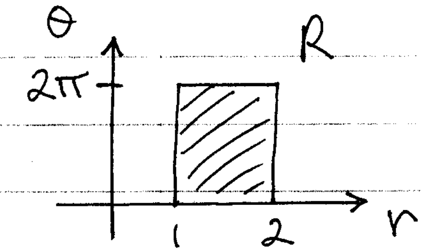
$$= \int_0^{2\pi}\left(\frac{16}{6}\sqrt{10} - \frac{9}{6}\sqrt{10}\right)d\theta = \frac{7}{6}\sqrt{10}\;\theta\Big|_0^{2\pi}$$

$$= \frac{7}{3}\sqrt{10}\,\pi$$

24.) S







$$S:\begin{cases} x=r\cos\theta & \text{for } 1\le r\le 2, \\ y=r\sin\theta & 0\le\theta\le 2\pi \\ z=r^2 \end{cases}$$

$$\vec{r}_\theta=(-r\sin\theta)\,\vec{i}+(r\cos\theta)\,\vec{j}+(0)\,\vec{k},$$

$$\vec{r}_r=(\cos\theta)\,\vec{i}+(\sin\theta)\,\vec{j}+(2r)\,\vec{k}, \text{ then}$$

$$\vec{r}_\theta\times\vec{r}_r=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ -r\sin\theta & r\cos\theta & 0\\ \cos\theta & \sin\theta & 2r\end{vmatrix}$$

$$=(2r^2\cos\theta-0)\,\vec{i}-(-2r^2\sin\theta-0)\,\vec{j}+(-r\sin^2\theta-r\cos^2\theta)\,\vec{k}$$

$$=(2r^2\cos\theta)\,\vec{i}+(2r^2\sin\theta)\,\vec{j}-r\underbrace{(\sin^2\theta+\cos^2\theta)}_{1}\,\vec{k}$$

$$=(2r^2\cos\theta)\,\vec{i}+(2r^2\sin\theta)\,\vec{j}+(-r)\,\vec{k};$$

$$|\vec{r}_\theta\times\vec{r}_r|=\sqrt{(2r^2\cos\theta)^2+(2r^2\sin\theta)^2+(-r)^2}$$

$$=\sqrt{4r^4\cos^2\theta+4r^4\sin^2\theta+r^2}$$

$$=\sqrt{4r^4\underbrace{(\cos^2\theta+\sin^2\theta)}_{1}+r^2}$$

$$= \sqrt{r^2(4r^2+1)} = r\sqrt{4r^2+1} \text{ ; then}$$

$$\text{Area } \mathcal{S} = \iint_{\mathcal{S}} 1\, dS = \iint_R |\vec{r}_\theta \times \vec{r}_\phi|\, dA$$

$$= \int_0^{2\pi}\int_1^2 r\sqrt{4r^2+1}\, dr\, d\theta = \int_0^{2\pi}\left(\frac{1}{8}\cdot\frac{2}{3}(4r^2+1)^{3/2}\Big|_{r=1}^{r=2}\right) d\theta$$

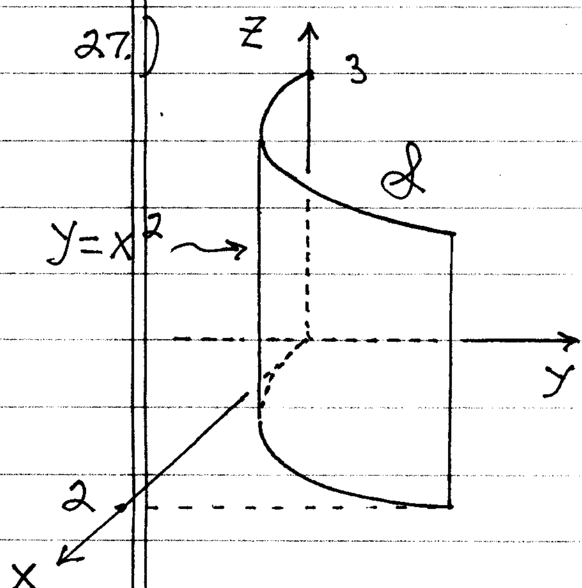
$$= \int_0^{2\pi}\left(\frac{1}{12}(17)^{3/2} - \frac{1}{12}(5)^{3/2}\right) d\theta$$

$$= \left(\frac{1}{12}(17)^{3/2} - \frac{1}{12}(5)^{3/2}\right)\cdot \theta\Big|_0^{2\pi}$$

$$= \left(\frac{1}{12}(17)^{3/2} - \frac{1}{12}(5)^{3/2}(2\pi)\right)$$

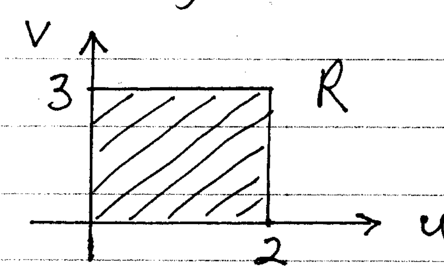
$$= \frac{1}{6}\pi\left((17)^{3/2} - (5)^{3/2}\right).$$

27.)



$$\mathcal{S}: \begin{cases} x = u \\ y = u^2 \\ z = v \end{cases}$$

for $0 \le u \le 2$, $0 \le v \le 3$



$$\vec{r}_u = (1)\,\vec{i} + (2u)\,\vec{j} + (0)\,\vec{k};$$
$$\vec{r}_v = (0)\,\vec{i} + (0)\,\vec{j} + (1)\,\vec{k};$$

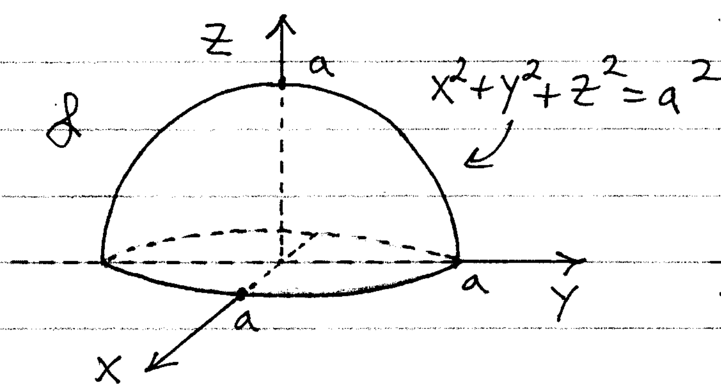
$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2u & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

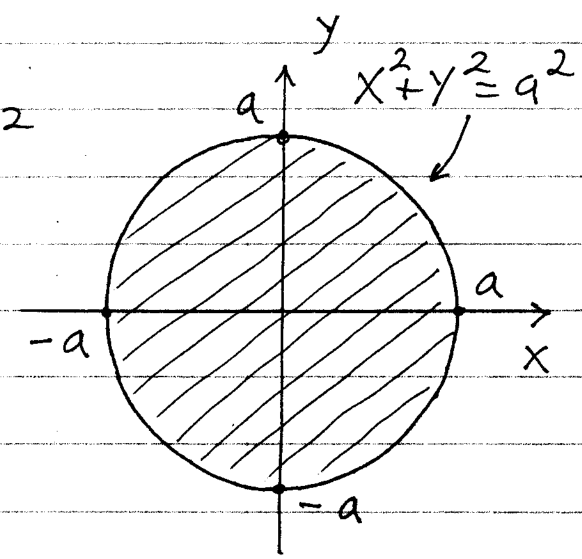
$$= (2u-0)\vec{i} - (1-0)\vec{j} + (0-0)\vec{k}$$
$$= (2u)\vec{i} + (-1)\vec{j} \;;$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{(2u)^2 + (-1)^2} = \sqrt{4u^2+1} \;; \text{ then}$$

$$\iint_{\mathcal{S}} x \, dS = \iint_{R} x \cdot |\vec{r}_u \times \vec{r}_v| \, dA$$
$$= \int_0^2 \int_0^3 u \cdot \sqrt{4u^2+1} \, dv \, du$$
$$= \int_0^2 \left(u\sqrt{4u^2+1} \cdot v \Big|_{v=0}^{v=3} \right) du$$
$$= \int_0^2 3u\sqrt{4u^2+1} \, du$$
$$= 3\,\frac{1}{8}\cdot\frac{2}{3}(4u^2+1)^{3/2}\Big|_0^2 = 3\,\frac{1}{12}(17)^{3/2} - 3\,\frac{1}{12}(1)^{3/2}$$
$$= \frac{1}{4}(17)^{3/2} - \frac{1}{4}$$

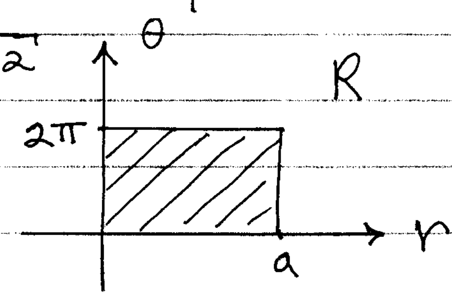
30.) $\mathcal{S}$





$$\mathcal{S}: \begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = \sqrt{a^2-(x^2+y^2)} = \sqrt{a^2-r^2} \end{cases}$$

for $0 \le r \le a$,
$0 \le \theta \le 2\pi$;



$$\vec{r}_\theta = (-r\sin\theta)\,\vec{i} + (r\cos\theta)\,\vec{j} + (0)\,\vec{k},$$

$$\vec{r}_r = (\cos\theta)\,\vec{i} + (\sin\theta)\,\vec{j} + \tfrac{1}{2}(a^2-r^2)^{\frac{-1}{2}}\cdot -2r\cdot \vec{k}$$

$$= (\cos\theta)\,\vec{i} + (\sin\theta)\,\vec{j} + \frac{-r}{\sqrt{a^2-r^2}}\cdot \vec{k};$$

$$\vec{r}_\theta \times \vec{r}_r = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -r\sin\theta & r\cos\theta & 0 \\ \cos\theta & \sin\theta & \frac{-r}{\sqrt{a^2-r^2}} \end{vmatrix}$$

$$= \left(\frac{-r^2\cos\theta}{\sqrt{a^2-r^2}}\right)\vec{i} - \left(\frac{-r^2\sin\theta}{\sqrt{a^2-r^2}}\right)\vec{j}$$

$$+ (-r\sin^2\theta - r\cos^2\theta)\,\vec{k}$$

$$= \left(\frac{-r^2\cos\theta}{\sqrt{a^2-r^2}}\right)\vec{i} + \left(\frac{r^2\sin\theta}{\sqrt{a^2-r^2}}\right)\vec{j} - r(\underbrace{\sin^2\theta + \cos^2\theta}_{1})\vec{k};$$

$$|\vec{r}_\theta \times \vec{r}_r| = \sqrt{\frac{r^4\cdot\cos^2\theta}{a^2-r^2} + \frac{r^4\sin^2\theta}{a^2-r^2} + r^2}$$

$$= \sqrt{\frac{r^4}{a^2-r^2}(\underbrace{\cos^2\theta + \sin^2\theta}_{1}) + r^2}$$

$$= \sqrt{\frac{r^4}{a^2-r^2} + r^2\cdot\frac{a^2-r^2}{a^2-r^2}}$$

$$= \sqrt{\frac{\cancel{r^4} + a^2r^2 - \cancel{r^4}}{a^2-r^2}} = \frac{ar}{\sqrt{a^2-r^2}};$$ ~~then~~

$$\iint_S z^2\,dS = \iint_R z^2\cdot|\vec{r}_\theta \times \vec{r}_r|\,dA$$

$$= \int_0^{2\pi}\int_0^{a} (a^2 - r^2) \cdot \frac{ar}{\sqrt{a^2-r^2}} \, dr \, d\theta$$

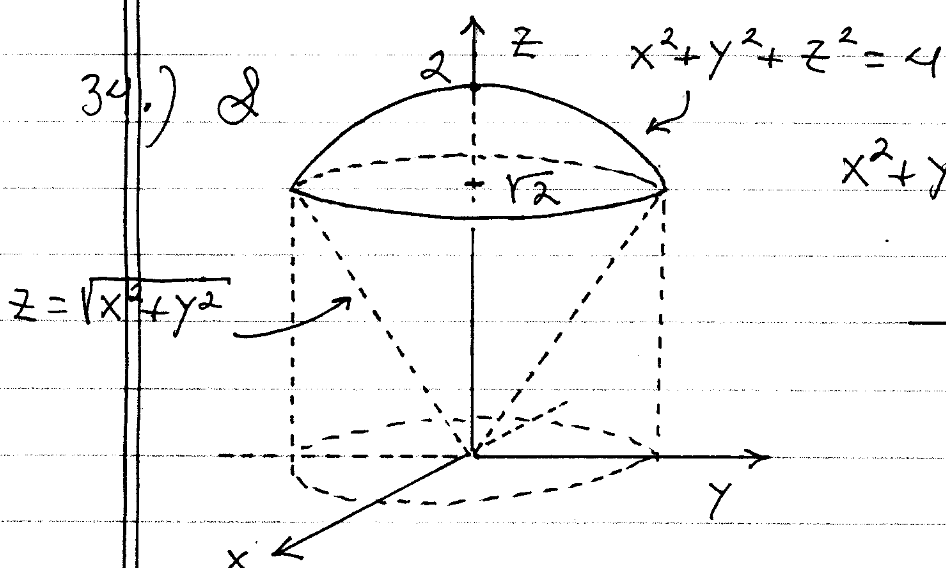
$$= \int_0^{2\pi}\int_0^{a} ar\sqrt{a^2-r^2} \, dr \, d\theta$$

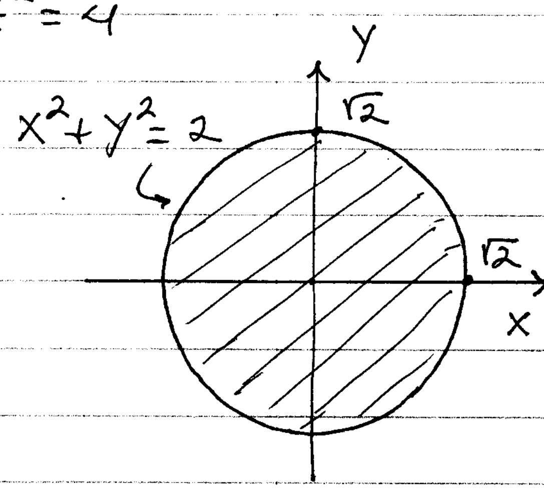
$$= \int_0^{2\pi} a \cdot \frac{-1}{\cancel{2}} \cdot \frac{\cancel{2}}{3}(a^2-r^2)^{3/2} \Big|_{r=0}^{r=a} \, d\theta$$

$$= \int_0^{2\pi} \left[a \cdot \frac{-1}{3}(0)^{3/2} - a\frac{-1}{3}(a^2)^{3/2} \right] d\theta$$

$$= \int_0^{2\pi} \frac{1}{3}a^4 \, d\theta = \frac{1}{3}a^4 \cdot \theta \Big|_0^{2\pi} = \frac{2}{3}a^4\pi$$

34.) a

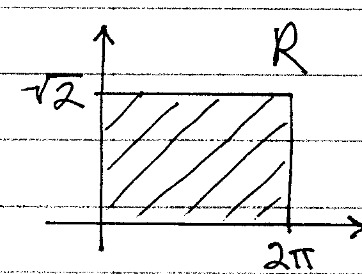




$$x^2+y^2+\left(\sqrt{x^2+y^2}\right)^2 = 4 \rightarrow$$

$$x^2+y^2+x^2+y^2 = 4 \rightarrow 2(x^2+y^2) = 4 \rightarrow$$

$$x^2+y^2 = 2 \quad \text{and} \quad z = \sqrt{2} \;;$$



$$a: \begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = \sqrt{4-(x^2+y^2)} = \sqrt{4-r^2} \end{cases}$$

for $0 \le r \le \sqrt{2}$,
$0 \le \theta \le 2\pi$

(SEE problem 30 solution.)

$$|\vec{r}_\theta \times \vec{r}_r| = \frac{2r}{\sqrt{4-r^2}} \text{ ; then}$$

$$\iint_{S} yz\, dS = \iint_{R} yz\, |\vec{r}_\theta \times \vec{r}_r|\, dA$$

$$= \int_0^{2\pi}\int_0^{\sqrt{2}} (r \sin\theta)\cdot \cancel{\sqrt{4-r^2}} \cdot \frac{2r}{\cancel{\sqrt{4-r^2}}}\, dr\, d\theta$$

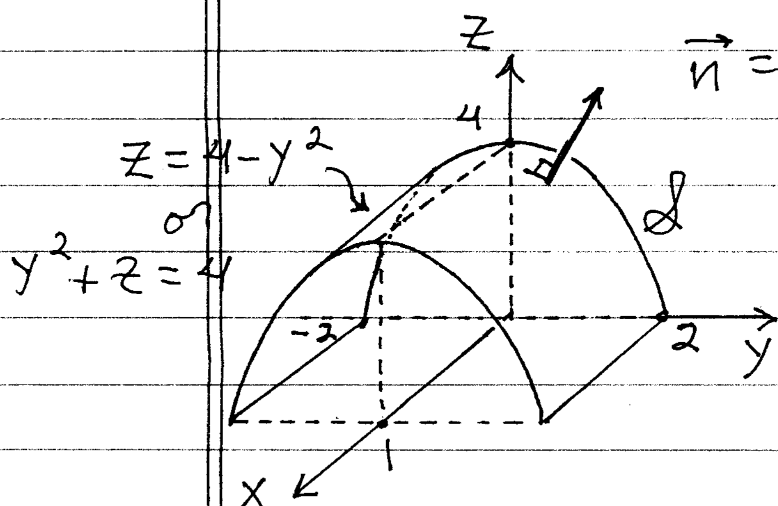
$$= \int_0^{2\pi}\int_0^{\sqrt{2}} 2r^2 \sin\theta\, dr\, d\theta$$

$$= \int_0^{2\pi} \left(\frac{2}{3} r^3 \sin\theta \Big|_{r=0}^{r=\sqrt{2}} \right) d\theta$$

$$= \int_0^{2\pi} \frac{2}{3} 2\sqrt{2} \sin\theta\, d\theta = -\frac{4}{3}\sqrt{2} \cos\theta \Big|_0^{2\pi}$$

$$= -\frac{4}{3}\sqrt{2} \left(\overset{1}{\cos 2\pi} - \overset{1}{\cos 0} \right) = 0$$

35.) $\vec{\nabla} f = (0)\vec{i} + (2y)\vec{j} + (1)\vec{k}$ and so

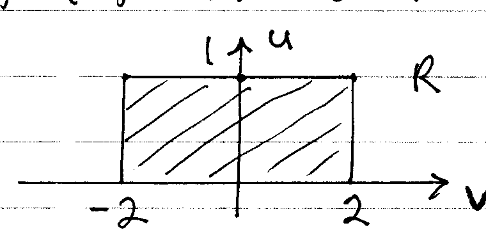


$$\vec{n} = \frac{\vec{\nabla} f}{|\vec{\nabla} f|} = \frac{(2y)\vec{j} + (1)\vec{k}}{\sqrt{(2y)^2 + (1)^2}}$$

$$= \frac{2y}{\sqrt{4y^2+1}}\vec{j} + \frac{1}{\sqrt{4y^2+1}}\vec{k} \ ,$$

$$\vec{F}(x,y,z) = (z^2)\vec{i} + (x)\vec{j} + (-3z)\vec{k} \ ;$$

$$S: \begin{cases} x = u \\ y = v \\ z = 4 - v^2 \end{cases} \quad \text{for } 0 \le u \le 1, \ -2 \le v \le 2,$$



$$\vec{r}_u = (1)\vec{i} + (0)\vec{j} + (0)\vec{k}, \quad \vec{r}_v = (0)\vec{i} + (1)\vec{j} + (-2v)\vec{k},$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 1 & -2v \end{vmatrix}$$

$$= (0)\vec{i} - (-2v-0)\vec{j} + (1-0)\vec{k} = (2v)\vec{j} + (1)\vec{k};$$

then $|\vec{r}_u \times \vec{r}_v| = \sqrt{(2v)^2 + (1)^2} = \sqrt{4v^2+1}$; then

$$\text{Flux} = \iint_S \vec{F}\cdot\vec{n}\, dS = \iint_S \left(\frac{2xy}{\sqrt{4y^2+1}} + \frac{-3z}{\sqrt{4y^2+1}} \right) dS$$

$$= \iint_R \frac{2xy-3z}{\sqrt{4y^2+1}} \cdot |\vec{r}_u \times \vec{r}_v|\, dA$$

$$= \int_0^1 \int_{-2}^2 \frac{2uv - 3(4-v^2)}{\cancel{\sqrt{4v^2+1}}} \cdot \cancel{\sqrt{4v^2+1}}\, dv\, du$$

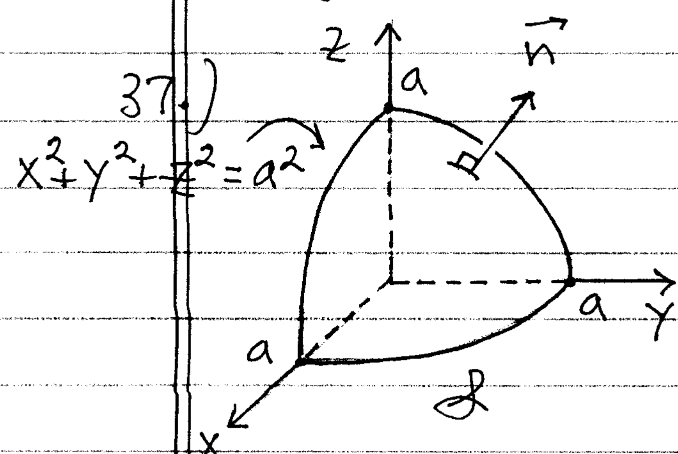
$$= \int_0^1 \int_{-2}^2 (2uv - 12 + 3v^2)\, dv\, du$$

$$= \int_0^1 (uv^2 - 12v + v^3) \Big|_{v=-2}^{v=2} du$$

$$= \int_0^1 ((\cancel{4u} - 24 + 8) - (\cancel{4u} + 24 - 8))\, du$$

$$= \int_0^1 -32\, du = -32u \Big|_0^1 = -32$$

37.)



$$\vec{\nabla} f = (2x)\vec{i} + (2y)\vec{j} + (2z)\vec{k},$$

$$|\vec{\nabla} f| = \sqrt{(2x)^2 + (2y)^2 + (2z)^2}$$

$$= \sqrt{4(x^2+y^2+z^2)}$$

$$= 2\sqrt{a^2}$$

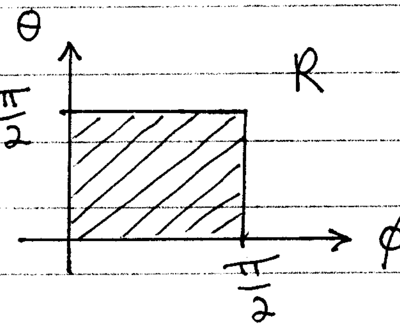
$$= 2a, \text{ so}$$

$$\vec{n} = \frac{\vec{\nabla} f}{|\vec{\nabla} f|} = \left(\frac{x}{a}\right)\vec{i} + \left(\frac{y}{a}\right)\vec{j} + \left(\frac{z}{a}\right)\vec{k} \;;$$

(Use spherical coordinates.)

$$\mathcal{S}: \begin{cases} x = a \sin\phi \cos\theta \\ y = a \sin\phi \sin\theta \\ z = a \cos\phi \end{cases}$$

for $0 \le \phi \le \frac{\pi}{2}$, $0 \le \theta \le \frac{\pi}{2}$



$$\vec{r}_\theta = (-a \sin\phi \sin\theta)\,\vec{i} + (a \sin\phi \cos\theta)\,\vec{j}\,,$$

$$\vec{r}_\phi = (a \cos\phi \cos\theta)\,\vec{i} + (a \sin\phi \cos\theta)\,\vec{j} + (-a \sin\phi)\,\vec{k}\,, \text{ and}$$

(See Parametrized Surfaces handout.)

$\ldots\ |\vec{r}_\theta \times \vec{r}_\phi| = a^2 \sin\phi$; then $\vec{F} = z\vec{k}$ and

$$\text{Flux} = \iint_{\mathcal{S}} \vec{F} \cdot \vec{n}\, dS = \iint_{\mathcal{S}} \frac{z^2}{a}\, dS$$

$$= \iint_R \frac{z^2}{a} \cdot |\vec{r}_\theta \times \vec{r}_\phi|\, dA = \int_0^{\pi/2}\!\int_0^{\frac{\pi}{2}} \frac{a^2 \cos^2\phi}{a} \cdot a^2 \sin\phi \, d\phi\, d\theta$$

$$= \int_0^{\frac{\pi}{2}}\!\int_0^{\frac{\pi}{2}} a^3 \cos^2\phi \sin\phi \; d\phi\, d\theta$$

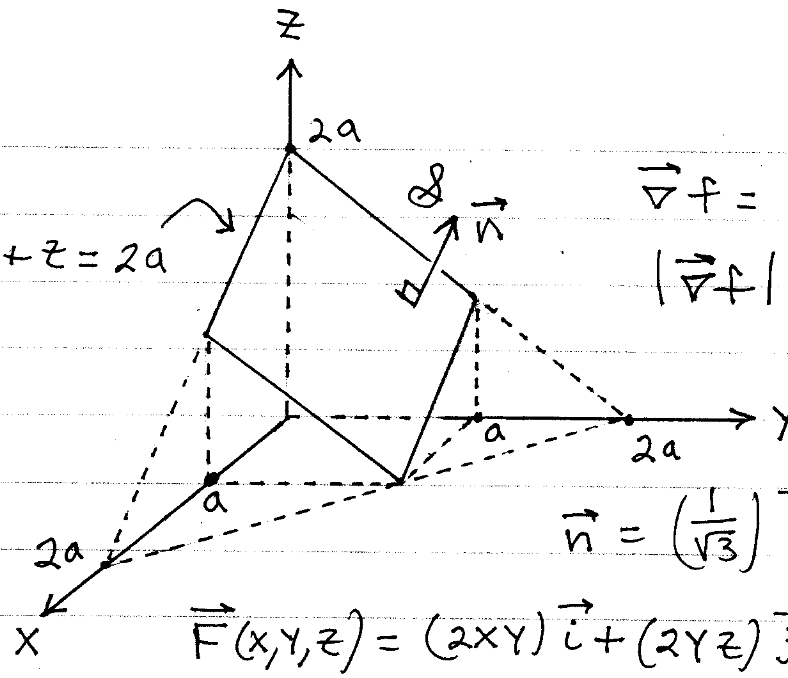
$$= \int_0^{\frac{\pi}{2}} \left(-\frac{1}{3} a^3 \cos^3\phi\right)\Big|_{\phi=0}^{\phi=\frac{\pi}{2}} d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[\left(\frac{-1}{3} a^3 \overset{0}{\cancel{\cos}}{}^3 \frac{\pi}{2}\right) - \left(\frac{-1}{3} a^3 \overset{1}{\cancel{\cos}}{}^3 0\right)\right] d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{3} a^3\, d\theta = \frac{1}{3} a^3\, \theta \Big|_0^{\frac{\pi}{2}} = \frac{1}{6} a^3 \pi$$

39.)

$x+y+z=2a$



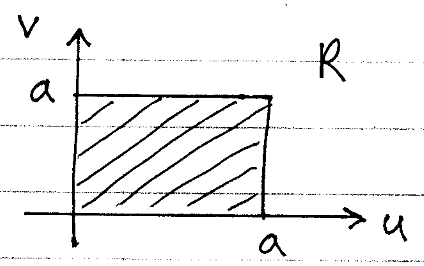
$\vec{\nabla} f = (1)\vec{i} + (1)\vec{j} + (1)\vec{k}$,

$|\vec{\nabla} f| = \sqrt{3}$;

$\vec{n} = \left(\frac{1}{\sqrt{3}}\right)\vec{i} + \left(\frac{1}{\sqrt{3}}\right)\vec{j} + \left(\frac{1}{\sqrt{3}}\right)\vec{k}$,

$\vec{F}(x,y,z) = (2xy)\vec{i} + (2yz)\vec{j} + (2xz)\vec{k}$;

$$S: \begin{cases} x = u \\ y = v \\ z = 2a - x - y = 2a - u - v \end{cases}$$



for $0 \le u \le a$, $0 \le v \le a$;

$\vec{r}_u = (1)\vec{i} + (0)\vec{j} + (-1)\vec{k}$,

$\vec{r}_v = (0)\vec{i} + (1)\vec{j} + (-1)\vec{k}$,

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix}$$

$= (0-{-1})\vec{i} - (-1-0)\vec{j} + (1-0)\vec{k}$

$= (1)\vec{i} + (1)\vec{j} + (1)\vec{k}$, $|\vec{r}_u \times \vec{r}_v| = \sqrt{3}$; then

$$\text{Flux} = \iint_S \vec{F} \cdot \vec{n}\, dS = \iint_S \frac{2}{\sqrt{3}}(xy + yz + xz)\, dS$$

$$= \iint_R \frac{2}{\sqrt{3}}\left(uv + v(2a-u-v) + u(2a-u-v)\right) |\vec{r}_u \times \vec{r}_v|\, dA$$

$$= \int_0^a \int_0^a \frac{2}{\cancel{\sqrt{3}}}\left(\cancel{uv} + 2av - \cancel{uv} - v^2 + 2au - u^2 - uv\right) \cancel{\sqrt{3}}\, dv\, du$$

$$= \int_0^a \int_0^a (4av - 2v^2 + 4au - 2u^2 - 2uv)\, dv\, du$$

$$= \int_0^a \left(2av^2 - \frac{2}{3}v^3 + 4auv - 2u^2v - uv^2 \right) \Big|_{v=0}^{v=a}$$

$$= \int_0^a \left[2a^3 - \frac{2}{3}a^3 + 4a^2u - 2au^2 - a^2u \right] du$$

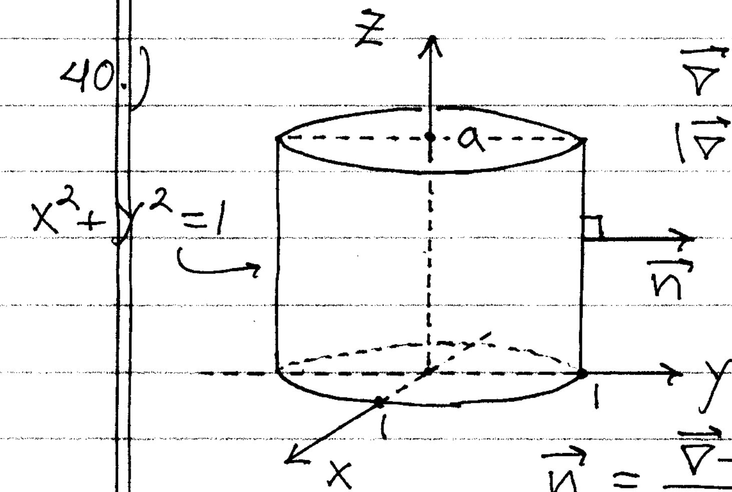
$$= \int_0^a \left[\frac{4}{3}a^3 + 3a^2u - 2au^2 \right] du$$

$$= \left(\frac{4}{3}a^3u + 3a^2 \cdot \frac{1}{2}u^2 - 2a \cdot \frac{1}{3}u^3 \right) \Big|_0^a$$

$$= \frac{4}{3}a^4 + \frac{3}{2}a^4 - \frac{2}{3}a^4$$

$$= \frac{8}{6}a^4 + \frac{9}{6}a^4 - \frac{4}{6}a^4 = \frac{13}{6}a^4$$

40.)



$\vec{\nabla} f = (2x)\vec{i} + (2y)\vec{j}$,

$|\vec{\nabla} f| = \sqrt{(2x)^2 + (2y)^2}$

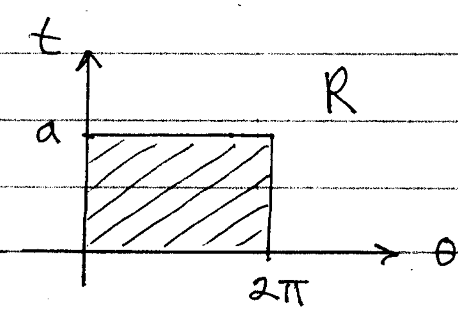
$= \sqrt{4x^2 + 4y^2}$

$= \sqrt{4(x^2 + y^2)}$

$= \sqrt{4(1)} = 2$, so

$$\vec{n} = \frac{\vec{\nabla} f}{|\vec{\nabla} f|} = (x)\vec{i} + (y)\vec{j}$$

$$S: \begin{cases} x = \cos\theta \\ y = \sin\theta \\ z = t \end{cases}$$



for $0 \le \theta \le 2\pi$, $0 \le t \le a$;

$\vec{r}_\theta = (-\sin\theta)\vec{i} + (\cos\theta)\vec{j} + (0)\vec{k}$,

$\vec{r}_t = (0)\vec{i} + (0)\vec{j} + (1)\vec{k}$;

$$\vec{r}_\theta \times \vec{r}_t = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= (\cos\theta - 0)\vec{i} - (-\sin\theta - 0)\vec{j} + (0-0)\vec{k}$$

$$= (\cos\theta)\vec{i} + (\sin\theta)\vec{j} \; ;$$

$$|\vec{r}_\theta \times \vec{r}_t| = \sqrt{\cos^2\theta + \sin^2\theta} = \sqrt{1} = 1 \; ;$$

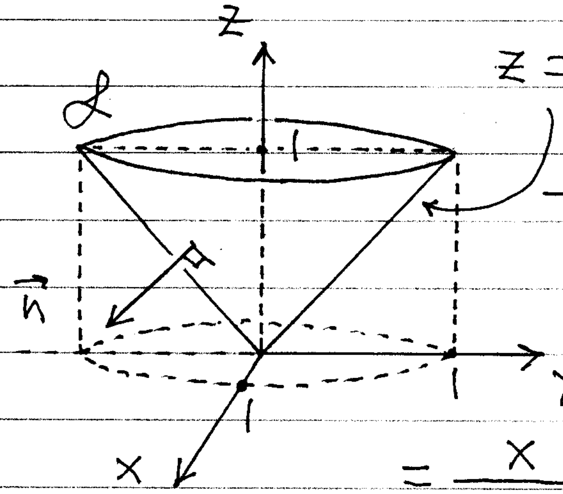
$\vec{F}(x,y,z) = (x)\vec{i} + (y)\vec{j} + (z)\vec{k}$; then

$$\text{Flux} = \iint_{S} \vec{F}\cdot\vec{n}\, dS = \iint_{S} (x^2+y^2)\, dS$$

$$= \iint_{R} (1)\, |\vec{r}_\theta \times \vec{r}_t|\, dA = \int_0^{2\pi}\int_0^{a} 1 \, dt\, d\theta$$

$$= \text{area } R = 2a\pi \, .$$

41.)



$$z = \sqrt{x^2+y^2} \longrightarrow 0 = \sqrt{x^2+y^2} - z$$

$$\longrightarrow \vec{\nabla} f = \tfrac{1}{2}(x^2+y^2)^{-1/2}\cdot 2x\cdot \vec{i} + \tfrac{1}{2}(x^2+y^2)^{-1/2}\cdot 2y \cdot \vec{j} + (-1)\vec{k}$$

$$= \frac{x}{\sqrt{x^2+y^2}}\vec{i} + \frac{y}{\sqrt{x^2+y^2}}\vec{j} + (-1)\vec{k} \; ;$$

$$|\vec{\nabla} f| = \sqrt{\left(\frac{x}{\sqrt{x^2+y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2+y^2}}\right)^2 + (-1)^2}$$

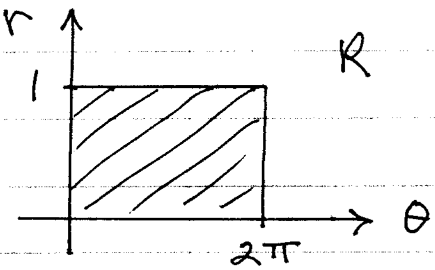
$$= \sqrt{\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} + 1}$$

$$= \sqrt{\frac{x^2+y^2}{x^2+y^2}+1} = \sqrt{1+1} = \sqrt{2} \text{ ; so}$$

$$\vec{n} = \frac{\vec{\nabla} f}{|\vec{\nabla} f|} = \frac{1}{\sqrt{2}}\left(\frac{x}{\sqrt{x^2+y^2}}\vec{i} + \frac{y}{\sqrt{x^2+y^2}}\vec{j} + (-1)\vec{k}\right);$$

$$\mathcal{S}: \begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = \sqrt{x^2+y^2} = \sqrt{r^2} = r \end{cases}$$

for $0 \le r \le 1$, $0 \le \theta \le 2\pi$;



$$\vec{r}_\theta = (-r\sin\theta)\vec{i} + (r\cos\theta)\vec{j} + (0)\vec{k},$$
$$\vec{r}_r = (\cos\theta)\vec{i} + (\sin\theta)\vec{j} + (1)\vec{k};$$

$$\vec{r}_\theta \times \vec{r}_r = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -r\sin\theta & r\cos\theta & 0 \\ \cos\theta & \sin\theta & 1 \end{vmatrix}$$

$$= (r\cos\theta - 0)\vec{i} - (-r\sin\theta - 0)\vec{j} + (-r\sin^2\theta - r\cos^2\theta)\vec{k}$$

$$= (r\cos\theta)\vec{i} + (r\sin\theta)\vec{j} + (-r)\vec{k};$$

$$|\vec{r}_\theta \times \vec{r}_r| = \sqrt{(r\cos\theta)^2 + (r\sin\theta)^2 + (-r)^2}$$

$$= \sqrt{r^2(\underbrace{\cos^2\theta + \sin^2\theta}_{1}) + r^2}$$

$$= \sqrt{2r^2} = \sqrt{2}\cdot r \text{ ; then}$$

$$\vec{F}(x,y,z) = (xy)\vec{i} + (-z)\vec{k} \quad \text{and}$$

$$\text{Flux} = \iint_{\mathcal{S}} \vec{F}\cdot\vec{n}\, dS = \iint_{\mathcal{S}} \frac{1}{\sqrt{2}}\left[\frac{x^2y}{\sqrt{x^2+y^2}} + z\right] dS$$

$$= \iint \frac{1}{\sqrt{2}}\left[\frac{x^2 y}{\sqrt{x^2+y^2}} + z\right] \cdot \left|\vec{r}_\theta \times \vec{r}_r\right| \, dA$$

$$= \overset{R}{\int_0^{2\pi}} \int_0^1 \cancel{\frac{1}{\sqrt{2}}}\left[\frac{(r^2\cos^2\theta)\cdot(r\sin\theta)}{\sqrt{r^2}} + r\right] \cancel{\sqrt{2}}\, r \, dr \, d\theta$$

$$= \int_0^{2\pi}\int_0^1 \left[\frac{r^{\cancel{3}\,2}\cos^2\theta\sin\theta}{\cancel{r}} + r\right] \cdot r \, dr \, d\theta$$

$$= \int_0^{2\pi}\int_0^1 (r^3\cos^2\theta\sin\theta + r^2) \, dr \, d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{4}r^4\cos^2\theta\sin\theta + \frac{1}{3}r^3\right)\Big|_{r=0}^{r=1} d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{4}\cos^2\theta\sin\theta + \frac{1}{3}\right) d\theta$$

$$= \left(\frac{1}{4}\cdot\frac{-1}{3}\cos^3\theta + \frac{1}{3}\theta\right)\Big|_0^{2\pi}$$

$$= \left(\frac{-1}{12}\overset{1}{\cancel{\cos^3 2\pi}} + \frac{2}{3}\pi\right) - \left(\frac{-1}{12}\overset{1}{\cancel{\cos^3 0}} - 0\right)$$

$$= \frac{2}{3}\pi$$