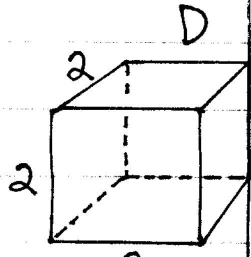
Section 16.8

5.) $\vec{F}(x,y,z) = (y-x)\vec{i} + (z-y)\vec{j} + (y-x)\vec{k}$,



$$\text{div}\,\vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$$

$$= (-1) + (-1) + (0) = -2;$$

then by Divergence Theorem

$$\text{Flux} = \iint_S \vec{F}\cdot\vec{n}\,dS = \iiint_D \text{div}\,\vec{F}\,dV$$

$$= \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} (-2)\,dV = -2\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} 1\,dV$$

$$= -2\,(\text{volume } D) = -2(8) = -16$$

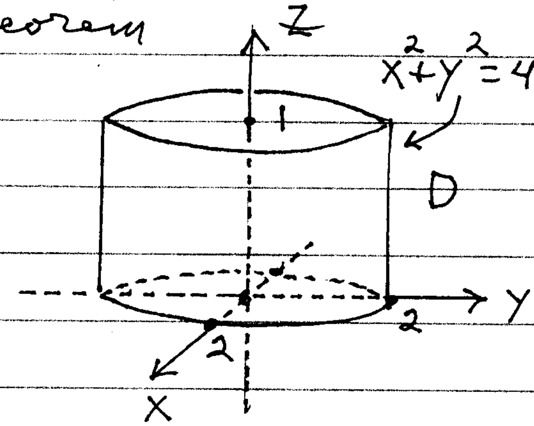
6.) c.) $\vec{F}(x,y,z) = (x^2)\vec{i} + (y^2)\vec{j} + (z^2)\vec{k}$,

$$\text{div}\,\vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$$

$$= 2x + 2y + 2z;$$

then by Divergence Theorem

$$\text{Flux} = \iint_S \vec{F}\cdot\vec{n}\,dS$$



$$= \iiint_D \text{div}\,\vec{F}\,dV$$

$$= \iiint_D (2x+2y+2z)\,dV$$

$$= \int_0^{2\pi}\int_0^{2}\int_0^{1} (2r\cos\theta + 2r\sin\theta + 2z)\,r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2\int_0^1 (2r^2\cos\theta + 2r^2\sin\theta + 2rz)\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^2 \Big[(2r^2\cos\theta + 2r^2\sin\theta) z + r z^2\Big]\Big|_{z=0}^{z=1} dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^2 (2r^2\cos\theta + 2r^2\sin\theta + r)\, dr\, d\theta$$

$$= \int_0^{2\pi} \left(\frac{2}{3}r^3\cos\theta + \frac{2}{3}r^3\sin\theta + \frac{1}{2}r^2\right)\Big|_{r=0}^{r=2} d\theta$$

$$= \int_0^{2\pi} \left(\frac{16}{3}\cos\theta + \frac{16}{3}\sin\theta + 2\right) d\theta$$

$$= \left(\frac{16}{3}\sin\theta - \frac{16}{3}\cos\theta + 2\theta\right)\Big|_0^{2\pi}$$

$$= \left(\frac{16}{3}\overset{0}{\cancel{\sin}}2\pi - \frac{16}{3}\overset{1}{\cancel{\cos}}2\pi + 4\pi\right)$$
$$- \left(\frac{16}{3}\overset{0}{\cancel{\sin}}0 - \frac{16}{3}\overset{1}{\cancel{\cos}}0 + 0\right) = 4\pi$$

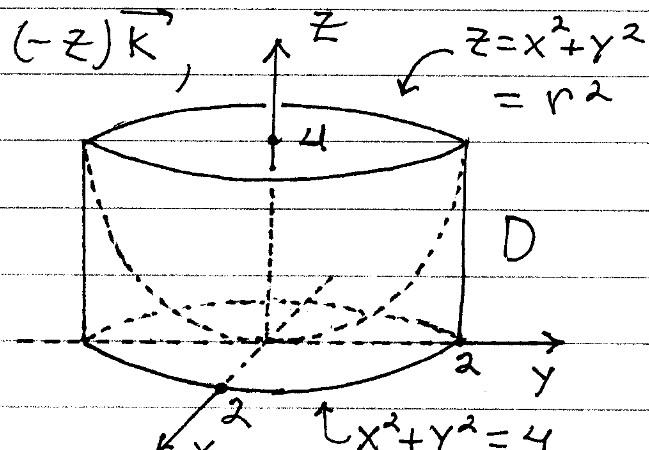
7.) $\vec{F}(x,y,z) = (y)\vec{i} + (xy)\vec{j} + (-z)\vec{k}$,

$\operatorname{div}\vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$

$= (0) + (x) + (-1) = x - 1$;

then by

Divergence Theorem



$$\text{Flux} = \iint_S \vec{F}\cdot\vec{n}\, dS = \iiint_D \operatorname{div}\vec{F}\, dV$$

$$= \iiint_D (x-1)\, dV = \int_0^{2\pi}\int_0^2\int_0^{r^2} (r\cos\theta - 1)\, r\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^2\int_0^{r^2} (r^2\cos\theta - r)\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^2 (r^2\cos\theta - r)\, z \Big|_{z=0}^{z=r^2} dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^2 (r^2\cos\theta - r)\, r^2\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^2 (r^4\cos\theta - r^3)\, dr\, d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{5}r^5\cos\theta - \frac{1}{4}r^4\right)\Big|_{r=0}^{r=2} d\theta$$

$$= \int_0^{2\pi} \left(\frac{32}{5}\cos\theta - 4\right) d\theta$$

$$= \left(\frac{32}{5}\sin\theta - 4\theta\right)\Big|_0^{2\pi}$$

$$= \left(\frac{32}{5}\overset{0}{\cancel{\sin 2\pi}} - 8\pi\right) - \left(\frac{32}{5}\overset{0}{\cancel{\sin 0}} - 0\right)$$

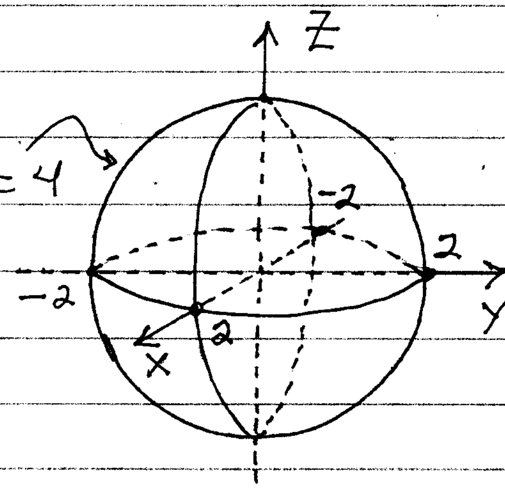
$$= -8\pi$$

8.) $\vec{F}(x,y,z) = (x^2)\,\vec{i} + (xz)\,\vec{j} + (3z)\,\vec{k}$;

$$\operatorname{div}\vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$$

$$= (2x) + (0) + (3) = 2x + 3;$$

$x^2 + y^2 + z^2 = 4$



by Divergence Theorem

$$\text{Flux} = \iint_S \vec{F}\cdot\vec{n}\, dS$$

$$= \iiint_D \operatorname{div}\vec{F}\, dV$$

$$= \iiint_D (2x+3)\, dV$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{2} (2\cdot\rho\sin\phi\cos\theta + 3)\,\rho^2\sin\phi \; d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{2} (2\rho^3\sin^2\phi\cos\theta + 3\rho^2\sin\phi)\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \left[\tfrac{1}{2}\rho^4\sin^2\phi\cos\theta + \rho^3\sin\phi\right]\Big|_{\rho=0}^{\rho=2} d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \left[8\cdot\tfrac{1}{2}(1-\cos 2\phi)\cos\theta + 8\sin\phi\right] d\phi\, d\theta$$

$$= \int_0^{2\pi} \left[4(\phi - \tfrac{1}{2}\sin 2\phi)\cos\theta - 8\cos\phi\right]\Big|_{\phi=0}^{\phi=\pi} d\theta$$

$$= \int_0^{2\pi} \Big[\{4(\pi - \tfrac{1}{2}\overset{0}{\cancel{\sin}\,2\pi})\cos\theta - 8\overset{-1}{\cancel{\cos}\,\pi}\}$$
$$- \{4(0 - \tfrac{1}{2}\overset{0}{\cancel{\sin}\,0})\cos\theta - 8\overset{1}{\cancel{\cos}\,0}\}\Big] d\theta$$

$$= \int_0^{2\pi} (4\pi\cos\theta + 8 + 8)\, d\theta$$

$$= (4\pi\sin\theta + 16\theta)\Big|_0^{2\pi}$$

$$= (4\pi\overset{0}{\cancel{\sin}\,2\pi} + 32\pi) - (4\pi\overset{0}{\cancel{\sin}\,0} + 0)$$

$$= 32\pi$$

14.) $F(x,y,z) = \dfrac{x}{\sqrt{x^2+y^2+z^2}}\vec{i} + \dfrac{y}{\sqrt{x^2+y^2+z^2}}\vec{j} + \dfrac{z}{\sqrt{x^2+y^2+z^2}}\vec{k}$,

$$\frac{\partial M}{\partial x} = \frac{\sqrt{x^2+y^2+z^2}\cdot(1) - x\cdot\frac{1}{\cancel{2}}(x^2+y^2+z^2)^{-1/2}\cdot\cancel{2}x}{x^2+y^2+z^2}$$

$$= \frac{\dfrac{\sqrt{x^2+y^2+z^2}}{1} - \dfrac{x^2}{\sqrt{x^2+y^2+z^2}}}{\dfrac{x^2+y^2+z^2}{1}}$$

$$= \frac{(x^2+y^2+z^2) - x^2}{\sqrt{x^2+y^2+z^2}} \cdot \frac{1}{x^2+y^2+z^2} = \frac{y^2+z^2}{(x^2+y^2+z^2)^{3/2}},$$

$$\frac{\partial N}{\partial y} = \frac{x^2+z^2}{(x^2+y^2+z^2)^{3/2}}, \quad \frac{\partial P}{\partial z} = \frac{x^2+y^2}{(x^2+y^2+z^2)^{3/2}},$$

so that

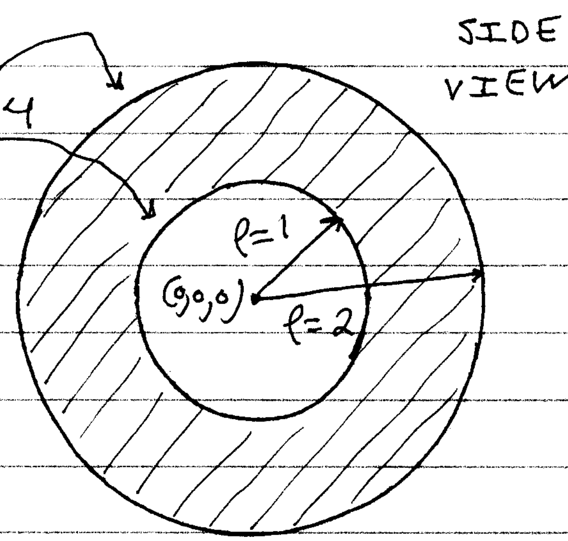
$$\text{div}\,\vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$$

$$= \frac{(y^2+z^2)+(x^2+z^2)+(x^2+y^2)}{(x^2+y^2+z^2)^{3/2}}$$

$$= \frac{2(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{3/2}} = \frac{2}{\sqrt{x^2+y^2+z^2}};$$

sphere : $x^2+y^2+z^2=4$

sphere : $x^2+y^2+z^2=1$



$$\text{Flux} = \iint_S \vec{F}\cdot\vec{n}\, dS$$

$$= \iiint_D \text{div}\,\vec{F}\, dV$$

$$= \iiint_D \frac{2}{\sqrt{x^2+y^2+z^2}}\, dV$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_1^2 \frac{2}{\sqrt{\rho^2}} \cdot \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_1^2 2\rho \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \left(\rho^2 \sin\phi \Big|_{\rho=1}^{\rho=2} \right) d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} (4\sin\phi - \sin\phi)\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} 3\sin\phi \; d\phi\, d\theta$$

$$= \int_0^{2\pi} \left(-3\cos\phi \Big|_{\phi=0}^{\phi=\pi} \right) d\theta$$

$$= \int_0^{2\pi} \left(-3\overset{-1}{\cancel{\cos}\pi} - {}^{-}3\overset{1}{\cancel{\cos}\, 0} \right) d\theta$$

$$= \int_0^{2\pi} 6\, d\theta = 6\theta \Big|_0^{2\pi} = 12\pi$$

16.) $\vec{F}(x,y,z) = \ln(x^2+y^2)\,\vec{i} + {}^{-}\left(\frac{2z}{x}\arctan\frac{y}{x}\right)\vec{j}$

$+\; z\sqrt{x^2+y^2}\cdot\vec{k}$;

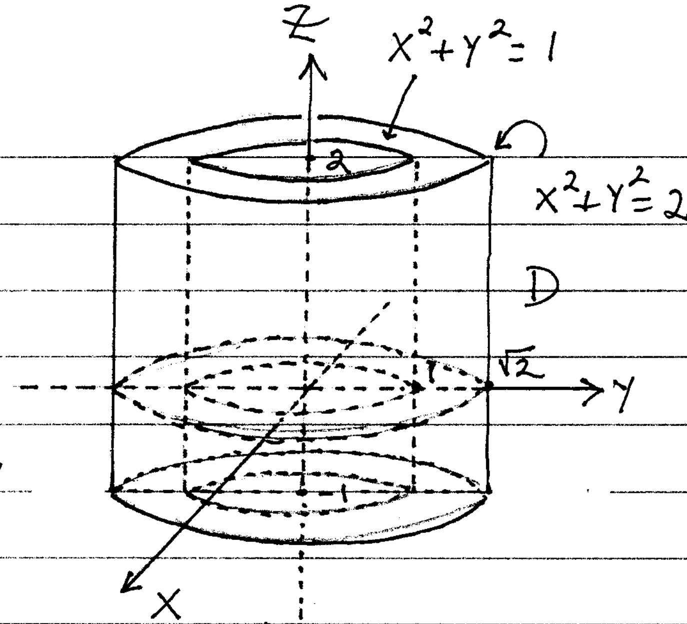
$$\frac{\partial M}{\partial x} = \frac{2x}{x^2+y^2},$$

$$\frac{\partial N}{\partial y} = \frac{-2z}{x}\cdot\frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{1}{x} = \frac{-2z}{x^2+y^2},$$

$$\frac{\partial P}{\partial z} = \sqrt{x^2+y^2}, \quad \text{then}$$

$$\operatorname{div}\vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$$

$$= \frac{2x + {}^{-}2z}{x^2+y^2} + \sqrt{x^2+y^2} \;;$$



$$\text{Flux} = \iint_{S} \vec{F}\cdot\vec{n}\, dS$$

$$= \iiint_{D} \operatorname{div}\vec{F}\, dV$$

$$= \iiint_{D} \left(\frac{2x + {}^{-}2z}{x^2+y^2} + \sqrt{x^2+y^2}\right) dV$$

$$= \int_0^{2\pi}\int_1^{\sqrt{2}}\int_{-1}^{2} \left(\frac{2r\cos\theta + {}^{-}2z}{r^2} + \sqrt{r^2}\right) r\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_1^{\sqrt{2}}\int_{-1}^{2} \left(2\cos\theta + \frac{{}^{-}2z}{r} + r^2\right) dz\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_1^{\sqrt{2}} \left(2\cos\theta\cdot z + \frac{{}^{-}1}{r} z^2 + r^2 z\right)\Big|_{z=-1}^{z=2} dr\, d\theta$$

$$= \int_0^{2\pi}\int_1^{\sqrt{2}} \Big[\left(4\cos\theta + \frac{{}^{-}4}{r} + 2r^2\right) - \left(-2\cos\theta + \frac{{}^{-}1}{r} - r^2\right)\Big] dr\, d\theta$$

$$= \int_0^{2\pi}\int_1^{\sqrt{2}} \left[6\cos\theta + \frac{{}^{-}3}{r} + 3r^2\right] dr\, d\theta$$

$$= \int_0^{2\pi} \left(6\cos\theta\cdot r + {}^{-}3\ln r + r^3\right)\Big|_{r=1}^{r=\sqrt{2}} d\theta$$

$$= \int_0^{2\pi} \Big(\left(6\sqrt{2}\cos\theta + {}^{-}3\ln\sqrt{2} + 2\sqrt{2}\right) - \left(6\cos\theta + {}^{-}3\cancel{\ln 1}^{\,0} + 1\right)\Big) d\theta$$

$$= \int_0^{2\pi} \left((6\sqrt{2}-6)\cos\theta + {}^{-}3\ln\sqrt{2} + (2\sqrt{2}-1)\right) d\theta$$

$$= (6\sqrt{2}-6)\cancel{\sin\theta\Big|_0^{2\pi}}^{\,0} + \left({}^{-}3\ln\sqrt{2} + (2\sqrt{2}-1)\right)\theta\Big|_0^{2\pi}$$

$$= \left(-3\ln\sqrt{2} + 2\sqrt{2} - 1\right) 2\pi$$

26.) If $\vec{F}(x,y,z) = a\vec{i} + b\vec{j} + c\vec{k}$, a, b, and c constants then

$$\operatorname{div} \vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$$

$$= (0) + (0) + (0) = 0, \text{ and}$$

so by Divergence Theorem

$$\text{Flux} = \iint_S \vec{F} \cdot \vec{n} \, dS$$

$$= \iiint_D \operatorname{div} \vec{F} \, dV$$

$$= \iiint_D 0 \, dV$$

$$= 0 .$$