1–Introduction to PDE

MATH 22C

1. Introduction To Partial Differential Equations

• Recall: A function f is an *input-output machine* for numbers:

$$y = f(t)$$

“Output” $y \in \mathcal{R}$ “Input” $t \in \mathcal{R}$

“Name of function” $\equiv f$

t=independent variable$\in \mathcal{R}$ (known)
y=dependent variable$\in \mathcal{R}$ (unknown)
f=name of function...

–PDE’s typically depends on time, so we use t as the input variable instead of x...

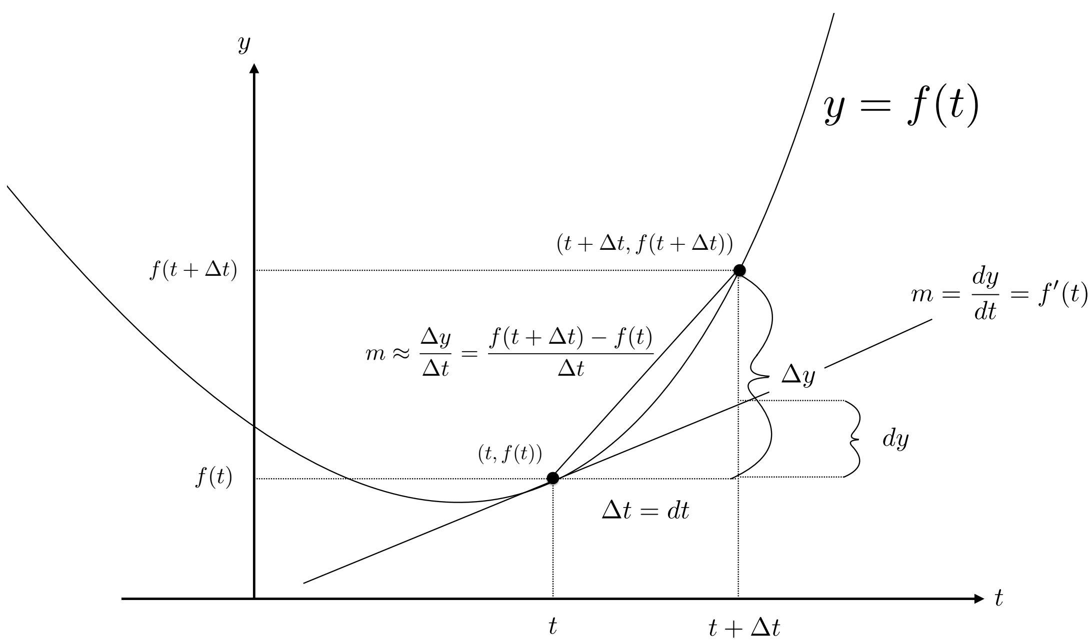
–Functions can be given by *exact formulas*...famous examples being: $y = (t)^2$, $y = e^{(t)}$, $y = \ln(t)$, $y = \sin(t)$, etc.

–But the solution of a *differential equation* in general has no such *exact* formula, so we are forced to talk about solutions with the general notation of functions $y = f(t)$. For example, if $f'(t) = f(t)^2$ we say f is a solution to $y' = y^2$.

–In fact, the transcendental functions that have formulas...polynomials, exponentials, logarithms, trig functions (together with their rational combinations and compositions)...are really special exact solutions of simple differential equations.

- Recall the *derivative* $f'(t)$ of a function:

$$\begin{aligned} f'(t) &= \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t} = m \\ &= \text{``slope of the line tangent to the graph of } f \text{ above the point } t\text{''} \end{aligned}$$



Notation:

$$y' = y_t = \frac{dy}{dt} = f'(t)$$

All different names for the same thing...

• An *ordinary differential equation* (ODE) is an equation involving an unknown function and its derivatives $y', y'', y''', ...etc.$

Examples: $y' = ky$, $y'' = a^2 y$, $y' = a^2 y^2$, etc.

• If a function depends on more than one independent variable, say $f(t, x)$ instead of just $f(t)$, then we can take partial derivatives:

$$\begin{aligned} \frac{\partial f}{\partial x}(x,t) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x, t) - f(x,t)}{\Delta x} \\ &= \text{“slope of the line tangent to the} \\ &\quad \text{graph of } f \text{ along } t = \textit{const.}, \text{ above the point } (x,t)\text{”} \end{aligned}$$

$$\begin{aligned} \frac{\partial f}{\partial t}(x,t) &= \lim_{\Delta t \to 0} \frac{f(x, t+\Delta t) - f(x,t)}{\Delta x} \\ &= \text{“slope of the line tangent to the} \\ &\quad \text{graph of } f \text{ along } x = \textit{const.}, \text{ above the point } (x,t)\text{”} \end{aligned}$$

Notation: $\frac{\partial y}{\partial t} = y_t = \frac{\partial f}{\partial t} = \frac{\partial f}{\partial t}(x,t)$

Whenever you see a “d” in an equation, it means there is only one independent variable...when you see ∂, it means there's more than one independent variable lurking...

• A *partial differential equation* (PDE) is an equation involving an unknown function of more than one variable (say $u = f(x,t)$ and its partial derives derivatives $u_t, u_x, u, u_{tt}, u_{xx}, ...etc$).

Q: *Why are PDE's so important?* ANS: The laws of science almost always come to us stated in terms of rates of

change...(F=ma, rates of chemical reactions, rates of population changes, etc.)...thus the *starting point* for problems in science are equations for rates of changes of things...solving them yields scientific predictions.
The PDE's we consider:

$u_t - cu_x = 0$ (Transport equation)—Linear
$u_t + uu_x = 0$ (Inviscid Burgers Equation)—Nonlinear
$u_t + uu_x = u_{xx}$ (Viscous Burgers Equation)—Nonlinear

The Linear Equations of Classical Physics:

$$u_{tt} - c^2 u_{xx} = 0 \quad \text{(WaveEquation)}$$
$$u_t - ku_{xx} = 0 \quad \text{(HeatEquation)}$$
$$u_{xx} + u_{yy} = 0 \quad \text{(LaplaceEquation)}$$

The Linear Equation of Quantum Mechanics:

$$-i\hbar u_t = Hu \qquad \text{(Schrodinger Equation)}$$
$$\text{(E.g., } Hu = \left\{ -\frac{\hbar^2}{\mu}\Delta + V \right\} u\text{)}$$

In fact–all these are model problems used to understand what is happening in *really hard* PDE's that are extremely important—

Eg, one of the Fundamental Equations of Science, the Navier-Stokes Equations:

• Plan for the class:

–Write down the Navier-Stokes Equation and see how impossibly complicated it is at first appearance...

–Review ODE's for background...get started on PDE's

–Interpret the listed PDE's as warmup problems for Navier-Stokes

–Study the listed PDE's one by one to get a sense of what they are saying, and use this to interpret the meaning of the Navier-Stokes Equations...

2. **The Navier-Stokes Equations:**

• The Navier-Stokes equations are a system of PDE's of Fundamental Importance to Science...
but far too complicated to understand all at once...

$$\rho_t + \mathrm{div}\,(\rho \mathbf{u}) = 0, \tag{1}$$

$$(\rho u_i)_t + \mathrm{div}\,(\rho u_i \mathbf{u} + p\mathbf{e_i}) = \xi \frac{\partial}{\partial x^i}(\mathrm{div}\ \mathbf{u}) + \mu \Delta u_i. \tag{2}$$

• Equations (1) and (2) express:

– Newton's Laws for Continuous Media (like air) *with* Friction.

–Equation (1) is the *Balance of Mass* or *Continuity Equation*

–Equation (2) is the *Balance of Momentum Equation*

• Here $\mathbf{x} = (x^1, x^2, x^3)$ and:

$$\begin{aligned} \rho &= \text{density} = \text{mass/volume} \\ \mathbf{u} &= (u_1, u_2, u_3) = \text{velocity}, \\ p &= \text{pressure}, \\ \mu &= \text{dynamic/shear viscosity} \\ \xi &= \text{bulk viscosity}. \end{aligned}$$

• Also:

$$
\begin{aligned}
\rho_t &= \frac{\partial \rho}{\partial t}, \\
\mathrm{div}\mathbf{F} &= \left(\frac{\partial F_1}{\partial x^1}\right) + \left(\frac{\partial F_2}{\partial x^2}\right) + \left(\frac{\partial F_3}{\partial x^3}\right), \\
\Delta f &= \frac{\partial^2 f}{\partial \left(x^1\right)^2} + \frac{\partial^2 f}{\partial \left(x^2\right)^2} + \frac{\partial^2 f}{\partial \left(x^3\right)^2}.
\end{aligned}
$$

• The equations express the principle that mass is conserved, and changes in momentum come from acceleration of the fluid in response to the gradient of the pressure, minus losses of momentum due to the friction of viscosity whose magnitude is measured by μ & ξ

• The Navier-Stokes Equations are EXTREMELY IMPORTANT.

– They are the starting point of Fluid Mechanics.

– They model fluid flow in cellular dynamics

– They model shock wave explosions in air.

– Most fluid flow problems are modeled by some modification of these equations, so if you know something about these equations, you have a starting point for modeling pretty much all fluids.

–They model flow of air so well that airplane designers use them to numerically simulate the flow of air over airplane shapes on a computer. (If you discover a new numerical algorithm that solves these equations significantly faster on a computer, you will become rich!)

–The shock waves that come off the wing of a plane that create sonic booms can be modeled by these equations. Away from the boundary layer, to very good approximation, you can take $\lambda = \mu = 0$, but on the plane's boundary, $\xi \& \mu$ dominate.

–Different fluids are modeled by adjusting the equation of state, that is, how the pressure p depends on the density ρ. (There is another variable that effects p, namely the temperature, and another equation for the time rate of change of the temperature, but for our purposes we assume p depends only on ρ, called a *barotropic* equation of state.)

• THE MAIN POINT TO UNDERSTAND: The Navier-Stokes equations are a fiercely nonlinear system of 4 coupled PDE's, far too complicated to understand all at once. No one is close to a complete mathematical understanding of the solutions that satisfy these equations. They are as complicated as the wheather!

–For example, it is not known to what degree the Navier-Stokes equations model the turbulent motion of fluids. There is a million dollar Clay Prize for the first person to prove mathematically that either

(1) Solutions $\rho(\mathbf{x}, t)$, $\mathbf{u}(\mathbf{x}, t)$ always exist starting from some initial configurations of the density and velocity of the fluid at some starting time, or

(2) Some starting configuration of density and velocity leads to a *singularity* where the density or velocity or one of their partial derviatives becomes infinite...indicating the onset of turbulence.

3. **Warmup Problems for the Navier-Stokes Equations:**

–If you neglect viscosity and set $\xi = \mu = 0$ in (2), you get the famous *compressible Euler equations*, which accurately describe the flow of a gas without friction or heat conduction, (first written down correctly by Leonard Euler in about 1750...),

$$\rho_t + \mathrm{div}\,(\rho\mathbf{u}) = 0, \tag{3}$$

$$(\rho u_i)_t + \mathrm{div}\,(\rho u_i\mathbf{u} + p\mathbf{e_i}) = 0. \tag{4}$$

If you assume zero bulk viscosity $\xi = 0$, (often a good assumption), assume $\rho = 1$ is constant and neglect the convective terms (with the x-derivatives), the second equation gives you the heat equation in each component of velocity,

$$u_t = \Delta u. \tag{5}$$

The heat equation isolates the pure effect of viscosity. Solutions of the heat equation typically decay as $t \to \infty$ to *time-independent* solutions with $u_t = 0$, so time independent solutions solve Laplace's equation

$$\Delta u = 0. \tag{6}$$

–Taking another direction, if you restrict the Navier-Stokes equations to one dimension by assuming everything depends only on x, so in particular $\mathbf{u} = u = dx/dt$, you get the 1-dimensional Navier-Stokes equations

$$\begin{aligned} \rho_t + (\rho u)_x &= 0, \\ (\rho u)_t + \left(\rho u^2 + p\right)_x &= \epsilon u_{xx}, \end{aligned}$$

which uses that $\frac{\partial}{\partial x}(\mathrm{div}\ \mathbf{u}) = u_{xx} = \Delta u$ so $\epsilon = \xi + \mu$.
–If you neglect viscosity and set $\epsilon = 0$ in the 1-D Navier-Stokes, you get the 1-D compressible Euler equations, the

setting for the mathematical theory of shock waves,

$$\rho_t + (\rho u)_x = 0, \tag{7}$$
$$(\rho u)_t + \left(\rho u^2 + p\right)_x = 0. \tag{8}$$

This system is the "Eulerian" version of the equations, meaning they describe the flow in physical space. If you linearize the equations about a constant density $\rho = \bar{\rho} + \delta\hat{\rho}$, then to leading order in δ, (i.e., for small amplitude solutions near $\bar{\rho}$, after a calculation to "linearize the equations", which we will do later on), the compressible Euler equations reduce to the wave equation in the perturbations of the density $\hat{\rho}$,

$$\hat{\rho}_{tt} - c^2\hat{\rho}_{xx} = 0. \tag{9}$$

The wave equation is the setting for the mathematical theory of sound waves, and this tells us that the sinusoidal waves that solve the wave equation, the modes of vibration in the theory of sound, are the solutions of the compressible Euler equations you get in the limit of weak waves. In fact, $c = \sqrt{p'(\rho)}$, so from this we can also calculate the speed of sound from knowledge of how the pressure depends on the density *alone.*

–Finally, one can show (7)-(8) is equivalent to the "Lagrangian" version of the equations, (based on moving with the particles), which is called the p-system, (so coined by Joel Smoller),

$$v_t - u_x = 0, \tag{10}$$
$$u_t + p(v)_x = 0, \tag{11}$$

where $v = 1/\rho$ is the specific volume, and u is the velocity. If you differentiate the first equation with respect to t and the second equation with respect to x you get

$$v_{tt} = u_{xt} = u_{tx} = -p(v)_{xx} = \left[-p'(v)v_x\right]_x,$$

which gives the nonlinear wave equation

$$v_{tt} - [p'(v)v_x]_x = 0.$$

In particular, if we take the *sound speed*

$$c \equiv \sqrt{-p'(v)},$$

then the nonlinear wave equation takes the form

$$v_{tt} - \left[c^2 v_x\right]_x = 0.$$

Now in the limit of "weak waves", we can assume the sound speed $c = constant$, in which case the nonlinear wave equation reduces to the linear wave equation

$$v_{tt} - c^2 v_{xx} = 0.$$

In particular, this is equivalent to the first order system

$$\begin{aligned} v_t - u_x &= 0, \\ u_t - c^2 v_x &= 0, \end{aligned}$$

as seen by differentiating the first equation with respect to x and the second equations with respect to t and setting

$$v_{tt} = u_{xt} = u_{tx} = c^2 v_{xx}.$$

So the linear wave equation emerges from the nonlinear theory of sound in the limit of weak waves.

In these notes, the nonlinear wave equation is introduced as the equation describing the nonlinear theory of sound neglecting friction and dissipation, and this reduces to the linear wave equation for weak signals. The heat equation is introduced to tell us how to incorporate dissipation, and Laplace's equation is introduced to describe the steady state solutions of the heat equation. Thus the main point of these notes is to describe the difference between linear and nonlinear PDE's through a comparison of the linear and nonlinear wave equation.

The wave equation comes up everywhere. It's ubiquitous. Indeed, we show in the next section that every nonlinear oscillator generated by a restoring force, oscillates sinusoidally according to the harmonic oscillator when the waves are sufficiently weak, so everywhere oscillators are generating sinusoidal oscillations according to the equation $\ddot{y} + k^2 y = 0$. Correspondingly, the wave equation is the simplest equation that propagates sinusoidal oscillations, moving in time, at the same speed in every direction. As a result, the wave equation has played a central role in the most fundamental discoveries of the 18'th, 19'th and 20'th centuries.

Indeed, in the middle of the 18'th century, in 1753, Leonard Euler derived the compressible Euler equations, linearized them to get the wave equation (the same equation his colleague *D′Alembert* obtained several years earlier to describe a vibrating string), and Euler thereby resolved the greatest question of his era: *What is sound?* In the middle of the next century, in 1853, Maxwell realized that he could get a wave equation out of the equations for electromagnetic fields if he added an additional equation to the equations he got from Faraday, and in so doing he used the wave equation to answer the greatest question of that era: *What is light?* And finally, at the beginning of the next century, in 1905, Albert Einstein made the bold leap that the wave equation from Maxwell's theory should be the same in every *inertial frame*, and hence the speed of light should also be constant in every *inertial frame*. By this he derived the spacetime transformations of special relativity from the invariances of the wave equation, thereby answering the greatest question of that era: *How are space and time entangled?* In these notes we will relive all of these discoveries from the point of view of the linear and nonlinear wave equation, the point of view of PDE's.

4. **The Compressible Euler Equations and Einstein's General Relativity:**

Writing out the four compressible Euler equations (3), (4) in a matrix form clarifies the equations:

$$Div_{t,\mathbf{x}} \begin{pmatrix} \rho & \rho u_1 & \rho u_2 & \rho u_3 \\ \rho u_1 & \rho u_1^2 + p & \rho u_1 u_2 & \rho u_1 u_3 \\ \rho u_2 & \rho u_2 u_1 & \rho u_2^2 + p & \rho u_1 u_3 \\ \rho u_3 & \rho u_3 u_1 & \rho u_3 u_2 & \rho u_3^2 + p \end{pmatrix} \equiv Div\, T = 0. \qquad (12)$$

This means take the t-derivative of the first column plus the x_1-derivative of the second column plus the x_2-derivative of the third column plus the x_3-derivative of the fourth column and set each row equal to zero.

Let's now count the number of equations and unknowns. Well, these are four equations in the five unknown functions $\rho(\mathbf{x},t), p(\mathbf{x},t), u_1(\mathbf{x},t), u_2(\mathbf{x},t), u_3(\mathbf{x},t)$. Now like algebraic equations, partial differential equations should have the same number of equations as unknowns to be *well posed*, (i.e. in order not to be inconsistent with no solutions, or to admit too many solutions). Thus (12) has one more unknown than equation. Thus, we need an "equation of state to close the equations", which in the case of a barotropic fluid, requires imposing $p = p(\rho)$. The *equation of state* represents the physics of the fluid that must be added to the equations. This is all we need to describe *in principle* how Albert Einstein discovered the equations of General Relativity.

To start, Einstein interpreted $\rho \equiv \rho c^2$ as the energy density, (based on $E = mc^2$), and conjecture that

$$T \equiv \begin{pmatrix} \rho & \rho u_1 & \rho u_2 & \rho u_3 \\ \rho u_1 & \rho u_1^2 + p & \rho u_1 u_2 & \rho u_1 u_3 \\ \rho u_2 & \rho u_2 u_1 & \rho u_2^2 + p & \rho u_1 u_3 \\ \rho u_3 & \rho u_3 u_1 & \rho u_3 u_2 & \rho u_3^2 + p \end{pmatrix} \qquad (13)$$

measured the energy and the flux of energy at a point in spacetime, and as such was the source of spacetime curvature. He called T the Stress Energy Tensor and set out to look for an equation for the gravitational field, (say of the sun, replacing Newton's force law), of the form $G = \kappa T$, where G was a 4×4 matrix measuring the curvature of spacetime, and κ was a coupling constant. Now since T is a symmetric 4×4 matrix, $T_{ij} = T_{ji}$, (note that the matrix (13) is symmetric about the diagonal!), he required that G should be a symmetric 4×4 matrix that describes the curvature. But a symmetric 4×4 matrix admits ten independent entries, (count them). Thus there are ten unknowns in G and four unknowns in T, so $G = \kappa T$ would produce ten equations in 14 unknowns. But he wanted the equations to be independent of spacetime coordinates, which he realized meant that four more equations could be freely specified to specify the coordinates system. (You'll just have to believe this part.) Thus there are 14 equations in 14 unknowns, and no more equations can be imposed to make the system well posed. But his theory could make no physical sense unless he could impose that energy and momentum be conserved. That is, asking that energy-momentum be conserved, he needed $Div\, T = 0$ according to the compressible Euler equations (12), and this then imposes four more equations—and he can't impose any more equations because he already has 14 equations in 14 unknowns. Thus by counting equations and unknowns, he realized there was no freedom left, and that he must require $Div\, G = 0$ at the start. This enabled him to guess the form of the curvature G (from the Riemann curvature tensor proposed by Riemann in 1854), by noticing that there was essentially only one "curvature tensor" constructible from the Riemann curvature tensor, which also satisfied the condition $Div\, G = 0$. By this line of reasoning he found the equations $G = 8\pi T$ of general relativity. This was the most difficult step in the discovery of the field equations, and it took him several years to get it straight, making several missteps along the way.

Now to understand this in detail requires several upper division and graduate classes in differential geometry, and the Stress Energy Tensor T in (12) isn't exactly correct. To be precise, it is the *relativistic* version of T in (13), (based on Einstein's Special Relativity of 1905), and the *relativistic* compressible Euler equations $Div\, T = 0$, that play the role of imposing conservation of energy as explained above. And also, to be precise, the ten free functions in G aren't the ten independent components of G, but really the ten free components of the gravitational metric in terms of which these entries are defined according to Riemann's formula for the curvature tensor. But in principle it is the same. This is how his argument goes.

Conclusion: The compressible Euler equations are absolutely fundamental, and their understanding is the starting point for arguably the greatest discovery of all time—The theory of General Relativity, first proposed by Albert Einstein in correct form in 1915, after nine years of struggle.