3-EXISTENCE THEOREMS for ODE's
MATH 22C

In this section we consider the initial value problem (ivp) for first order autonomous systems of ODE's of the form

$$\mathbf{y}' = f(\mathbf{y}), \tag{1}$$

$$\mathbf{y}(t_0) = \mathbf{y}_0. \tag{2}$$

Here $\mathbf{y}(\mathbf{t}) = (x(t), y(t))$ denotes an unknown curve in the plane parameterized by t. The equation (1) states that

(i) the tangent vector to the solution curve $\mathbf{y}'(t)$ should be tangent to the known vector field $f(\mathbf{y}) = (f_1(\mathbf{y}), f_2(\mathbf{y}))$ at each point, and

(ii) the speed

$$\frac{ds}{dt} = \|\mathbf{y}'(t)\| = \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}$$

of the curve should equal the length

$$\|f(\mathbf{y})\| = \sqrt{f_1(\mathbf{y})^2 + f_2(\mathbf{y})^2}$$

of the vector field $f(\mathbf{y})$ at each point. (In fact, the existence theorems are the same for general non-autonomous systems of n equations $\mathbf{y}' = f(\mathbf{y}, t)$ where $\mathbf{y}(\mathbf{t}) = (y_1(t), ..., y_n(t))$, but to keep it easy to visualize, we'll restrict to scalar and 2×2 systems, and we'll assume autonomous, the case when f does not depend explicitly on t.)

1. The fundamental existence theorem for first order scalar ODE's

• Consider first the (ivp) for a scalar first order autonomous ODE in one unknown function $y = f(t)$:

$$y' = f(y) \tag{3}$$

$$y(t_0) = y_0. \tag{4}$$

–We always assume f is a continuous function, i.e., its graph is an unbroken surface in the sense that at all points we have

$$\lim_{y_n \to y} f(y_n) = f(y).$$

The most interesting thing is that continuity of f is not enough, but we don't need f to be differentiable either. All we need is that f be *Lipschitz continuous* as the following theorem asserts.

Theorem 1. *Suppose f is Lipschitz continuous in y. Then a unique solution $y(t)$ exists for all t.*

Definition 2. *$f(y)$ is* Lipschitz continuous *in y if there exists a constant K such that*

$$|f(y_2) - f(y_1)| \leq K|y_2 - y_1|. \tag{5}$$

Now we know from fundamental ODE (3) $\dot{y} = y^2$ that solutions do not always exist for all t when the equation is nonlinear. So we know we can only get a *local* existence theorem that can apply to nonlinear equations in general. Here is the modification to (4) that generalizes:

Theorem 3. *Suppose f is* Lipschitz continuous *for all y_1, y_2 in some interval $[y_0 - \delta, y_0 + \delta]$. Then a unique solution $y(t)$ of*

$$y'(t) = f(y(t)),$$
$$y(t_0) = y_0.$$

exists for all t in some interval $t \in (t_0 - \epsilon, t_0 + \epsilon)$.

• **Example:** $y' = ky$

– We check that $f(y) = ky$ is *Lipschitz continuous* in y. For this we write:

$$|f(y_2) - f(y_1)| = |ky_2 - ky_1| = |k(y_2 - y_1)| = |k||y_2 - y_1|.$$

– Thus by Definition 2, $f(y)$ is Lipschitz continuous with uniform $K = |k|$ which is independent of t, so Theorem 4 tells us that a solution $y(t)$ satisfying $y' = ky$, $y(t_0) = y_0$ exists for all time $-\infty < t < \infty$.

– We already know this, since we have a formula for the solution given by

$$y(t) = y_0 e^{k(t-t_0)}.$$

• **Example:** $y' = y^2$

– $y' = y^2 = f(y,t)$ means $f(y,t) = f(y) = y^2$, and again f doesn't depend on t (except through the unknown function $y(t)$).

– We check that $f(y) = y^2$ is *Lipschitz continuous* in y. For this we write:

$$|f(y_2)-f(y_1)| = |y_2^2-y_1^2| = |(y_2+y_1)(y_2-y_1)| = |y_2+y_1||y_2-y_1|.$$

– Now for y_1 and y_2 sufficiently close to y_0, say

$$y_1, y_2 \in [y_0 - \delta, y_0 + \delta] \subset (-2y_0, 2y_0),$$

we have $|y_2 + y_1| \leq 4y_0$. Thus we can take $K = 4y_0$ for y sufficiently close to y_0

– Therefore: The assumptions of Theorem 3 hold, so we conclude from it that a unique solution $y(t)$ satisfying $y'(t) = y(t)^2$, $y(0) = y_0$ exists in some interval $t \in (t_0 - \epsilon, t_0 + \epsilon)$.

–This confirms what we already know, since we have a formula for the solution $y' = y^2$ given by

$$y(t) = \frac{1}{\frac{1}{y_0} - (t - t_0)} = \frac{y_0}{1 - y_0(t - t_0)}.$$

– For example, in the case when the initial condition is taken at $t_0 = 0$, $y(0) = y_0$, we have the formula

$$y(t) = \frac{1}{\frac{1}{y_0} - t}.$$

Thus, for example, if $y_0 > 0$, we know the solution only exists so long as

$$\frac{1}{y_0} - t > 0,$$

or

$$y < \frac{1}{y_0}.$$

Thus solutions $y(t)$ exist only so long as

$$-\infty < t < \frac{1}{y_0},$$

–This confirms: the ϵ guaranteed by Theorem 3 must satisfy

$$\epsilon < \frac{1}{y_0}.$$

–That is: the interval of existence shrinks to zero as $y_0 \to \infty$.

– **Conclude:** When ODE's are nonlinear, we can only expect a short time *local* existence theorem. In fact, Theorem 3 is the best general existence theorem we have.

• **Instructive Example:** $y' = \sqrt{y}$

–We can solve this as a separable equation: $\frac{y'}{\sqrt{y}} = 1$, so integrating both sides gives

$$\int_{y_0}^{y(t)} \frac{y'}{\sqrt{y}} dy = \int_0^t d\xi = t,$$

where

$$\int_0^t \frac{y'}{\sqrt{y}} dy = 2\left[\sqrt{y(t)} - \sqrt{0}\right] = 2\sqrt{y(t)},$$

so $2\sqrt{y(t)} = t$, or $y(t) = t^2/4$.

–This leads to the following conundrum:

Both $y = \frac{t^2}{4}$ and $y(t) = 0$ solve the initial value problem $y' = \sqrt{y}$, $y(0) = 0$. But if Theorem 3 applies, then there is not a *unique* solution.

–The resolution is that $f(y) = \sqrt{y}$ is NOT Lipschitz continuous in a interval around $y_0 = 0$.

–To see this, use the

Mean Value Theorem: If $f(y)$ is differentiable on (a, b) and continuous on $[a, b]$, then $f(b) - f(a) = f'(y_*)(b - a)$ for some $y_* \in (a, b)$. Applying this to $f(y) = \sqrt{y}$ when $a = y_1, b = y_2$, we get

$$|f(y_2) - f(y_1)| = |f'(y_*)(y_2 - y_1)| = \frac{1}{2\sqrt{y_*}}|y_2 - y_1|,$$

and we see that the constant $K = \frac{1}{2\sqrt{y_*}}$ is unbounded for y_1 and y_2 sufficiently near $y_0 = 0$, contradicting the condition for Lipschitz continuity in an interval around y_0.

–**Conclude:** When $f(y)$ is not Lipshitz continuous in an interval around y_0, we cannot expect $y' = f(y, t)$, $y(t_0) = y_0$ to have a unique solution in an interval about $t = 0$.

We can now establish an easy to check criteria for existence and uniqueness of solutions.

Theorem 4. *If $f(y)$ is continuously differentiable on (a, b), then a unique local solution $y(t)$ of the initial value problem (3), (4) exists for every t_0 and every $y_0 \in (a, b)$.*

Proof: By Theorem 3 it suffices to prove that f is Lipschitz continuous in some open ball about y_0. But by the mean value theorem,

$$|f(y_2) - f(y_1)| = f'(y^*)|y_2 - y_1|.$$

Now since f' is assumed continuous, and a continuous function takes on its maximum and minimum value on a closed interval $[y_0 - \delta, y_0 + \delta] \subset (a, b)$, we can set K equal to the maximum of $|f'(y)|$ on this interval. It follows that the conditions of Theorem 3 are met. $\square$

Proof: Note that for $y' = ky$ and $y' = y^2$ the function f is continuously differentiable, but not so for $f(y) = \sqrt{y}$ at $y = 0$.

2. The fundamental existence theorem for first order systems of ODE's

–**Recall:** A second order equation can always be written as a first order system: For example, consider $x'' + a^2 x = 0$, or more generally, the general second order ODE

$$x'' = g(x, x', t).$$

Write it as a first order system by taking $y = x'$ so

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} v \\ g(x, y, t) \end{pmatrix}$$

–In this case, you can let

$$\mathbf{y}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

be a vector, and write the system in the same form as the scalar ODE's, namely,

$$\mathbf{y}' = f(\mathbf{y}, t) \tag{6}$$

where

$$f(\mathbf{y}, t) = \begin{pmatrix} -y \\ g(x, y, t) \end{pmatrix}$$

Note that the first system will be autonomous ($f = f(\mathbf{y})$) iff the original equation is autonomous ($g = g(x, x')$). To keep the visualization simple, restrict now to general autonomous first order 2×2 systems of the form

$$\mathbf{y}' = f(\mathbf{y}),$$

where f is any continuous function of $\mathbf{y}$,

$$f(\mathbf{y}) = \begin{pmatrix} f_1(x, y) \\ f_2(x, y) \end{pmatrix},$$

so the system is

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} f_1(x, y) \\ f_2(x, y) \end{pmatrix},$$

a system of two coupled ODE's, the equations coupled through the function f. That is, you have to know $x(t)$ to solve the $y(t)$ equation, and you have to know $y(t)$ to solve the $x(t)$ equation. Since they have to be solved together at once, existence of solution is a bit subtle.

–The basic existence theorems are *EXACTLY THE SAME* for systems, except you have to change the absolute value $|\cdot|$ to the Euclidean norm $\|\cdot\|$:

$$\|\mathbf{y}\| = \sqrt{x^2 + y^2}.$$

Here are the theorems...

Theorem 5. *Suppose f is Lipschitz continuous in $\mathbf{y}$. Then a unique solution $\mathbf{y}(t)$ of (8), (9) exists for all t.*

Definition 6. $f(\mathbf{y})$ *is* Lipschitz continuous *if there exists a constant K such that*

$$\|f(\mathbf{y}_2) - f(\mathbf{y}_1)\| \leq K\|\mathbf{y}_2 - \mathbf{y}_1\|, \tag{7}$$

for all $\mathbf{y}_1, \mathbf{y}_2 \in \mathcal{R}$.

But again, solutions of nonlinear ODE's don't always exist for all t, as the Ricotti system $\dot{x} = x^2$, $\dot{y} = y^2$ tells us, so the general result is again a short time *local* existence theorem:

Theorem 7. *Suppose* $f(\mathbf{y})$ *is* Lipschitz continuous *in some interval containing the initial data* $\mathbf{y}_0$*. Then a unique solution* $\mathbf{y}(t)$ *of (6) exists in some interval* $t \in (t_0 - \epsilon, t_0 + \epsilon)$*, satisfying*

$$\mathbf{y}'(t) = f(\mathbf{y}(\mathbf{t})), \tag{8}$$

$$\mathbf{y}(t_0) = \mathbf{y}_0. \tag{9}$$

By an argument similar to the proof of Theorem 8, the following sufficient condition for existence and uniqueness of solution holds.

Theorem 8. *If* $f(\mathbf{y})$ *is continuously differentiable, then a unique local solution* $\mathbf{y}(t)$ *exists for every* $\mathbf{y}_0$*.*

–Recall that in the last section our PDE application for the existence and uniqueness theorem (7) was that, if the (ivp) for an autonomous 2×2 system has a unique solution, then shock waves cannot form when the transport equation is linear. That is, recall that shock waves form when two characteristic curves intersect. In the general linear transport equation, the characteristic curves along which the solutions of the linear PDE $a(x,y)u_x + b(x,y)u_y = 0$ are constant, solve the autonomous ODE

$$\dot{\mathbf{y}} = f(\mathbf{y}),$$

where

$$f(x,y) = \begin{pmatrix} a(x,y) \\ b(x,y) \end{pmatrix}.$$

(We changed from (x,t) to (x,y) to fit the notation of Theorem 7.) Now we showed that because we can replace t with $t-t_0$ in an autonomous system, if two solution cross, then we can shift the time so that they cross at the same time, and so there would be two solutions with the same initial value (9). We conclude then from Theorem 7 that if $(a(x,y),b(x,y)$ is Lipschitz continuous in an open ball around every point $\mathbf{y_0}\in\mathcal{R}$, (a very mild condition!), then the solution of the initial value problem is *unique* at every initial value $\mathbf{y}_0$, and hence characteristics cannot cross at any $\mathbf{y}_0$. By Theorem 8, we need only verify that a and b are continuously differentiable to rule out the shock waves.

Conclusion: By Theorem 7, shock waves are a *nonlinear* phenomenon.

–In fact, Theorems 5, 7, hold for systems of arbitrary size, say $\mathbf{y}(t)=(y_1(t),...,y_n(t))\in\mathcal{R}^n$ with

$$\|\mathbf{y}\|=\sqrt{y_1^2+\cdots+y_n^2}\,,$$

so you only have to remember these two theorems, and they apply to all systems of ODE's, of all orders.

–No such unifying theory exists for PDE's!