4-THE LINEAR WAVE EQUATION
MATH 22C

In previous sections we considered the linear advection equation

$$u_t + cu_x = 0,$$

the equation that describes wave propagation to the right at speed c. The point is that

$$u(x,t) = f(x - ct)$$

solves the equation because

$$u_t = -cf'(x - ct) \quad and \quad u_x = f'(x - ct),$$

so

$$u_t + cu_x = -cf'(x - ct) + f'(x - ct) = 0;$$

and this represents wave propagation to the right because

$$u = f(x - ct) = const.$$

when the input of f is constant,

$$x - ct = const,$$

i.e., along straight lines of speed

$$dx/dt = c.$$

The reason we know that every solution of the equation

$$u_t + cu_x = 0$$

is a right moving wave of form

$$u(x,t) = f(x - ct)$$

for some function f is because we know the initial value problem

$$u_t + cu_x = 0$$
$$u(x,0) = f(x)$$

has a unique solution, and we just showed

$$u(x,t) = f(x - ct)$$

solves (1), so this must be it! (We haven't actually proven that the initial value problem has a unique solution, but being first order, there should be one initial condition just like a first order ODE. We'll return to the initial value problem later.)

Similarly, the equation

$$u_t - cu_x = 0,$$

is the equation that describes wave propagation to the left at speed c because

$$u(x,t) = f(x + ct)$$

solves the equation because

$$u_t = cf'(x + ct) \;\; and \;\; u_x = f'(x + ct)$$

so

$$u_t - cu_x = cf'(x - ct) - f'(x - ct) = 0,$$

and this represents wave propagation to the left because

$$u = f(x + ct) = const.$$

when

$$x + ct = const,$$

i.e., along straight lines of speed

$$dx/dt = c.$$

• Now the speed of sound in air is $c \approx 1100ft/s$. approximately constant, but real sound wave propagation, say along a long shock tube of gas, involves wave propagation in both directions—to the right *and* to the left. The simplest equation that admits the simultaneous propagation of waves both to the right and to the left at sound speed c is the wave equation

$$u_{tt} - c^2 u_{xx} = 0.$$

Lets see that this is true.

–Consider first a wave $u(x,t) = f(x - ct)$ moving to the right at speed c. (You can think of $f(x-ct) = sin(x-ct)$, a sinusoidal oscillation moving to the right at speed $dx/dt = c$). In this case

$$u_t = -cf'(x - ct), \qquad u_{tt} = (-c)^2 f''(x - ct),$$

$$u_x = f'(x - ct), \qquad u_{xx} = f''(x - ct),$$

so

$$u_{tt} - c^2 u_{xx} = c^2 f''(x - ct) - c^2 f''(x - ct) = 0.$$

Conclude: Right going waves $u = f(x - ct)$ solve the wave equation.

–Consider then a wave $u(x,t) = f(x+ct)$ moving to the left at speed c. In this case, since we differentiate with respect to t twice, the answer is the same,

$$u_{tt} = c^2 f''(x + ct), \qquad u_{xx} = f''(x - ct),$$

so again

$$u_{tt} - c^2 u_{xx} = c^2 f''(x - ct) - c^2 f''(x - ct) = 0.$$

Sometimes this is explained as follows. View

$$u_t + cu_x = 0$$

as the *differential operator* operating on the function $u(x,t)$ to get zero,

$$\left\{\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right\} u(x,t) = \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0,$$

so that the operator is

$$\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}.$$

Thus we can view the wave equation as the product of the left wave operator and the right wave operator, namely

$$u_{tt} - c^2 u_{xx} = 0$$

can be written as

$$\left\{\frac{\partial^2}{\partial t^2} - c\frac{\partial^2}{\partial x^2}\right\} u(x,t) = \frac{\partial^2 u}{\partial t^2} - c\frac{\partial^2 u}{\partial x^2} = 0,$$

which factors (like a difference between two squares) into

$$\left\{\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right\}\left\{\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right\} u(x,t) = 0.$$

This uses that mixed partial derivatives commute so the middle term cancels, and so we can also write it as

$$\left\{\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right\}\left\{\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right\} u(x,t) = 0.$$

Thus to see that the right going waves $u = f(x - ct)$ solve the wave equation, we use the first one

$$u_{tt} + c^2 u_{xx} = \left\{\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right\}\left\{\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right\} f(x - ct) = 0,$$

which is zero because the right going advection equation operates first; and to see the left going waves $u = f(x + ct)$ solve the wave equation, we use the second one

$$u_{tt} + c^2 u_{xx} = \left\{\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right\}\left\{\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right\} f(x + ct) = 0,$$

which is zero because the left going advection equation operates first. In this sense, we can see that the wave equation admits both left and right going waves because it is the simplest equation that contains the left and right going advection equations as factors.

• **Conclude:** The wave equation is the simplest equation that propagates waves in *both* directions. In fact, this holds more generally. In 3-dimensions, the wave equation is

$$u_{tt} + c^2 (u_{xx} + u_{yy} + u_{zz}) = 0.$$

Using the notation that the Laplacian operator Δ is defined as the sum of second partials,

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2},$$

the wave equation is usually written

$$u_{tt} + c^2 \Delta u = 0.$$

Turns out, the wave equation $u_{tt} + c^2 \Delta u = 0$ is the simplest equation such that solutions *propagate in every direction at speed* c. It's not so easy to see this in 2- and 3-dimensions, but we can see it all clearly in one dimension, the case when $u_{tt} - c^2 u_{xx} = 0$. This is an excellent model for ("linear"="weak") sound wave propagation in a 1-d shock tube.

For the wave equation in one dimension, we can prove the following:

Theorem 1. *Any smooth solution $u(x,t)$ of the wave equation*

$$u_{tt} - c^2 u_{xx} = 0,$$

is the sum of left and right going waves propagating at speed c. That is,

$$u(x,t) = u_1(x,t) + u_2(x,t),$$

where

$$u_1(x,t) = f(x + ct),$$

and

$$u_2(x,t) = g(x - ct).$$

It is instructive to give the

Proof: First note that the wave equation

$$u_{tt} - c^2 u_{xx} = 0$$

is *linear* and *homogeneous* (i.e., *linear* because it is a sum of terms of the form $a(x,t)u^{(k)}$, and *homogeneous* because $u = 0$ is a solution, that is, there is no inhomogeneous term of form $b(x,t)$ not multiplied by u or a derivative of u.) Thus the principle of superposition holds: sums and multiples of solutions are also solutions, meaning the solution space is a *Vector Space*. Thus, let u_1 and u_2 be arbitrary left and right going waves,

$$u_1(x,t) = f(x+ct),$$

and

$$u_2(x,t) = g(x-ct).$$

Since we have shown that each of these separately solves the wave equation, we know by superposition that the sum

$$u(x,t) = f(x+ct) + g(x-ct),$$

must also be a solution of the wave equation. I.e.,

$$u_{tt} = c^2 f''(x+ct) + c^2 g''(x-ct),$$

and

$$u_{xx} = f''(x+ct) + g''(x-ct),$$

so

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= c^2 \left\{ f''(x+ct) + g''(x-ct) \right\} \\ &\quad - c^2 \left\{ f''(x+ct) + g''(x-ct) \right\} = 0. \end{aligned}$$

Conclude: The sum of a left going wave and a right going wave solves the wave equation. We now prove the hard part, that conversely, every solution of the wave equation takes the form of a left going plus a right going wave. That is, we prove that if $u(x,t)$ solves the wave equation $u_{tt} - c^2 u_{xx} = 0$, then there exist functions $f(x+ct)$ and $g(x-ct)$ such that

$$u(x,t) = f(x+ct) + g(x-ct).$$

Here's the strategy. We use that fact that the initial value problem

$$u_{tt} + c^2 u_{xx} = 0 \tag{1}$$
$$u(x,0) = h(x) \tag{2}$$
$$u_t(x,0) = k(x) \tag{3}$$

has a unique solution. The wave equation is a *second* order equation, so just as second order ODE's like $\ddot{y} + a^2 y = 0$ requires two initial conditions $y(0)$ and $\dot{y}(0)$, so also the second order wave equation $u_{tt} + c^2 u_{xx} = 0$ requires two initial conditions $u(x,0)$ and $u_t(x,0)$. (In Chapter 6 we'll return later to prove existence and uniqueness of solutions of the initial value problem for the wave equations by means of the *Energy Method*.) Assuming this, if for any given initial data $h(x)$ and $k(x)$ we can find function f and g such that

$$u(x,t) = f(x+ct) + g(x-ct)$$

satisfies (2) and (3), then we know this *must* be the one and only solution of the initial value problem (1), (2), (3); hence every solution must be of this form.

So all we need is to find f and g such that

$$u(x,0) = f(x) + g(x) = h(x), \tag{4}$$
$$u_t(x,0) = cf'(x) - cg'(x) = k(x). \tag{5}$$

Differentiating (6), and multiplying (7) by $1/c$ gives,

$$f'(x) + g'(x) = h'(x), \tag{6}$$
$$f'(x) - g'(x) = \tfrac{1}{c}k(x). \tag{7}$$

which provides two equations in the two unknowns f' and g', which we can solve by taking (6)+(7) and (6)-(7),

$$2f'(x) = h'(x) + \tfrac{1}{c}k(x), \tag{8}$$
$$2g'(x) = h'(x) - \tfrac{1}{c}k(x). \tag{9}$$

We obtain f and g by integrating (8), (9):

$$\begin{aligned} f(x) &= \frac{1}{2}\int_0^x \left[h'(\xi) + \frac{1}{c}k(\xi)\right] d\xi + f(0), \\ &= \frac{1}{2}h(x) + \frac{1}{2c}\int_0^x k(\xi)d\xi - \frac{1}{2}h(0) + f(0), \end{aligned} \tag{10}$$

$$\begin{aligned} g(x) &= \frac{1}{2}\int_0^x \left[h'(\xi) - \frac{1}{c}k(\xi)\right] d\xi + g(0) \\ &= \frac{1}{2}h(x) - \frac{1}{2c}\int_0^x k(\xi)d\xi - \frac{1}{2}h(0) + g(0). \end{aligned} \tag{11}$$

Now f and g as defined by (10), (11) meet the initial condition (7) by construction, but to meet (6) we need

$$f(x) + g(x) = h(x) - h(0) + f(0) + g(0). \tag{12}$$

must equal $h(x)$, so we need the final condition

$$f(0) + g(0) = h(0).$$

Note that f and g are only determined to within a constant because the decomposition of $u(x,t)$ into right and left moving waves is only determined to within a constant, i.e.,

$$\begin{aligned} u(x,t) &= f(x+ct) + g(x-ct) \\ &= \{f(x+ct) + C\} + \{g(x-ct) - C\}, \end{aligned}$$

the free constant C being arbitrary.

Conclude: Since the initial value problem

$$u_{tt} + c^2 u_{xx} = 0$$
$$u(x, 0) = h(x)$$
$$u_t(x, 0) = k(x)$$

has a *unique* solution, and we can always solve this with the sum of a right and a left moving wave, we *must* have that the general solution looks like

$$u(x, t) = f(x + ct) + g(x - ct),$$

the sum of a left and a right going wave of speed c. □

• **Summary:** The wave equation is the simplest equation that imposes everything propagates at speed c. Our next project is to show that when you *linearize* the compressible Euler equations (Newton's laws for sound waves neglecting friction) about the rest point $\rho = \bar{\rho}$, $v = 0$, you get the wave equation. Thus, small amplitude solutions propagate with a constant speed, the sound speed. We will use this to determine the speed of sound.

In a second topic, we will discuss Maxwell's equation for the propagation of electric and magnetic fields, and show that each component solves the wave equation with a speed that can be calculated to be the speed of light. By this route, Maxwell conjectured that light was really electromagnetic radiation, a proposal that remained controversial until Heinrich Hertz generated electromagnetic radiation from spinning magnets some two decades later.

In a third topic, we will trace Einstein's discovery of special relativity. Einstein conjectured that Maxwell's equation, and hence the wave equation, should be correct in all inertial coordinate systems. That is, electromagnetic waves solve the wave equation, but he conjectured that, unlike the compressible Euler equations, there is no background density ρ_0, no *ether* medium, for the waves to propagate through. Thus he proposed that spacetime itself must be resonantly tuned to the electromagnetic waves in the sense that the *inertial* coordinate systems are not related by the Galilean transformations that preserve Newton's laws, but rather by the transformation that preserve the wave equation. Asking that the wave equation $u_{tt} - c^2 u_{xx} = 0$ be the same in every inertial coordinate system is the same as asking that the *speed of light* c be the same in each inertial coordinate system. The spacetime transformations that leave the wave equation invariant are the Lorentz transformations of special relativity, and Einstein revolutionized physics by proposing that these replace Galilean transformations as the fundamental coordinate transformations that preserve the physics. All of this fundamental physics is based on the *wave equation*!