# 5-LINEARIZING EQUATIONS ABOUT REST POINTS:

The Speed of Sound, the Speed of Light
and
The Fundamental Difference Between Them

**MATH 22C**

## 1. Points of equilibrium=rest points

• *An equilibrium point* or *rest point* of an equation is a constant state solution of the equation. At the constant state, "all forces are in balance" in the sense that the dynamics imposed by the equation keep the state constant.

– Consider first the case of a general *autonomous* first order system of ODE's, ($f = f(\bar{\mathbf{u}})$ depends on $t$ only through the unknown $\mathbf{u}$),

$$\mathbf{u}' = f(\mathbf{u}), \tag{1}$$

where $\mathbf{u} = (u_1, ..., u_n)$ can be a vector with any number of components. Thus a rest point $\bar{\mathbf{u}}$ is a constant state solution where $f(\bar{\mathbf{u}}) = 0$ for all $t$, so that $\bar{\mathbf{u}}' = 0$.

**Definition 1.** *A* **rest point** *or* **equilibrium point** *of system (1) is a constant state* $\bar{\mathbf{u}}$ *satisfying*

$$f(\bar{\mathbf{u}}) = 0. \tag{2}$$

• For example, if the ODE describes the dynamics of the populations of some animal species living on an island, $u_k$=population of species $k$, then the equilibrium state $\bar{\mathbf{u}}$ is a point of stable populations. If you perturb the state, then $f(\mathbf{u})$ becomes nonzero, so $\mathbf{u}'(t)$ becomes nonzero, and the populations drift away from "perfect balance" $\bar{\mathbf{u}}$. The biggest question then is, once perturbed, will the populations return to $\bar{\mathbf{u}}$, or will they drift away toward another equilbrium point $\hat{\mathbf{u}}$?

**Definition 2.** *If the equations return any sufficiently small perturbation* $\bar{\mathbf{u}}$ *back to* $\bar{\mathbf{u}}$*, then we say the* $\bar{\mathbf{u}}$ *is a stable equilibrium point. If some arbitrarily small peturbation makes it drift away (perhaps toward another equilibrium point), then we say* $\bar{\mathbf{u}}$ *is an unstable equilibrium point.*

–The notion of stable vs unstable equilibria is an important concept for understanding phenomena in nature even without doing any mathematics at all. For example, consider the following explanation:

"*In 2006, the economy was pretty stable with a steady 4.5 percent unemployment rate. The banking crisis disturbed the equilibrium point, and the economy drifted off toward another equilibrium point where the unemployment rate was 9.5 percent. We do not know whether lowering interest rates and quantitative easing will get us back to the old equilibrium point.*"

The presumption in this statement is that the unemployment rate $r$ is evolving according to some really complicated ODE $\mathbf{u}' = f(\mathbf{u})$ with $r$ one of the variables $u_k = r$, and that we perturbed one equilibrium point $\bar{\mathbf{u}}$ with $\bar{u}_k = r = 4.5$, and the ODE's evolved to another one $\hat{\mathbf{u}}$ with $\hat{u}_k = r = 9.5$. The last statement then translates into *We do not know whether changing the function* $f$ *[by lowering interest rates and quantitative easing] to* $\mathbf{u}' = f(\mathbf{u}) + g(\mathbf{u})$ *will produce a new ODE whose solutions will take us back to the old equilibrium point.*

This paradigm pervades the modern way of thinking about how things work.

• The formal method for determining the stability of a rest point of an ODE is to "linearize" the equations about the rest point, and then study the stability of the linearized equations.

## 2. Linearizing a nonlinear ODE at a rest point

• The formal method for linearizing a nonlinear ODE at a rest point $\bar{\mathbf{u}}$ is as follows:

**(1)** Start with constant state $\bar{\mathbf{u}}$ such that $f(\bar{\mathbf{u}}) = 0$ and the ODE is solved by

$$0 = \bar{\mathbf{u}}' = f(\bar{\mathbf{u}}) = 0.$$

**(2)** Write

$$\mathbf{u}(t) = \bar{\mathbf{u}} + \delta \mathbf{v}(t),$$

and look for the equation $\mathbf{v}(t)$ solves in the limit $\delta \to 0$.

**(3)** Plug $\mathbf{u}(t) = \bar{\mathbf{u}} + \delta \mathbf{v}(t)$ into the ODE for $\mathbf{u}$.

**(4)** Taylor expand all nonlinear functions about $\bar{\mathbf{u}}$, and multiply everything out.

**(5)** Throw away all terms of order $O(\delta^2)$ and higher, [that is, any term multiplied by $\delta^2, \delta^3, ...$]

**(6)** What's left is a linear equation for perturbation $\mathbf{v}(t)$. This equation is called the "linearization of the ODE about rest point $\bar{\mathbf{u}}$."

• The method works for ODE's of any size, and you can linearize around any known solution $\mathbf{u}(t)$ of $\mathbf{u}' = f(\mathbf{u})$, not just constant state solutions $\mathbf{u}(t) = \bar{\mathbf{u}} = const.$—it just gets more complicated. By the same procedure you can also linearize around both constant state and more general solutions of PDE's, but for PDE's $\mathbf{v}(\mathbf{x}, t)$ will depend on $\mathbf{x}$ and $t$, and the linearized equation will be a PDE.

• So far we looked at the two linear ODE's $y' = ky$, $y'' + a^2 y = 0$ and the nonlinear equation $y' = y^2$.

For examples, we now linearize $y' = f(y)$ (the general case of $y' = y^2$) and $y'' + \frac{g}{L}\sin y = 0$ about their rest points, and determine their stability.

## 3. The case of first order scalar equations $y' = f(y)$

• A rest point $y = \bar{y} \in \mathcal{R}$ of the autonomous first order scalar equation $y' = f(y)$, is a point where $f(\bar{y}) = 0$.

**Example 1:** Assume the equation is: $y' = y^2 - 1$.

• The nonlinear function is $f(y) = y^2 - 1$, so the rest points satisfy $y^2 - 1 = 0$, and so $\bar{y} = -1, 1$.

–Linearizing about $\bar{y} = 1$, we plug

$$y(t) = 1 + \delta v(t)$$

into the equation

$$y' = y^2 - 1$$

to obtain

$$\begin{aligned}(1 + \delta v(t))' &= f(1 + \delta v(t)) = (1 + \delta v(t))^2 - 1 = 1 + 2\delta v(t) + \delta^2 v(t) - 1 \\ &= 2\delta v(t) + \delta^2 v(t).\end{aligned}$$

–**Note** that the constant term cancels out (this always happens because $\bar{y}$ solves the equations exactly!), and what is left is an order $\delta$ term and an order $\delta^2$ term.

– Following step **(5)**, we then throw away all terms higher order than $O(\delta)$, and obtain

$$\delta v' = 2\delta v(t),$$

which after canceling the $\delta$ on both sides gives

$$v' = 2v(t). \tag{3}$$

The resulting equation (3) is called "the equation obtained by linearizing the nonlinear ODE $y' = y^2 - 1$ about the rest point $\bar{y} = 1$."

– The linear equation (3) is our fundamental equation for the exponential $v' = kv$ with $k = 2$, so it has the solution

$$v(t) = v_0 e^{2t}.$$

We conclude that, sufficiently close to the rest point $\bar{y} = 1$, (that is, neglecting errors of order $\delta^2 v(t)^2$), solutions of the nonlinear ODE evolve like

$$y(t) = \bar{y} + \delta v(t) = 2 + \delta v_0 e^{2t}.$$

In particular, since $k = 2 > 0$, the exponential grows in forward time, so solutions move *away* from $\bar{y} = 1$, and we conclude that $\bar{y} = 1$ is an **unstable** rest point.

• Linearizing about the other rest point $\bar{y} = -1$ of ODE $y' = y^2 - 1$, we plug

$$y(t) = -1 + \delta v(t)$$

into the equation

$$y' = y^2 - 1$$

to obtain

$$\begin{aligned}(-1 + \delta v(t))' &= f(1 + \delta v(t)) \\ &= (-1 + \delta v(t))^2 - 1 \\ &= 1 - 2\delta v(t) + \delta^2 v(t) - 1 \\ &= -2\delta v(t) + \delta^2 v(t).\end{aligned}$$

–**Note again** that the constant term cancels out (because $\bar{y}$ solves the equations exactly), and what is left is an order $\delta$ term and an order $\delta^2$ term. In math analysis we refer to a term of order $\delta^n$ as $O(\delta^n)$ or $O(1)\delta^n$, read aloud as *Big Oh of* $\delta^n$, and "*Big Oh of One Times* $\delta^n$", respectively. In particular $O(1)$ denotes an error bounded by a constant as $\delta \to 0$, so the error notation $O(1)\delta^n$ is convenient because then we can *cancel* powers of $\delta$ when they appear in every term of the equation.

– Following step **(5)**, we throw away all terms higher order than $O(\delta)$, and obtain

$$\delta v' = -2\delta v(t),$$

which after canceling the $\delta$ on both sides gives

$$v' = -2v(t). \tag{4}$$

The resulting equation (4) is "the equation obtained by linearizing the nonlinear ODE $y' = y^2 - 1$ about the rest point $\bar{y} = -1$."

– The linear equation (4) is our fundamental equation $v' = kv$ for the exponential but with $k = -2$, so it has the solution

$$v(t) = v_0 e^{-2t}.$$

We conclude that, sufficiently close to the rest point $\bar{y} = -1$, (neglecting errors of order $\delta^2 v(t)$), solutions of the nonlinear ODE evolve like

$$y(t) = \bar{y} + \delta v(t) = 2 + \delta v_0 e^{-2t},$$

and since $k = -2 < 0$, the exponential *decreases* in forward time, so solutions move *back toward* the rest point $\bar{y} = -1$ in forward time, and we conclude that $\bar{y} = -1$ is a **stable** rest point.

• From the example, the general picture emerges: If we linearize $y' = f(y)$ about a rest point $\bar{y}$ where $f(\bar{y}) = 0$, we set $y(t) = \bar{y} + \delta v(t)$ and get a linearized equation for $v(t)$...but since this linear equation is scalar, the only first order linear equation we get is $v' = kv$ for some $k$ (no $t$ term because $f(y)$ doesn't depend on $t$!), so the sign of $k$ tells us the stability of the rest point!

• For example, consider the ODE $y' = f(y)$ where $f$ is the nonlinear function with the following graph:

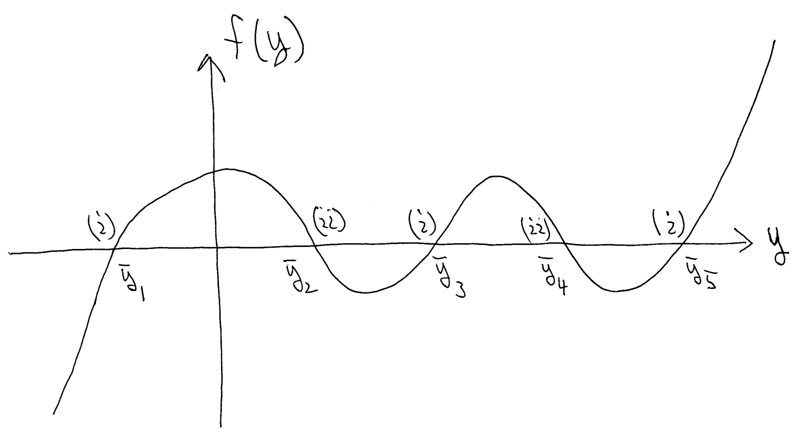


FIGURE 1. A nonlinear function with five rest points.

– Then (generically) the rest points come in two kinds: (i) Where the graph crosses the $x$-axis going **up** from left to

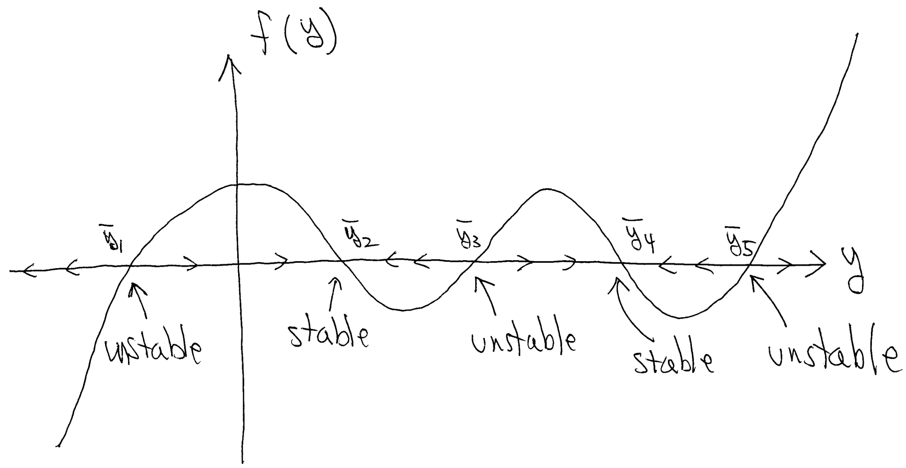


FIGURE 2. The phase portrait for a nonlinear function with five rest points.

right; (ii) Where the graph crosses the $x$-axis going **down** from left to right.

– Now using Taylor's theorem to expand $f(\bar{y}+\delta v(t))$ about rest point $\bar{y}$, we obtain

$$f(\bar{y}+\delta v(t)) = f(\bar{y}) + f'(\bar{y})\delta v(t) + O(\delta^2 v(t)^2).$$

Using this we can obtain a general formula for the linearization of the ODE $y' = f(y)$ about a rest point $\bar{y}$. Namely, substituting $\bar{y}+\delta v(t)$ for $y$ in the ODE $y' = f(y)$ gives

$$(\bar{y}+\delta v(t))' = f(\bar{y}+\delta v(t)) = f(\bar{y}) + f'(\bar{y})\delta v(t) + O(\delta^2 v(t)^2).$$

Using that $f(\bar{y}) = 0$ and throwing away the higher order terms in $\delta$, we obtain the linearized equation

$$v' = f'(\bar{y})\, v,$$

which is **stable** if

$$f'(\bar{y}) < 0, \qquad \text{[case(i)]},$$

and **unstable** if

$$f'(\bar{y}) > 0, \qquad \text{[case(ii)]}.$$

– The stability result above is confirmed by phase diagram for solutions of $y' = f(y)$ in Figure 2. The point is that the sign of the derivative at a rest point is determined by the sign of $f$ on either side of the rest point, and so we can graph the directions of solution curves without having to know any exact formulas for solutions. We have proven the following theorem:

**Theorem 3.** *At a non-degenerate rest point $\bar{y}$ of an autonomous scalar ODE $y' = f(y)$, (a value of $\bar{y}$ where $f(\bar{y} = 0$ but $f'(\bar{y}) \neq 0$), the solution enters or leaves the rest point* exponentially*. That is, $y(t) = \bar{y} + \delta v(t) + O(1)\delta^2$ where*

$$v(t) = v_0 e^{f'(\bar{y})t}.$$

## 4. Linearizing about a rest point of the nonlinear pendulum

• For a nonlinear $2 \times 2$ example, we consider the nonlinear pendulum. To derive the equations from Newton's laws, consider a mass $m$ swinging on a massless frictionless pendulum of length $L$. Balancing the forces leads to the nonlinear equation ,

$$\ddot{\theta} + \frac{g}{L}\sin\theta = 0. \tag{5}$$

Our purpose here is simply to describe the linearization process at rest points, so we refer to Strogatz for a complete derivation and phase portrait description of solutions of the nonlinear pendulum.

Now $\bar{\theta} = 0$ is a rest point, so to find the behavior of solutions near $\theta = 0$ we linearize the equations about the rest point. For this, let $\theta(t) = \bar{\theta} + \delta v(t)$, plug this into (5), expand in powers of $\delta$ using Taylors theorem, collect the $O(1)\delta$ term, and throw higher order terms away, to recover a linear equation for the perturbation $v(t)$. Thus,

$$\begin{aligned} 0 &= \left\{\bar{\theta} + \delta v\right\}'' + \frac{g}{L}\sin\left\{\bar{\theta} + \delta v\right\} \\ &= \delta\ddot{v} + \frac{g}{L}\sin\delta v \\ &= \delta\ddot{v} + \frac{g}{L}\delta v + O(1)\delta^3 \end{aligned}$$

where we used that $\bar{\theta} = 0$ together with the Taylor expansion

$$\sin\delta v = \delta v - \frac{1}{6}(\delta v)^3 + \cdots .$$

Throwing away higher order terms and canceling the remaining $\delta$ yields the linearized equations for the perturbation $v$:

$$\ddot{v} + \frac{g}{L}v = 0. \tag{6}$$

But this is our harmonic oscillator $\ddot{y} + a^2 y = 0$ with $y \equiv v$ and $a^2 = g/L$. Thus solutions of the linearized equations are the sines and cosines

$$v(t) = A\sin\sqrt{\frac{g}{L}}t + B\cos\sqrt{\frac{g}{L}}t.$$

From this we can conclude that for small perturbations from the rest point $\bar{\theta} = 0$, the solutions look like

$$\theta(t) = v(t),$$

which are sinusoidal oscillations of period $2\pi\sqrt{\frac{L}{g}}$. In particular, at the linearized level, the oscillations are independent of the mass and the amplitude, thereby explaining why pendulum clocks were so popular *in the old days!*

The second order nonlinear pendulum (5) can be written as a first order nonlinear autonomous system

$$\dot{\mathbf{y}} = f(\mathbf{y}),$$

where $\mathbf{y} = (x, y) \equiv (\theta, \dot{\theta})$ and

$$f(\mathbf{y}) = \begin{pmatrix} y \\ -\sqrt{\frac{g}{L}}\sin x \end{pmatrix} = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix}.$$

For such systems we already know that two solutions cannot cross in the $(x, y)$-plane, so this means that the structure of solutions is determined by their structure at the rest points, and this is given by the linearized equations at the rest point. Working generally, at a rest point $\bar{y} = (\bar{x}, \bar{y})$, $f(\bar{y}) = 0$, which is the vector equivalent of $f_1(\bar{x}, \bar{y}) = 0 = f_2(\bar{x}, \bar{y})$, and to linearize we set $\mathbf{y}(t) = \bar{\mathbf{y}} + \delta\mathbf{v}(t)$ which is the vector form of

$$y(t) = \bar{y} + \delta v_1(t), \tag{7}$$

$$x(t) = \bar{x} + \delta v_2(t). \tag{8}$$

The linearized equations at rest point $\bar{\mathbf{y}}$ are always of the constant coefficient homogeneous form

$$\dot{v} = Av,$$

where $A$ is the $2\times 2$ matrix $A$ whose first row is $\nabla f_1(\bar{\mathbf{y}})$, and second row $\nabla f_2(\mathbf{y})$. For completeness, the phase portrait of solutions of the nonlinear pendulum is given in Figure 3.

In general, the elementary solutions of the linearized equations $\dot{v} = Av$ are given by $v(t) = Re^{\lambda t}$ where $(R, \lambda)$ is an eigen-pair of $A$, $(AR = \lambda R)$, because these solve $\dot{v} = Av$

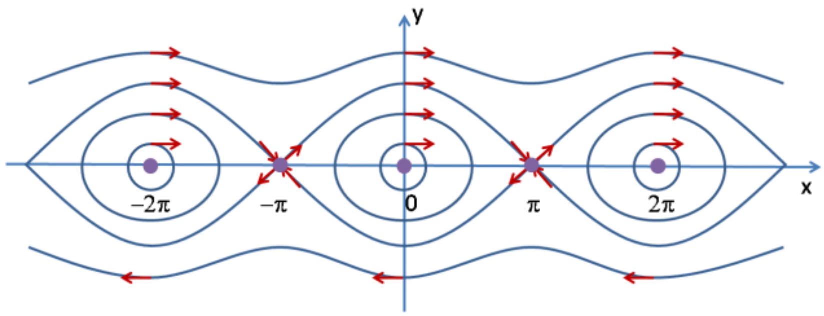


FIGURE 3. Phase portrait of solutions of the nonlinear pendulum in the $(\theta, \dot{\theta})$-plane.

*exactly*. It follows then, that the stability of a rest point $\bar{\mathbf{y}}$ is determined by the real part $a$ of the eigenvalues $\lambda = a + ib$. Indeed, a rest point is stable (in the sense that every small perturbation of the rest point returns the solution to the rest point in positive time) if and only if the real part of every eigenvalue is negative, so that the exponential in $Re^{(a+ib)t}$ decays to zero as $t \to 0$. In particular, we see from this that in *general*, solutions near rest points of ODE's evolve (to leading order in $\delta$) according to the exponential function $e^{\lambda t}$. A systematic working out of this theory is a central topic of MAT119A.
These examples lead to the following definitions:

**Definition 4.** *A rest point is called asymptotically stable if sufficiently small perturbations of the rest point evolve back* *to the rest point. The rest point is called marginally stable or just stable if sufficiently small perturbations of the rest point stay small for all time. The rest point is unstable if there exists a small but fixed ball open ball centered at the rest point such that there exist arbitrarily small perturbations starting in that ball that leave the open ball in positive time.*

Our examples show that for scalar ODE's, the rest points $\bar{y}$ solving $\dot{y} = f(y)$ (so $f(\bar{y}) = 0$) are *asymptotically stable* when $f'(\bar{y}) < 0$, and *unstable* when $f'(\bar{y}) > 0$. On the other hand, the rest point at $\theta = 0, \dot{\theta} = 0$ is *marginally stable.* In fact, we cannot determine from the linearized equations that the rest point is marginally stable, because arbitrarily small errors of order $\delta^2$ could take the solutions away from $\theta = 0$. However, we can see from the phase portrait of the full nonlinear pendulum in Figure 3 that the rest point is actually marginally stable.

Linearizing partial differential equations (PDE's) goes the same way, except it is much more difficult to determine stability or instability of equilibrium points of PDE's because it is difficult to calculate the "eigenvalues". However, linearizing complicated PDE's is one of the most important ways of understanding what they mean!

## 5. Linearizing PDE's

• Consider first the nonlinear inviscid Burgers equation, the case of the transport equation when the sound speed $c$ depends on the solution in the simplest way: Namely, $c = u$,

$$u_t + uu_x = 0.$$

Again, this says that $u$ is constant along (straight) lines of speed $u$—that is, the speed is the same as the value of the solution along the characteristic curve. When the density of a gas, say air in a shock tube, is given by $\rho(x,t) = u(x,t)$ then asking that $u$ be a solution of Burgers is the simplest model for the *nonlinear* effects in propagation of sound waves down the tube. In fact, the formation of shock waves represents the *breaking* of the sound waves, something like the breaking of waves in the ocean, and leads to the dissipation of the sound wave signal.

Now $u = \bar{u}$ solves the Burgers equations, (think of it as a constant density solution which propagates no waves). Let's now linearize the Burgers equation about the rest point $u = \bar{u}$ to see what solutions do to leading order when the constant state is perturbed by a small amount $\delta v$.

To find the equations for the perturbations $v$, we linearize Burgers equation $u_t + uu_x = 0$ about the constant state $\bar{u}$. The procedure follows exactly the steps **(1)-(6)** above, only this time $v = v(x,t)$ depends on $x$ and $t$, not just $t$, because Burgers is a PDE, not an ODE. So as before, set $u(x,t) = \bar{u} + \delta v(x,t)$, plug this into the Burgers equation, expand all nonlinear functions, throw away all terms of order $O(1)\delta^2$, and what is left is $\delta$ times the linearized equations for $v$.

So plugging $\bar{u} + \delta v$ into $0 = u_t + uu_x$ gives

$$\begin{aligned} 0 &= (\bar{u} + \delta v(x,t))_t + (\bar{u} + \delta v(x,t))(\bar{u} + \delta v(x,t))_x \\ &= \delta v_t + (\bar{u} + \delta v)\delta v_x \\ &= \delta(v_t + \bar{u}v_x) + \delta^2 v v_x. \end{aligned}$$

Neglecting the $O(1)\delta^2$ term and dividing by $\delta$ we are led to the linearized equations for the perturbation $v(x,t)$:

$$v_t + \bar{u}v_x = 0.$$

Since the constant state $\bar{u}$ is fixed, we recognize this a transport to the right with speed $\bar{u}$. This makes sense because the Burgurs equation give transport to the right at speed

$u$, so for solutions $u(x,t) \approx \bar{u}$, it should be transport to the right at approximately, (that is, to leading order in $\delta$), transport to the right at speed $\bar{u}$. The linearized equations confirm this.

The natural question to ask then is: Are the constant solutions of the Burgers equation stable? But the perturbations solve the transport equation which keep things constant along characteristics, so the perturbations do not decay back to the constant state. That is, we know the linearized equations keep small perturbations small, which indicates at the linearized level, the constant states are *marginally stable*. Thus the best we can expect is that the constant states of Burgers are marginally stable. To verify this, we'd have to go back to solutions of the nonlinear equations to check the nonlinear evolution of the perturbations. Since the nonlinear equations can create shock waves, the perturbations will only be defined up until the shocks form. In fact, at shocks, the derivatives of the solutions tend to infinity, but the solution still keeps its values near the constant state $\bar{u}$. Thus whether you want to call the constant state "nonlinearly stable" depends on whether you include the derivative of the solution in your stability criterion. We say the stability depends on the norm in which you measure the difference! This makes PDE's a lot harder then ODEs!

• Consider next the problem of linearizing around a general, possibly non-constant solution $\bar{u}(x,t)$ of Burgers equation. For this, assume $\bar{u}(x,t)$ is a known solution that exactly solves

$$\bar{u}_t + \bar{u}\bar{u}_x = 0.$$

We ask: how do small perturbations of $\bar{u}(x,t)$ propagate, to leading order in $\delta$? To find the leading order linear equations for the perturbation $v(x,t)$, write

$$u(x,t) = \bar{u}(x,t) + \delta v(x,t),$$

plug this into Burgers equation $u_t + uu_x = 0$ and obtain

$$\begin{aligned} 0 &= (\bar{u}(x,t) + \delta v(x,t))_t + (\bar{u}(x,t) + \delta v(x,t))(\bar{u}(x,t) + \delta v(x,t))_x \\ &= \bar{u}_t + \delta v_t + (\bar{u} + \delta v)(\bar{u}_x + \delta v_x \\ &= \bar{u}_t + \bar{u}\bar{u}_x + \delta v_t + \delta \bar{u}_x v + \delta \bar{u} v_t + \delta^2 v v_x. \end{aligned}$$

Now the zero order terms in $\delta$ vanish, namely $\bar{u}_t + \bar{u}\bar{u}_x = 0$, because $\bar{u}(x,t)$ solves Burgers exactly, (the reason we always linearize around *solutions* is so that the zero order terms in $\delta$ will vanish!), so neglecting the $O(1)\delta^2$ term and dividing by $\delta$ we are led to the linearized equations for the perturbation $v(x,t)$ around a general solution $\bar{u}(x,t)$ of Burgers:

$$v_t + \bar{u} v_t + \bar{u}_x v = 0. \tag{9}$$

Now $\bar{u}$ and $\bar{u}_x$ are treated as known functions of $(x,t)$, (they are *linear* coefficients), so the equation is still linear homogeneous in the unknown function $v$, but the last term is new; its $\bar{u}_x(x,t)$=known function of $(x,t)$ times undifferentiated $v$, and is called a *zero order term* or *source term* because the unknown function $v$ comes in undifferentiated. This term only appears in the linearized equations when the solution $\bar{u}$ is non-constant.

• Consider now the equation (9). The first two terms we can understand. These describe a transport equation to the right with variable speed $c(x,t) = \bar{u}(x,t)$ a known function. The terms involving the derivative of $v$ still describe the *characteristics* of the PDE (9), since the derivatives provide the speed of propagation. So as before, the characteristic curves are curves moving at speed $c$ in the $(x,t)$ plane. Just as before, these curves are given by solutions of the ODE

$$\dot{x}(t) = \bar{u}(x,t) = c(x,t), \tag{10}$$

which can in principle be determined without knowing the solution $v$ because $\bar{u}(x,t)$ is the known function we are linearizing about. Just for fun, lets see if the solution $v(x,t)$ is constant along characteristics $(x(t),t)$ like the variable speed transport equation $u_t + c(x,t)u_x = 0$.

$$\frac{d}{dt}v(x(t),t) = v_t + v_x x'(t) = v_t + v_x \bar{u}(x,t) = -\bar{u}_x v. \quad (11)$$

That is, $v(x,t)$ is not constant along the characteristic because of the extra zero order term in the equation (9). So this time setting the speed of the curve $x'(t) = \bar{u}(x,t)$ doesn't make $v$ constant along solution curves. That is, when the source term present, asking that $v$ be constant along characteristics is too much to ask. But in fact, all we need ask is that the solution $v$ reduce to an ODE along such curves. In this case, since $v_t + v_x x'(t) = -\bar{u}_x v$, (11) gives us

$$\frac{d}{dt}v(x(t),t) = -\bar{u}(x(t),t)v(x(t),t). \quad (12)$$

Thus we can solve (10) by an ODE using the known function $\bar{u}(x,t)$, then, given $x(t)$, we can solve the ODE (12) for $v$ along the characteristic curve $(x(t),t)$. This gives us a procedure for solving the PDE by solving for its values along any characteristic curve, a process by which we only need solve ODE's! Putting this together, we could summarize: To solve the linear transport equation with source terms (9), solve the ODE's

$$\begin{pmatrix} x \\ v \end{pmatrix}' = \begin{pmatrix} \bar{u}(x,t) \\ -\bar{u}_x(x,t)v \end{pmatrix} = f(x,v,t),$$

which is a non-autonomous ODE for the unknowns $(x,v)$ whose solution provides the values of the unknown $v$ along characteristics.

The lesson, then, is that when a first order linear equation has zero order terms, it is too much to expect the solution to be constant along characteristic curves, but we can ask that the solution reduce to an ODE along the curve, ODE's being easier to solve than PDE's. By this procedure, the underlying characteristic curve $\dot{x} = c(x,t)$ is solved by an ODE using the *known* function $\bar{u}$, and then we can solve for the values of the unknown $v$ along any such characteristic by solving the ODE (12). In fact, the method of characteristics can be applied in the same way to general first order nonlinear PDE's.

**Conclude:** Linear PDE's are MovingCordinate=22C-Ch5obtained from nonlinear PDE's by *linearization*. You linearize around a given solution of the nonlinear equation, and the linearized equations describe the evolution of small changes from the given solution. Our example was the nonlinear Burgers equation. When we linearized around a constant state, we got the linear transport equation. When we linearized around a general solution of the Burgers equation, we got a general homogeneous first order linear equation. The linearized equations could be solved by the method of characteristics, namely, the characteristic curves are curves of speed $c = \bar{u}(x,t)$, and along these curves the PDE reduces to an ODE. Since characteristics for a linear equation never intersect in the $xt$-plane, and since the speed $\bar{u}(x,t)$ becomes discontinuous if the solution $\bar{u}$ (which we linearize around) develops a shock wave, the ODE's for the characteristics are no longer Lipschitz, $\bar{u}_x$ becomes infinite, and we can no longer expect the linearized equations for Burgers to approximate the nonlinear evolution after shock waves form.

We now give an important example in which the linearized equation is not a first order transport equation, but rather the second order linear wave equation.

## 6. Linearizing the Compressible Euler Equations and the Determination of the Speed of Sound.

• Recall that the equations that describe the fully nonlinear motion of a gas with friction modeled by viscosity are the Compressible Navier-Stokes equations

$$\rho_t + div(\rho\mathbf{u}) = 0,$$
$$(\rho u^i)_t + div(u_i\mathbf{u} + p\mathbf{e}_i) = +\mu\Delta u_i + \xi(div\mathbf{u})_{x_i}.$$

Here $\rho$ is the density=mass/volume, $\mathbf{u} = (u^1, u^2, u^3)$ is the velocity, $p$=pressure, and $\mu$ and $\xi$ are the shear and bulk viscosities. This is the fundamental model expressing *Newton's laws of motion for a continuous media, with friction.*

Note that the Navier-Stokes equations consist of four equations in the five unknown functions $\rho, u^1, u^2, u^3, p$, and so we must be given the *equation of state* $p = p(\rho)$ in order to *close the equations*, that is, have the same number of equations and unknowns. It turns out that it is not so difficult to derive these equations from first principles and see that they must be correct, but at first the equations themselves look impossibly complicated to understand. So we understand them by breaking them down into warmup problems.

As a first simplification we neglect the viscosity so the equations reduce to the Compressible Euler Equations

$$\rho_t + div(\rho\mathbf{u}) = 0,$$
$$(\rho u_i)_t + div(u_i\mathbf{u} + p\mathbf{e}_i) = 0,$$

four equations which describe Newton's laws for a continuous media *without friction.* This is the setting for the study of gas dynamics and the theory of shock waves.

As a second simplification, restrict to one dimensional motion along a shock tube. (There really are shock tubes, the largest about three feet in diameter, and in them scientists create one dimensional shock waves moving down the tube, and can compare them with theoretical predictions from the equations.)

$$\rho_t + (\rho u)_x = 0, \tag{13}$$

$$(\rho u)_t + (\rho u^2 + p(\rho))_x = 0. \tag{14}$$

Note that we have assumed we know the equation of state $p = p(\rho)$, that is, we know $p$ as a known function of density $\rho$.[1] Note that the first equation would be a Burgers type transport equation if $u$ in the first equation were $\rho$, but then this would mean transport in one direction, so we know it isn't right as a gas sends sound waves in both directions. So the coupling of the first equation to the second through $u$ must make it more complicated.

We next ask what the linearized theory of sound should be by asking for the equations that describe the perturbations from still air to leading order in $\delta$, i.e., perturbations from a constant density $\rho_0$, assuming we observe from a location fixed with respect to the fluid particles, so that $u_0 = 0$. Note that the velocity $u$ measures the velocity of a speck of dust (a *fluid particle*), as it is carried through the fluid, and this *does not* measure the sound speed, the speed of *wave* propagation through the gas.

So we follow our usual procedure for linearization about the constant state $\rho = \rho_0$, $u = u_0 = 0$. Set $\rho = \rho_0 + \delta\hat{\rho}$,

[1] In reality, the pressure depends on the temperature as well, so we are assuming constant temperature here. For the case of an ideal gas of $n$-molecules, the equation of state can be derived from first principles, and it's called the *polytropic equation of state*. This is pretty much the only case where the full equation of state can be determined exactly from first principles. In other cases we can view the equation of state as given to us by experiments.

$u = u_0 + \delta\hat{u} = \delta\hat{u}$, plug these into the compressible Euler equations (13), (14), multiply everything out, throw away terms of order $\delta^2$, divide the $\delta$ from the $O(1)\delta$ term, and thereby obtain a linear equation for the perturbations $\hat{\rho}$ and $\hat{u}$. Start then with equation (13):

$$\begin{aligned} 0 &= [\rho_0 + \delta\hat{\rho}]_t + [(\rho_0 + \delta\hat{\rho})(\delta\hat{u})]_x \\ &= \delta\hat{\rho}_t + \delta\rho_0\hat{\rho}_x + \delta^2(\hat{\rho}\hat{u}))_x, \end{aligned}$$

so discarding the $O(1)\delta^2$ terms and dividing by $\delta$ gives the first linearized equation

$$\hat{\rho}_t + \rho_0\hat{u}_x = 0. \tag{15}$$

Substituting next into the second Euler equation (14) and using Taylor's theorem on the pressure we obtain:

$$\begin{aligned} 0 &= [(\rho_0 + \delta\hat{\rho})(\delta\hat{u})]_t + [(\rho_0 + \delta\hat{\rho})(\delta\hat{u})^2 + p(\rho_0 + \delta\hat{\rho})]_x \\ &= \delta\rho_0\hat{u}_t + \delta^2\left[\hat{\rho}\hat{u}\right]_t + [\delta^2\rho_0\hat{u}^2 + \delta^3\hat{\rho}\hat{u}^2 + p(\rho_0) + p'(\rho_0)\delta\hat{\rho} + O(1)\delta^2]_x \\ &= \delta\rho_0\hat{u}_t + \delta p'(\rho_0)\hat{\rho}_x + O(1)\delta^2, \end{aligned}$$

so discarding the $O(1)\delta^2$ terms and dividing by $\delta$ gives the second linearized equation

$$\rho_0\hat{u}_t + p'(\rho_0)\hat{\rho}_x = 0. \tag{16}$$

**Conclude:** Equations (15) and (16) are the equations obtained by linearizing the compressible Euler equations (13), (14) about the constant states $\rho_0$, $u_0 = 0$.

To see what the linearized equations (15) and (16) mean, differentiate (15) with respect to $t$ and (16) with respect to $x$, solve for $\rho_0\hat{u}_{tx}$ in the second equation and substitute it into the first to obtain

$$\rho_0\hat{\rho}_{tt} + \rho_0 p'(\rho_0)\hat{\rho}_{xx} = 0,$$

so that dividing by the constant $\rho_0$ gives the wave equation in $\hat{\rho}$!

$$\hat{\rho}_{tt} + p'(\rho_0)\hat{\rho}_{xx} = 0. \tag{17}$$

Now we can understand it. Equation (17) tells us that, to leading order in $\delta$, the perturbations of a constant density solution of the compressible Euler equations, as observed in a frame fixed with respect to the fluid, evolve according the the wave equation

$$\hat{\rho}_{tt} + c^2 \hat{\rho}_{xx} = 0.$$

with speed

$$c = \sqrt{p'(\rho_0)}.$$

**Conclude:** In the weak wave limit, perturbations $\hat{\rho}(x, t)$ of the still fluid $\rho_0$ look like

$$\hat{\rho}(x, t) = f(x + ct) + g(x - ct),$$

that is, they consist of the superposition of a sound wave moving to the right plus a sound wave moving to the left, all waves moving at speed $c$. An important consequence of the linearization is that the speed of sound $c$ is *different from* the speed $u_0$, the velocity of the underlying fluid. In fact, this tells us that in the weak wave limit, the sound waves move at speed $u_0 \pm c$, and to get the wave equation, we have taken $u_0 = 0$; i.e., our coordinate frame has to be fixed relative to the fluid.

Note that along the way we have discovered some pretty significant physics. Namely, the speed of sound relative to the fluid speed $u_0$ is given by

$$c = \sqrt{p'(\rho_0)}.$$

We got this from understanding the PDE's. When Euler first did this in about 1750, he had solved arguably the biggest open problem in physics since Newton's Principia in 1787: namely, how do you derive the speed of sound from first principles. He did it by first deriving the nonlinear equations, then linearizing them to get the wave equation in the density, and then seeing that the resulting wave equation gave a formula for the speed of sound in terms of the pressure, $c = \sqrt{p'(\rho_0)}$.

Having come this far, it is instructive to linearize the full 3-dimensional compressible Euler equation about the still fluid solution $\mathbf{u} = 0$ and $\rho = \rho_0$. In this case, the same procedure leads to the 3-dimensional wave equation $\hat{\rho}_{tt} - p'(\rho_0)\Delta\hat{\rho} = 0$, where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the Laplacian, so the equation is

$$\hat{\rho}_{tt} - c^2 \left\{ \frac{\partial^2 \hat{\rho}}{\partial x^2} + \frac{\partial^2 \hat{\rho}}{\partial y^2} + \frac{\partial^2 \hat{\rho}}{\partial z^2} \right\} = 0$$

with

$$c = \sqrt{p'(\rho)}.$$

The only vector identity we'll need is that the Laplacian is the divergence of the gradient. That is, recall that the gradient is defined as the vector operator

$$\nabla \equiv \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right),$$

making it easy to express the divergence of a vector function as

$$div\mathbf{v} = \nabla \cdot \mathbf{v} = \frac{\partial}{\partial x}(v_1) + \frac{\partial}{\partial y}(v_2) + \frac{\partial}{\partial z}(v_3)$$

which leads to,

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \nabla \cdot \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) = \nabla \cdot \nabla = div\nabla.$$

To carry out the linearization, start with the compressible Euler equations (Newton's laws without friction for the continuum)

$$\begin{aligned} \rho_t + div(\rho\mathbf{u}) &= 0, \\ (\rho u_i)_t + div(u_i\mathbf{u} + p\mathbf{e}_i) &= 0, \end{aligned} \tag{18}$$

and follow our usual procedure for linearization: Set $\rho = \rho_0 + \delta\hat{\rho}$, $u = (u_1, u_2, u_3) = \mathbf{u}_0 + \delta\hat{\mathbf{u}} = \delta\hat{\mathbf{u}}$ because we linearize around the constant density $\rho = \rho_0$ and constant velocity $u_0 = 0$. Plug (substitute) this into the equations, Taylor expand the pressure, and throw away every term with a power $\delta^2$ or higher. Plugging in we obtain:

$$(\rho_0 + \delta\hat{\rho})_t + div([\rho_0 + \delta\hat{\rho}]\,\delta\hat{\mathbf{u}}) = 0,$$
$$([\rho_0 + \delta\hat{\rho}]\,\delta\hat{u}_i)_t + div(\hat{u}_i\delta^2\mathbf{u} + p(\rho_0 + \delta\hat{\rho})\mathbf{e}_i) = 0.$$

Discarding the zero derivatives and obvious $O(\delta^2)$ terms gives

$$\delta\hat{\rho}_t + \delta div(\rho_0\hat{\mathbf{u}}) = 0,$$
$$\delta\rho_0(\hat{u}_i)_t + div(p(\rho_0 + \delta\hat{\rho})\mathbf{e}_i) = 0.$$

Taylor expanding $p$ gives

$$p(\rho_0 + \delta\hat{\rho}) = p(\rho_0) + \delta p'(\rho_0)\hat{\rho} + O(\delta^2),$$

so plugging this in, throwing away the $O(\delta^2)$ terms, and dividing by $\delta$ gives the linearized equations

$$\hat{\rho}_t + \rho_0\,div\,(\hat{\mathbf{u}}) = 0,$$
$$\rho_0(\hat{u}_i)_t + p'(\rho_0)\,div\,(\hat{\rho}\mathbf{e}_i) = 0.$$

Now

$$div\,(\hat{\rho}\mathbf{e}_i) = \frac{\partial\hat{\rho}}{\partial x_i},$$

so the three equations

$$\rho_0(\hat{u}_i)_t + p'(\rho_0)\,div\,(\hat{\rho}\mathbf{e}_i) = 0$$

for $i = 1, 2, 3$ are equivalent to the single vector equation

$$\rho_0\hat{\mathbf{u}}_t + p'(\rho_0)\,\nabla\,\hat{\rho} = 0.$$

Thus our linearized equations are

$$\hat{\rho}_t + \rho_0\,div\,(\hat{\mathbf{u}}) = 0, \tag{19}$$
$$\rho_0\hat{\mathbf{u}}_t + p'(\rho_0)\,\nabla\,\hat{\rho} = 0. \tag{20}$$

To derive the wave equation from these, take the time derivative of the first equation (21), and the divergence of the second equation (22), and use $div\nabla = \Delta$ to obtain

$$\hat{\rho}_{tt} + \rho_0\, div\, (\hat{\mathbf{u}}_t) = 0, \tag{21}$$

$$\rho_0 div\, (\hat{\mathbf{u}}_t) + p'(\rho_0)\, \Delta\, \hat{\rho} = 0. \tag{22}$$

Subtracting the two equation we obtain the second order linearized equation

$$\hat{\rho}_{tt} - p'(\rho_0)\, \Delta\, \hat{\rho} = 0. \tag{23}$$

**Conclude:** When we linearize the 3-dimensional compressible Euler equations around the constant state solution corresponding to still air $\mathbf{u} = 0$ at constant density $\rho_0$, we obtain the 3-dimensional wave equation with sound speed $c = \sqrt{p'(\rho)}$. Since the still air has no preferred direction, from the physics we conclude that the wave equation in three dimensions must be the equation that propagates everything at the same speed $c$ in every direction! Said differently, being linear, the solutions of the wave equation must be the superposition of waves moving at speed $c$ in all the different directions. This can be proven mathematically in a more advanced class on PDE's.

## 7. Albert Einstein's Discovery of Special Relativity by means of the Wave Equation.

• In the last section we linearized the compressible Euler equations about the fixed background solution $\rho = \rho_0$, $u = u_0 = 0$, and found that the linearized equations, the equations that describe small perturbations $\hat{\rho}(x,t)$ from the constant state $\rho_0$ according to $\rho(x,t) = \rho_0 + \delta\hat{\rho}(x,t)$, solve the wave equation

$$\hat{\rho}_{tt} - c^2 \hat{\rho}_{xx} = 0. \tag{24}$$

In particular, if $\hat{\rho}$ solves (24), then so does

$$\rho(x,t) = \rho_0 + \delta\hat{\rho}(x,t).$$

The wave equation, then, describes weak sound waves that propagate down a 1-dimension shock tube, and is the starting point for the modern theory of music, explaining the superposition of sound waves and the consequent theory of harmonics. The wave equation tells us that small amplitude sound waves moving down the shock tube solve the wave equation, and hence always consist of a left going wave $f(x+ct)$ superimposed with a right going wave $g(x-ct)$. Indeed, we showed that every solution $\rho(x,t)$ of the wave equation decomposes into

$$\rho(x,t) = f(x+ct) + g(x-ct).$$

Because the wave equation is linear and homogeneous, the right and left going waves can be further decomposed into the harmonic oscillations that compose the wave by superposition.

Most important in obtaining the wave equation by linearizing about the constant state was the assumption that the velocity $u_0 = 0$. This imposes the condition that the observer who sees the perturbations described by the wave equation, be *fixed* with respect to the *rest frame* of the fluid. That is, if we were to linearize about the constant state $\rho = \rho_0$, $u = u_0 \neq 0$, so that the constant density $\rho_0$ is moving down the shock tube at velocity $u_0$, then it would be the same as if the observer were moving along the shock tube at speed $-u_0$, and the fluid were fixed. Thus we could ask, would be get the wave equation if the observer were moving with a velocity $v = -u_0$ with respect to the fluid?

But we can know immediately without calculation that the answer must surely be no! The moving observer can *never* get the wave equation as description of the linearized waves moving down the tube *unless* $u_0 = 0$. Indeed, the wave equation says there are waves moving in both directions at speed $c$, so if the wave equation describes the waves in the rest frame of the fluid, then the moving observer, say moving to the right, will be speeding toward the right going waves and speeding away from the left going waves, so the right and left going waves cannot be moving at the same speed according to the moving observer.

**Conclude:** Only the observer fixed with respect to the medium of propagation will see the linearized waves described by the wave equation.

We can see this mathematically by trying to linearize the compressible Euler equations (13),(14) about the constant state $\rho = \rho_0$, $u = u_0 \neq 0$. Following our procedure, let $\rho(x,t) = \rho_0 + \delta\hat{\rho}(x,t)$, $u(x,t) = u_0 + \delta\hat{u}(x,t)$, plug these into equations (13),(14), expand, throw away all terms of order $\delta^2$, divide by $\delta$, and obtain two linear equations for the perturbations $\hat{\rho}$ and $\hat{u}$. Since we only want the linearized equation in $\rho$, lets use the trick of differentiating (13) with respect to $x$ and (14) with respect to $t$ at the start, substitute the expression for $(\rho u)_{xt}$ from the second equation into the first to obtain the exact Euler equation

$$\rho_{tt} = (\rho u^2 + p)_{xx}. \tag{25}$$

Now linearize (25) by substituting $\rho(x,t) = \rho_0 + \delta\hat{\rho}(x,t)$, $u(x,t) = u_0 + \delta\hat{u}(x,t)$, expand, and throw away terms second order in $\delta$. To make the algebra transparent, multiply out the first term on the right hand side of (25) and collect the $O(\delta^2)$ as follows:

$$\begin{aligned}
\left\{\rho u^2\right\}_{xx} &= \left\{(\rho_0 + \delta\hat{\rho})(u_0 + \delta\hat{u})^2\right\}_{xx} \\
&= \left\{(\rho_0 + \delta\hat{\rho})(u_0^2 + 2u_0\delta\hat{u}) + O(\delta^2))\right\}_{xx} \\
&= \left\{(\rho_0 u_0^2 + \delta u_0^2\hat{\rho} + 2\rho_0 u_0\delta\hat{u} + O(\delta^2)\right\}_{xx} \\
&= \delta u_0^2\hat{\rho}_{xx} + 2\rho_0 u_0\delta\hat{u}_{xx} + O(\delta^2).
\end{aligned}$$

For the second term on the right hand side of (25), Taylor expand the pressure:

$$\begin{aligned} p(\rho_0 + \delta\hat{\rho})_{xx} &= \left(p(\rho_0) + \delta p'(\rho_0)\hat{\rho} + O(\delta^2)\right)_{xx} \\ &= \delta p'(\rho_0)\hat{\rho}_{xx} + O(\delta^2). \end{aligned}$$

Thus our linearization procedure of substituting $\rho(x,t) = \rho_0 + \delta\hat{\rho}(x,t)$ and $u(x,t) = u_0 + \delta\hat{u}(x,t)$ into (25), using our expressions for the two terms on the right hand side, dropping all $O(\delta^2)$ terms, and dividing by $\delta$, leads to the linearized equation for the rest point $\rho = \rho_0$, $u = u_0$:

$$\hat{\rho}_{tt} = 2\rho_0 u_0 \hat{u}_{xx} + \left(u_0^2 + p'(\rho_0)\right)\hat{\rho}_{xx}. \tag{26}$$

Now this is not an equation in $\hat{\rho}$ alone due to the presence of the $\hat{u}_{xx}$ term. To eliminate this, linearize (13) to obtain

$$\hat{\rho}_t + u_0\hat{\rho}_x + \rho_0\hat{u}_x = 0,$$

so differentiating with respect to $x$ and solving for $\hat{u}_{xx}$ gives

$$\hat{u}_{xx} = -\frac{\hat{\rho}_{tx} + u_0\hat{\rho}_{xx}}{\rho_0},$$

an expression for $\hat{u}_{xx}$ in terms of known constants $\rho_0, u_0$ and derivatives of $\hat{\rho}$. Substituting this into (26) gives

$$\begin{aligned} \hat{\rho}_{tt} &= -2\rho_0 u_0 \frac{\hat{\rho}_{tx} + u_0\hat{\rho}_{xx}}{\rho_0} + \left(u_0^2 + p'(\rho_0)\right)\hat{\rho}_{xx} \\ &= (-u_0^2 + p'(\rho_0))\hat{\rho}_{xx} - 2u_0\hat{\rho}_{tx}, \end{aligned} \tag{27}$$

yielding the more complicated wave equation

$$\hat{\rho}_{tt} - (p'(\rho_0) - u_0^2)\hat{\rho}_{xx} + 2u_0\hat{\rho}_{tx} = 0. \tag{28}$$

Equation (28) describes the linearized waves when the still air of constant density $\rho_0$ is moving by the observer at a constant velocity $u_0$. When $u_0 = 0$, we recover the wave equation. Thus the mathematics confirms what we knew from the physics must be true: linearizing the equation about the constant state $\rho_0$, $u_0$, does not produce the wave equation when $u_0 \neq 0$!

**Conclude:** The linearized waves that propagate according to the wave equation as recorded by an observer *fixed* with respect to the rest frame of the fluid, turn out to solve the more complicated equation (28) in the frame of the *moving* observer.

Actually, we can obtain this result more directly by just taking the wave equation

$$\hat{\rho}_{tt} - p'(\rho_0)\hat{\rho}_{xx} = 0,$$

correct in the frame moving with the fluid, and see what this equation transforms over to in the frame of the moving observer. To be specific, lets assume (without loss of generality) that $u_0 < 0$, so the observer see's the fluid going by to the *left* at the constant velocity $u_0 < 0$. It follows that a different observer, moving with the fluid, would see the original observer moving to the *right* at speed $v = -u_0 > 0$. Now let $x$ denote coordinate distance along the shock tube in the frame *fixed* with the fluid medium (the unbarred observer), and let $\bar{x}$ denote the coordinate distance measured along the shock tube by the (barred) observer moving at velocity $v$ to the right relative to the fixed unbarred observer, and lets assume that $x$-axis and $\bar{x}$-axis coincide at $t = 0$. Then the fluid at position $\bar{x} = x$ at $t = 0$, will be at $\bar{x} = x - vt$ at time $t$. It follows that

$$\bar{x} = x - vt,$$

must give the $\bar{x}$-coordinate in terms of the $x$-coordinate at time $t$, (see Figure 4).

Assuming then that $\rho(x,t) = \rho_0 + \delta\hat{\rho}(x,t)$ represents the density that solves the wave equation in the frame of the fluid (in $(x,t)$-coordinates),

$$\hat{\rho}_{tt} - p'(\rho_0)\hat{\rho}_{xx} = 0, \tag{29}$$

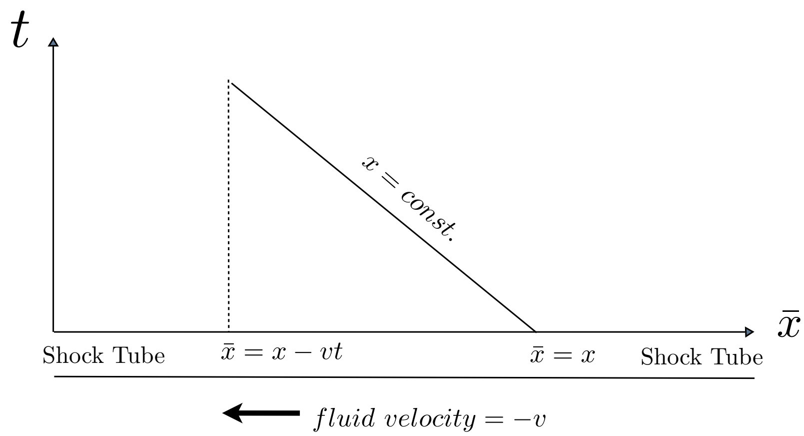


FIGURE 4. The coordinate $x$ stays constant along the moving fluid particles.

the solution in $(\bar{x}, t)$ coordinates would be

$$\bar{\rho}(\bar{x}, t) = \bar{\rho}(x - vt, t) = \rho(x, t).$$

That is, the corrections in the barred frame are described by a different function $\bar{\rho}$ of $(\bar{x}, t)$ than $\hat{\rho}$ is of $(x, t)$ (so we put a bar over it), but it should be the same function if we substitute $\bar{x} = x - vt$. We find the equation for $\bar{\rho}(x, t)$, then, by substituting $\bar{\rho}(x - vt, t)$ for $\rho(x, t)$ in (29). Using this together with the chain rule we can calculate

$$\rho_t = \bar{\rho}_t - v\bar{\rho}_{\bar{x}}; \quad \rho_{tt} = \bar{\rho}_{tt} - 2v\bar{\rho}_{\bar{x}t} + v^2\bar{\rho}_{\bar{x}\bar{x}},$$

and

$$\rho_x = \bar{\rho}_{\bar{x}}; \quad \rho_{xx} = \bar{\rho}_{\bar{x}\bar{x}}.$$

Thus we have

$$0 = \rho_{tt} - p'(\rho_0)\rho_{xx} = \bar{\rho}_{tt} - 2v\bar{\rho}_{\bar{x}t} + v^2\bar{\rho}_{\bar{x}\bar{x}} - p'(\rho_0)\bar{\rho}_{\bar{x}\bar{x}},$$

so the right hand side gives the equation $\bar{\rho}$ satisfies,

$$\bar{\rho}_{tt} - (p'(\rho_0) - v^2)\rho_{\bar{x}\bar{x}} - 2v\bar{\rho}_{\bar{x}t} = 0. \tag{30}$$

Comparing this with (28) (replacing $\bar{x}$ with $x$ because $\bar{x}$ is now the frame in which the fluid moves) using that $v = -u_0$, we see equation (30) is exactly the equation we obtained by linearizing about the constant state $\rho_0$, $u_0 \neq 0$—and its not the wave equation!

**Conclude:** The wave equation describes linearized waves moving in two directions at the same speed through a medium, but it is only correct in the frame fixed with the medium.

• We now are in a position to understand how Albert Einstein used the wave equation to discover the theory of special relativity and the relativity of time.

The story starts back in the year 1861 when J.C. Maxwell published *On physical lines of force.* In this paper he proposed that light consisted of the propagation of electromagnetic waves. The route by which he came to this conclusion is as follows: He started with Faraday's proposal that electric and magnetic fields were described by vector fields $\mathbf{E} = (E_1, E_2, E_3)$ and $\mathbf{B} = (B_1, B_2, B_3)$ that varied in space and time. For example, the electric field is the force per charge exerted on a particle at a point in the field, a vector because the force has a magnitude and a direction. Maxwell then derived the partial differential equations that describe the time evolution of these fields. Three of the equations he obtained from the physical laws that Faraday had discovered earlier, but to *close the equations*, (something like we closed the Euler equation by setting $p = p(\rho)$), he guessed and proposed a fourth equation, and the resulting four equations are named *Maxwell's Equations.*

Now this is when it gets really interesting. As a consequence of his equations, (for our purposes here the exact form of Maxwell's equations is not important), each component of $\mathbf{E}$ and $\mathbf{B}$ solves the wave equation! That is, in one dimension, each component, say $E$, evolves in time according to the equation

$$E_{tt} - \epsilon_0 \mu_0 E_{xx} = 0.$$

That is, each component solves the wave equation with speed

$$c = \sqrt{\epsilon_0 \mu_0}.$$

Thus Maxwell's equations imply waves of electromagnetic fields propagate in both directions at speed $c = \sqrt{\epsilon_0 \mu_0}$. Now for the purposes here it also doesn't matter exactly what the constants $\epsilon_0$ and $\mu_0$ measure, the important point is that they are physical constants, (called the *permittivity* and *permeability* of empty space), derivable from experiments using only magnets, currents and charges. But when Maxwell calculated $c$ using the best values he could find for $\epsilon_0$ and $\mu_0$ at that time, he discovered that

$$\sqrt{\epsilon_0 \mu_0} \approx c = \text{speed of light}.$$

This then led Maxwell to the bold proposal that light consisted of the propagation of electromagnetic fields. This proposal remained controversial for some 26 years, until Heinrich Hertz proved Maxwell right in 1887 by generating electromagnetic radiation from spinning magnets.

But now still, there is a real problem. As we saw above, the wave equation is only valid in the frame fixed with respect to the medium through which the waves are propagating. But in Maxwell's derivation, no medium was ever assumed at the start. So it was conjectured that there must be some mysterious *ether*, filling all of space, and this invisible ether provided the medium, like the density in the compressible Euler equations, through which the electromagnetic waves were propagating.

Enter now Albert Einstein in 1905. While working in the patent office in Bern Switzerland, Einstein's earlier musings about the nature of time developed into the idea that perhaps the wave equation was correct in every inertial coordinate system. That is, he proposed that the inertial frames in which the laws of physics take the same form, were not related by $\bar{x} = x - vt$ as Newton proposed, but rather there was a change in time as well, and it all worked out to make the wave equation valid for *every* inertial observer. If so, then an observer moving with velocity $v$ would see the electromagnetic waves moving to the right and to the left at the speed of light $c$, just like every other observer. If so, then the wave equation Maxwell found was *resonantly tuned*, so to speak, with the way space and time were entangled. Turning this around, Einstein realized that the wave equation was then a doorway to the discovery of a mysterious and surprising connection between time and space. But the big question remained: *How do you derive the transformation between the inertial frames from the starting principle that the wave equation is correct in every frame?* We now relive Einstein's discovery by retracing his steps to one of the greatest discoveries of all time. Here is the shortest route to the answer.

To start, instead of the Galilean transformation $x = \bar{x} + vt$, consider the transformation

$$\begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} \begin{pmatrix} \bar{t} \\ \bar{x} \end{pmatrix}, \tag{31}$$

where $\theta$ is a constant to be determined later. Recall that

$$\cosh\theta = \frac{e^{\theta} + e^{-\theta}}{2}, \qquad \sinh\theta = \frac{e^{\theta} - e^{-\theta}}{2}$$

constructed so that

$$\cosh^2\theta - \sinh^2\theta = 1, \tag{32}$$

as is easily verified. So assume that $E(t,x)$ solves the wave equation in the inertial coordinate frame $(t,x)$. (We always put $t$ first in relativity!) As a preliminary, note first that we can without loss of generality assume $c=1$. That is, if we change units of time so $\tau = ct$, (this gives time the dimensions of length), then

$$E_t = E_\tau \frac{d\tau}{dt} = cE_\tau, \quad E_{tt} = c^2 E_{\tau\tau},$$

so

$$E_{\tau\tau} - E_{xx} = 0,$$

thereby converting $c=1$ in the new units for time. So assuming $c=1$ and writing the function $E(t,x)$ in terms of $(\bar{t},\bar{x})$ gives

$$E(t,x) = E(\bar{t}\cosh\theta + \bar{x}\sinh\theta, \bar{t}\sinh\theta + \bar{x}\cosh\theta) \equiv \bar{E}(\bar{t},\bar{x}).$$

That is, the solution in the barred frame should be the solution from the unbarred frame with the values for $(t,x)$ substituted by their expressions in terms of $(\bar{t},\bar{x})$. We now find the equation that $\bar{E}$ satisfies in $(\bar{t},\bar{x})$-coordinates. By the chain rule for partial derivatives,

$$\begin{aligned} \bar{E}_{\bar{t}} &= E_t \cosh\theta + E_x \sinh\theta, \\ \bar{E}_{\bar{t}\bar{t}} &= E_{tt}\cosh^2\theta + 2E_{xt}\sinh\theta\cosh\theta + E_{xx}\sinh^2\theta, \end{aligned}$$

and similarly,

$$\begin{aligned} \bar{E}_{\bar{x}} &= E_t \sinh\theta + E_x \cosh\theta, \\ \bar{E}_{\bar{x}\bar{x}} &= E_{tt}\sinh^2\theta + 2E_{tx}\cosh\theta\sinh\theta + E_{xx}\cosh^2\theta, \end{aligned}$$

from which we conclude that in $(\bar{t},\bar{x})$-coordinates, $E$ satisfies

$$\begin{aligned} \bar{E}_{\bar{t}\bar{t}} - \bar{E}_{\bar{x}\bar{x}} &= E_{tt}\left(\cosh^2\theta - \sinh^2\theta\right) - E_{xx}\left(\cosh^2\theta - \sinh^2\theta\right) \\ &= E_{tt} - E_{xx} = 0. \end{aligned} \tag{33}$$

That is, $\bar{E}$ satisfies the wave equation in transformed coordinates $(\bar{t},\bar{r})$ if and only if $E$ satisfies the wave equation in

$(t,x)$-coordinates. Here, again, $\bar{E}(\bar{t},\bar{x})$ is just the function you get by substituting into $E(t,x)$ the values of $(t,x)$ in terms of $(\bar{t},\bar{x})$ as given in (31). [2]

**Conclude:** The coordinate changes (31), based on hyperbolic sines and cosines, are the sought after changes in $x$ and $t$ that preserve the wave equation. It remains only then to interpret the hyperbolic angle $\theta$ physically.

So consider finally the coordinate change (31) multiplied out as (c.f. Figure 5),

$$t = \bar{t}\cosh\theta + \bar{x}\sinh\theta \quad (34)$$

$$x = \bar{t}\sinh\theta + \bar{x}\cosh\theta. \quad (35)$$

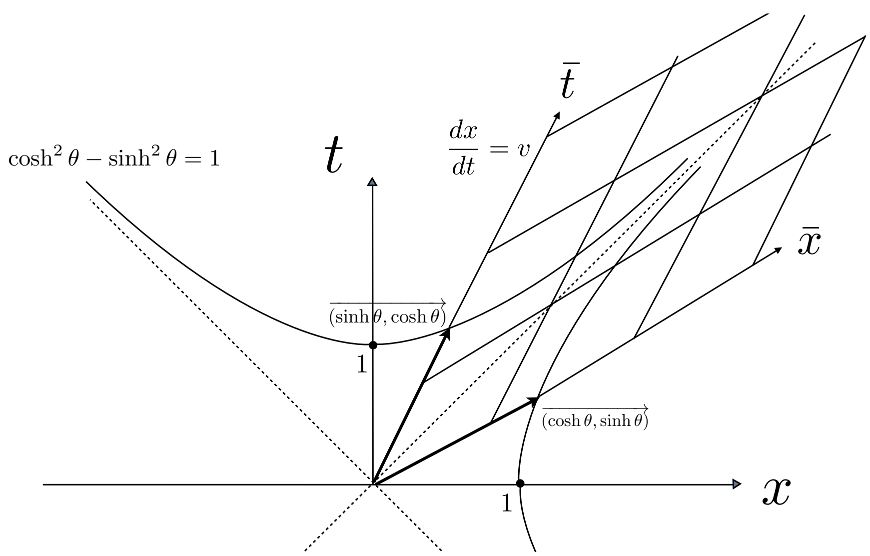


FIGURE 5. The $(\bar{t},\bar{x})$ Coordinate System.

[2]Note that if you wondered how you might guess the transformation (31) in the first place, consider that if you assumed an arbitrary matrix transformation of the form

$$\begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \bar{t} \\ \bar{x} \end{pmatrix}$$

at the start, you would be led at the step (33) to the requirement that $b = c$, $a^2 - b^2 = 1$, $d^2 - c^2 = 1$, which would lead you directly to hyperbolic sines and cosines by the identity $\cosh^2\theta - \sinh^2\theta = 1$.

Setting $\bar{x} = 0$ we see that the observer fixed at the origin in the barred coordinate system moves along the curve

$$\begin{aligned} t &= \bar{t}\cosh\theta, \\ x &= \bar{t}\sinh\theta, \end{aligned}$$

which we can write in vector form as

$$\begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} \cosh\theta \\ \sinh\theta \end{pmatrix} \bar{t}. \tag{36}$$

That is, he moves along the curve tangent to the vector $(\cosh\theta, \sinh\theta)$ in the $(t, x)$-plane as his time $\bar{t}$ increases. The velocity of the barred observer as measured in the frame of the unbarred observer is thus the change in $x$ divided by the change in $t$ along the observer's direction vector $(\cosh\theta, \sinh\theta)$,

$$v = \frac{\sinh\theta}{\cosh\theta} = \tanh\theta.$$

We can now derive the Lorentz transformations of special relativity as Einstein did by writing the coordinate change (34), (35) in terms of $v$ instead of $\theta$. Dividing the fundamental identity $\cosh^2\theta - \sinh^2\theta = 1$ by $\cosh^2\theta$ yields the identity

$$1 - \tanh^2\theta = \frac{1}{\cosh\theta},$$

which by $v = \tanh\theta$ gives

$$\cosh\theta = \frac{1}{\sqrt{1 - v^2}}.$$

Then also

$$\sinh\theta = \sqrt{\cosh^2\theta - 1} = \frac{v}{\sqrt{1 - v^2}}.$$

Substituting these into (35) gives the final form of the Lorentz transformations, the *boosts*, which are the foundation of Einstein's theory of special relativity,

$$t = \frac{1}{\sqrt{1-v^2}}\bar{t} + \frac{v}{\sqrt{1-v^2}}\bar{x} \tag{37}$$

$$x = \frac{v}{\sqrt{1-v^2}}\bar{t} + \frac{1}{\sqrt{1-v^2}}\bar{x}. \tag{38}$$

Putting the units of $c$=*speed of light* back in would give the spacetime transformations

$$t = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}}\bar{t} + \frac{\left(\frac{v}{c}\right)}{\sqrt{1-\left(\frac{v}{c}\right)^2}}\bar{x} \tag{39}$$

$$x = \frac{\left(\frac{v}{c}\right)}{\sqrt{1-\left(\frac{v}{c}\right)^2}}\bar{t} + \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}}\bar{x}. \tag{40}$$

**Conclude:** To resolve the problem in Maxwell's theory that the wave equation, and hence the speed of light bound, could only hold in the frame fixed with the medium through which the wave propagated (called the *ether*, thought to be like the still fluid $\rho_0$ in sound wave propagation), Einstein conjectured that there was no such ether medium, and the wave equation was correct and applied in every inertial frame. This meant that the inertial frames themselves had to be modified to make this true. The resulting coordinate changes that preserve the wave equation have been derived, as Einstein derived them, in (39), (40). Einstein then proposed that the time $\bar{t}$ in the barred frame really was the time at which the barred observer was aging, not the time $t$ of any other rest frame. In fact, he called it the theory of relativity, because there is no universal rest frame, only frames at rest *relative to the observer*. By this line of thinking Einstein was led to propose that time, like distance, is not universal, but is rather a *metrical quantity* that elapses according to the length of the curve traversed in spacetime. The ultimate prediction is that a moving observer will age more slowly than his/her stationary brother, and if the observer goes off near the speed of light and then returns, he/she can return moments later to meet his/her brother's great, great, great, great, grandchildren. All of this, forever more, is based on the PDE which D'Alambert proposed in 1748 to describe the motions of a vibrating string, the *wave equation.*

Note finally, that at the level of the fundamental equations, the wave equation is not an approximate equation for wave propagation in electromagnetism as it is for the compressible Euler equations, it starts out linear—electromagnetic radiation propagates exactly according to the linear, homogeneous equation $E_{tt} - c^2 E_{xx} = 0$. This applies at the *fundamental level* of Maxwell's equations, "fundamental" meaning at the starting point of the theory, without including effects from other sources. For sound waves, the fundamental equations are the (fiercely) nonlinear compressible Euler equations, and the theory of sound/music based on the wave equation emerges in the weak signal limit by linearization—for light waves, the fundamental equations are linear at the start! Thus for electromagnetic radiation, the principle of superposition holds *exactly.* This then explains why we can fill the airways with so many transmissions of cell phone signals all at once, and they don't interfere with each other. Indeed, by the principle of superposition, (which holds because the wave equation is *linear* and *homogeneous*), we can add up all the signals at one end, propagate them through the airways by the wave equation, and then pull them apart at the other end to recover the exact signal sent out at the start. In contrast, for sound waves moving through the air, the wave equation only holds approximately for weak signals, and for large amplitude signals, or after a long time, the nonlinearities of the compressible Euler equations take over, and shock wave dissipation attenuates, and eventually destroys the signal.