7–The Nonlinear Wave Equation and the Interaction of Waves

MATH 22C

1. Introduction

Just as the nonlinear advection equation looks the same as the linear advection equation $u_t + cu_x = 0$ except that the speed of sound c depends on the solution u, so also the nonlinear wave equation is the linear wave equation $u_{tt} - c^2 u_{xx} = 0$ under the assumption that the speed of sound c depends on the solution u as well. Now for the advection equation, the solution, being a single wave

$$u(x,t) = f(x - ct)$$

moving to the right, is constant along curves $x - ct = const.$ These so called characteristic curves are the *sound waves* moving at the speed of sound c in the (x,t)-plane. Thus sound wave propagation occurs along curves $(x(t),t)$ satisying

$$\dot{x} = c,$$

as they should. When c depends on u, for example $c = u$ is the Burger's equation, then solutions $u(x,t)$ still remain constant along the characteristics

$$\dot{x} = c(u),$$

but since the speeds depend on the solution, characteristics can intersect and create shock waves. But the connection between the sound waves in the linear advection equation $u_t + cu_x = 0$ and the nonlinear advection equations, (say Burgers equation $u_t + uu_x = 0$) is clear: The nonlinear wave moves the constant value u along lines of speed u in the (x,t) plane, thereby creating an approximate wave of speed c when u takes values near the the constant value c.

Now the wave equation is a more realistic model for sound wave propagation than the advection equation because it is obtained by linearizing the compressible Euler equations about a constant state. The advection equations $u_t + cu_x = 0$ and $u_t - cu_x = 0$ admit the right and left going waves separately, but the wave equation admits the superposition of both waves, and we have shown that all solutions of the wave equation decompose into a left and right going wave of form

$$u(x, t) = f(x + ct) + g(x - ct).$$

But a solution of the wave equation, being in general a superposition of *both* left and right going waves, is only constant along the characteristic curves $dx/dt = \pm c$ when the wave is a pure left ($g = 0$) or pure right ($f = 0$) going wave. In general, the superposition mixes up the right and left moving waves and the solution will not be constant along the characteristics $dx/dt = \pm c$. To understand the characteristics in a way that generalizes to nonlinear wave equations in which the speed c depends on the solution, we need a more general framework that breaks the solution down into identifiable components. The goal is to see that the nonlinear wave equation *also* admits nonlinear waves that propagate to the right and to the left, but being *nonlinear*, superposition *fails*, and a general solution does not exactly decompose into a sum of left and right going waves as do solutions of the linear (constant c) wave equation. To understand what we might call the *nonlinear superposition* of left and right going waves created by the nonlinear wave equation, we develop a more general method that treats the wave equation as a first order system, and is based on finding Riemann invariants: functions r and s of the unknowns that are constant along sound waves, that is, constant along the characteristic curves in the (x,t)-plane. By this we can explain the propagation of left and right going nonlinear waves along the characteristics $ds/dt = \pm c$, when c depends on the solution.

To write the wave equation as a first order system, assume

$$w_{tt} - c^2 w_{xx} = 0,$$

view w as a potential, and set $w_t = u$, $w_x = v$ to get the 2×2 first order system

$$v_t - u_x = 0, \tag{1}$$
$$u_t - c^2 v_x = 0, \tag{2}$$

valid whether c is constant, or c depends on the unknowns u, v. The first equation (1) expresses that mixed partials $w_{xt} = w_{tx}$ are equal, and the second equation (2) expresses the wave equation. Equations (1), (2) take the matrix form

$$\begin{pmatrix} v \\ u \end{pmatrix}_t + \begin{pmatrix} 0 & -1 \\ -c^2 & 0 \end{pmatrix} \begin{pmatrix} v \\ u \end{pmatrix}_x = 0,$$

a general first order system of the form

$$U_t + AU_x = 0, \tag{3}$$

where $U = (v, u)$ and A is a 2×2 matrix. The main goal of this section is to explain how the eigenpairs (λ, R) of A decouple the nonlinear wave equation into its left and right moving component waves.

All of this is worth the trouble because we show that, when rewritten as an equivalent system in a coordinate frame *moving with the fluid*, the compressible Euler equations take the *exact* form of the nonlinear wave equation $u_{tt} - c^2 u_{xx} = 0$, where c depends on the solution. (This is the so called Lagrangian formulation of the compressible Euler equations.) That is, since the sound speed is the speed of sound waves *relative to the fixed fluid*, we know we can only get the linear wave equation (with wave propagation at speed c in *both* directions) by linearizing the equation in a coordinate frame fixed with the medium of propagation. (We assume no relativity to make the speeds the same in every frame!) In a general solution of the nonlinear equations, the fluid velocity u varies from point to point, so we can't get a global frame fixed with the medium of propagation. Thus we do the next best thing. To turn the Euler equations into the nonlinear wave equation, we transform x and t so that *at each point in the solution* we are *moving with the fluid*. The transformation thus depends on the solution. The resulting equations obtained by transforming the original compressible Euler equations

$$\rho_t + (\rho u)_x = 0, \tag{4}$$
$$(\rho u)_t + (\rho u^2 + p)_x = 0, \tag{5}$$

by a solution dependent transformation that *moves with the medium* at every point, results in the equivalent system of two equations

$$v_t - u_x = 0, \tag{6}$$
$$u_t + p(v)_x = 0, \tag{7}$$

in the two unknowns

$$v = 1/\rho = \text{specific volume}, \tag{8}$$
$$u = \text{velocity}. \tag{9}$$

The original equations (4), (5) are called the *Eulerian formulation of the equations* and the equivalent transformed equations (6), (7) are called the *Lagrangian formulation of the equations*. The Lagrangian measure of distance x appearing in (6), (7), constructed to measure the distance between fluid particles as constant, is different from the Eulerian x appearing in (4), (5), as will be clarified in the the derivation of (6), (7) below. In the Lagrangian version, the equation of state expresses the pressure $p = p(v)$ as a function of specific volume $v = 1/\rho$, and since this is the only nonlinear function in the equations, these equations are called the p-system, (so named by Joel Smoller). The p-system provides the simplest realistic quantitative model for shock wave propagation down a one dimensional shock tube. Comparing (6), (7) with (1), (2), we see that the the p-system agrees with the nonlinear wave equation written as a first order system if we make the identification

$$c = \sqrt{-p'(v)}.$$

Recall that for the linearized Euler equations

$$\rho_{tt} - c^2\rho_{xx} = 0,$$

the sound speed was

$$c = \sqrt{\frac{dp}{d\rho}} = \sqrt{\frac{dp}{dv}\ \frac{dv}{d\rho}} = \frac{\sqrt{-p'(v)}}{\rho},$$

the extra factor of ρ in the denominator accounting for the difference between the Eulerian and Lagrangian version of the equations.

Finally, we can reverse the steps and recover the nonlinear wave equation from the p-system (6),(7) as follows. By the first equation (6) we have

$$Curl_{(x,t)}(v,u) = (u_x - v_t)\mathbf{k} = 0,$$

so from vector calculus we know that this is the condition required for the vector field $(v(x,t), u(x,t))$ to be the gradient of some function $w(x,t)$; i.e., there exists a potential function w such that

$$\nabla_{(x,t)}w = (w_x, w_t) = (v, u).$$

Thus $v = w_x$ and $u = w_t$. Equation (7) thus gives us the equation for the potential w,

$$w_{tt} + p'(v)w_{xx} = 0. \tag{10}$$

Note that $p'(v) < 0$ because pressure *increases* with density, and *decreases* with inverse density $v = 1/\rho$. Thus $p'(v) = -|p'(v)|$, and (10) really is the wave equation

$$w_{tt} - c^2 w_{xx} = 0 \tag{11}$$

with nonlinear sound speed $c = c(v) = \sqrt{-p'(\rho)} > 0$. Thus again, the speed of sound for the p-system, written as a nonlinear wave equation, is $c = \sqrt{-p'(v)}$, and the p-system written in terms of the potential is the nonlinear wave equation $w_{tt} - c^2 w_{xx} = 0$.

2. The p-system

We now show that the compressible Euler equations (4), (5), which describe the nonlinear wave motion of the density and velocity of a gas propagating down a one dimensional shock tube, really can be written as the nonlinear wave equation (6), (7). That is, recall that we obtained the wave equation $\rho_{tt} - c^2\rho_{xx} = 0$ by linearizing the compressible Euler equation in a frame fixed with respect the fluid; i.e., about a constant state $\rho = \rho_0$, $u = u_0 = 0$. In this frame, the velocity u_0 is zero, so the gas is not moving. Only in this frame can we get the wave equation because the wave equation has two equal sound speeds $\pm c$ moving in both directions, something we would not see if the fluid were say moving to the right with velocity $u_0 > 0$. The same situation occurs for the fully nonlinear Euler equations (4), (5). That is, since the nonlinear wave equation also has nonlinear sound speeds $\pm c(v)$ depending on the value v in the solution, sound waves move in both directions at the same speed at each point of a solution, so we can only get the nonlinear wave equation in a coordinate system fixed with respect to the fluid *at every point of the fluid.* To achieve this, for each given solution $\rho(x,t)$, $u(x,t)$, we make a change from x-coordinate to ξ-coordinate so that

$$x = x(\xi, t),$$

moves with the fluid.

Before constructing the change of variable $x = x(\xi, t)$ that accomplishes this, we first note that (5) can be written in the convenient form

$$u_t + uu_x + p_x/\rho = 0. \tag{12}$$

To see this, use the product rule on (5) to obtain

$$\begin{aligned}(\rho u)_t + (\rho u^2 + p)_x &= \rho_t u + \rho u_t + (\rho u)_x u + \rho u u_x + p_x \\ &= u(\rho_t + (\rho u)_x) + \rho(u_t + uu_x + p_x/\rho),\end{aligned}$$

so since

$$u_t + (\rho u)_x = 0$$

by (6), we obtain (12) by dividing by ρ. Thus we can take the compressible Euler equations to be the system

$$\rho_t + (\rho u)_x = 0, \tag{13}$$

$$u_t + uu_x + p_x/\rho = 0. \tag{14}$$

One payoff for writing the second equation in this form is to see that when the pressure is zero, then $p_x = 0$ as well, and the second equation reduces exactly to the scalar Burgers equation. Thus solution to Burgers is a real problem for fluids.

Our purpose now is to define a change of coordinates

$$x = x(\xi, t)$$

such that at each fixed ξ,

$$x = x(\xi, t)$$

gives x as a function of t moving with the fluid particle at every point. We then show that when we write (6), (7) in terms of (ξ, t) we obtain the p-system

$$v_t - u_\xi = 0, \tag{15}$$

$$u_t + p(v)_\xi = 0. \tag{16}$$

which is exactly (6), (7) upon renaming ξ as x.

To this end, assume the velocity field $u(x, t)$ for a solution of the compressible Euler equations (13), (14) is given. That is, $u(x, t)$ is the velocity of the so-called *fluid particle* at position x at time t, the velocity of a speck of dust moving with the fluid. Keep in mind that $u(x, t)$ is *different* from the sound speed c. I.e., we've seen that sound wave propagates at a speed $\pm c$ *relative to the fluid particles.* For example, the speck of dust starting at $x = 0$ at time $t = 0$ moves along a curve $x_0(t)$ at speed $\dot{x}_0(t) = u(x_0(t), t)$, so $x_0(t)$ solves the initial value problem

$$\dot{x} = u(x(t), t),$$
$$x(0) = 0.$$

Similarly, the dust particle starting at $x = a$ at time $t = 0$ moves along a curve $x_a(t)$ at speed $\dot{x}_0(t) = u(x(t), t)$, so $x_a(t)$ solves the initial value problem

$$\dot{x} = u(x(t), t),$$
$$x(0) = a.$$

Thus we can define the general solution $x_a(t) = x(a, t)$ as the position at time t of particle that started at position $x = a$ at time $t = 0$. In particular, the particles are moving at speed u, so we know that for fixed a (that fixes the particle),

$$\dot{x}_a(t) = u,$$

where u is the speed at position $x = x_a(t)$, time t. In terms of partial derivatives, this is expressed by

$$\frac{\partial x}{\partial t}(a,t) = u(x(a,t),t).$$

Now for the clever part. We define ξ to be a function of the initial positions a by

$$\xi = \int_0^a \rho(x,0)dx. \tag{17}$$

That is, ξ is the function of a equal to the amount of mass between $x = 0$ and $x = a$ at time $t = 0$. If you fix a, you fix ξ as well, and vice versa. Fixing a is the *same* as fixing ξ. So we can write the particle path as $x_\xi(t) = x(\xi,t)$, and by the same argument

$$\dot{x}_\xi(t) = u,$$

where u is the speed at position $x = x_\xi(t)$, time t. In terms of partial derivatives this is expressed by

$$\frac{\partial x}{\partial t}(\xi,t) = u(x(\xi,t),t).$$

In particular, for any function of $f(x,t)$, we can write

$$f(x(\xi,t),t)$$

to express it as a function (ξ,t), and so by differentiating f with respect to t holding ξ fixed we obtain the rate at which f changes *along the particle path*. That is, by the chain rule

$$\frac{\partial}{\partial t} f(x(\xi,t),t) = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial t} = f_t + f_x u.$$

This is called the *material derivative* of f because it is the rate at which f changes as measured in a frame *moving with the material*. Note that all we need is that ξ is a function of

a to compute the time derivative of f holding ξ fixed (the material derivative), but to compute

$$\frac{\partial}{\partial \xi} f(x(\xi,t),t) = \frac{\partial f}{\partial x}\,\frac{\partial x}{\partial \xi},$$

we need the formula for $\frac{\partial}{\partial \xi} x(\xi,t)$.

To complete the task of deriving the p-system, we now show that, when written in terms of the transformed variable ξ, the compressible Euler equations (13), (14) reduce to the p-system, that is, they take the form of the nonlinear wave equation. For this we need to compute $\frac{\partial}{\partial \xi} x(\xi,t)$ which we do by use of the following fact which states that the total mass between two particle paths is constant in time:

$$\xi = \int_0^a \rho(x,0)dx = \int_{x_0(t)}^{x_\xi(t)} \rho(x,t)dx \equiv M_\xi(t), \qquad (18)$$

where the $\equiv$ defines $M_\xi(t)$. The first equality in (18) is intuitively clear because $M_\xi(t)$ is the total mass between the particle path $x_0(t)$ and $x_\xi(t)$ at time t, and mass is conserved, which implies M_ξ is constant with $M_\xi(t) = M_\xi(0) = \xi$. Thus to verify the first equality in (18) it suffices to prove that $\dot{M}_\xi(t) = 0$. In fact, (18) follows directly from (13). Before proving this, we point out that the main purpose of (18) is that from this we can calculate $\partial x/\partial \xi$ by taking the partial derivative of (18) with respect to ξ on both sides of the first equality sign to get

$$1 = \frac{\partial}{\partial \xi}\int_{x_0(t)}^{x_\xi(t)} \rho(x,t)dx = \rho\frac{\partial x}{\partial \xi}, \qquad (19)$$

which we can solve for $\partial x/\partial \xi$ to get

$$\frac{\partial x}{\partial \xi} = \frac{1}{\rho},$$

and its reciprocal

$$\frac{\partial \xi}{\partial x} = \rho,$$

To verify the first equality in (18) we need only to show that $M'(t) = 0$, (because then $M_\xi(t) = M_\xi(0) = \xi$). But by the chain rule for integrals,

$$\begin{aligned} M'_\xi(t) &= \frac{d}{dt}\int_{x_0(t)}^{x_\xi(t)} \rho(x,t)dx = \frac{d}{dt}\int_{x(0,t)}^{x(\xi,t)} \rho(x,t)dx \\ &= \rho(x(\xi,t),t)x'_\xi(t) - \rho(x(0,t)x'_0(t) + \int_{x(0,t)}^{x(\xi,t)} \rho_t(x,t)dx. \end{aligned}$$

But the speed of the particle paths is u so

$$x'_\xi(t) = u(x(\xi,t),t), \tag{20}$$

$$x'_0(t) = u(x(\xi,0),0), \tag{21}$$

and using (13) in the form $\rho_t = -(\rho u)_x$ gives

$$\begin{aligned} \int_{x(0,t)}^{x(\xi,t)} \rho_t(x,t)dx &= -\int_{x(0,t)}^{x(\xi,t)} (\rho u)_x(x,t)dx \\ &= -\left\{(\rho u)(x(\xi,t),t) - (\rho u)(x(\xi,0),0)\right\}. \end{aligned}$$

Substituting (21)-(22) into the last line of (20) gives

$$\begin{aligned} M'(t) &= \{(\rho u)(x(\xi,t),t) - (\rho u)(x(0,t),t)\} \\ &\quad - \{(\rho u)(x(\xi,t),t) - (\rho u)(x(0,t),t)\} = 0, \end{aligned}$$

verifying (18) as claimed.

We now have everything we need to show that when ξ is taken in place of x, the compressible Euler equations convert to the p-system, a first order version of the nonlinear wave equation. All we need is

$$x = x(\xi,t),$$

with the partial derivative formula

$$\xi_x \equiv \frac{\partial \xi}{\partial x}(x,t) = \rho,$$

and material derivative formulas

$$\rho_t + u\rho_x = \frac{\partial}{\partial t}\rho(\xi,t), \tag{22}$$

$$u_t + uu_x = \frac{\partial}{\partial t}u(\xi,t). \tag{23}$$

So here it is:

To transform equation (13) write

$$\begin{aligned} 0 = \rho_t + (\rho u)_x &= (\rho_t + \rho_x u) + \rho u_x \\ &= \frac{\partial}{\partial t}\rho(\xi,t) + \rho u_\xi(\xi,t)\xi_x \\ &= \rho_t(\xi,t) + \rho^2 u_\xi(\xi,t), \end{aligned}$$

which we can rewrite as

$$\frac{1}{\rho^2}\rho_t + u_\xi = 0,$$

which gives the desired first equation (6) of the p-system

$$v_t - u_\xi = 0,$$

since $v = 1/\rho$ makes

$$v_t = \left(\frac{1}{\rho}\right)_t = -\frac{\rho_t}{\rho^2}.$$

To transform equation (5) write

$$0 = u_t + uu_x + p_x/\rho = \frac{\partial}{\partial t}u(\xi,t) + \frac{\partial p}{\partial \xi}\,\xi_x\,\frac{1}{\rho} = u_t(\xi,t) + p_\xi(\xi,t),$$

which is

$$u_t + p(v)_\xi = 0,$$

in agreement with (7) upon renaming ξ as x.

Conclude: The compressible Euler equations when written in terms of the Lagrangian variable ξ transform into the nonlinear wave equation (6), (7).

3. Riemann Invariants and the Interaction of Nonlinear Waves

We have written the wave equation $w_{tt} - c^2 w_{xx} = 0$ as a first order system

$$v_t - u_\xi = 0, \tag{24}$$

$$u_t - c^2 v_x = 0, \tag{25}$$

with $u = w_t$ and $v = w_x$, and we found that the compressible Euler equations in Lagrangian coordinate ξ (which we now write as x) reduce to the p-system

$$v_t - u_\xi = 0,$$

$$u_t + p(v)_\xi = 0,$$

where u is the velocity and $v = 1/\rho$ is called the *specific volume.* Writing

$$p(v)_x = p'(v)v_x = -c^2 v_x,$$

with

$$c = \sqrt{-p'(v)}$$

we see that the p-system takes the form of the wave equation (24), (25), but it's now the nonlinear wave equation because c depends on the unknown v. Before describing solutions of the linear and nonlinear wave equations, we consider two specific equations of state $p = p(v)$ of interest.

• First, recall the ideal gas law relating the pressure p of a gas of N molecules in a container of volume V

$$pV = NRT,$$

where T is the temperature (in degrees Kelvin so $T > 0$) and R is the universal gas constant. In this case, if we divide by the total mass M of all N molecules in the container we get

$$p\frac{V}{M} = \frac{N}{M}RT = KT,$$

where K is constant, and V/M is then specific volume v. Assuming the temperature is constant,

$$KT = \sigma^2 > 0$$

results in the isothermal equation of state

$$p(v) = \frac{\sigma^2}{v}.$$

In this case

$$p'(v) = -\frac{\sigma^2}{v^2}, \tag{26}$$

and the sound speed $c(v)$ in (24), (25) is

$$c(v) = \frac{\sigma}{v}.$$

Of course this is the speed of sound dx/dt as measured in the Lagrangian frame moving with the fluid. In the Eulerian frame (physical space) the speed of sound $\sqrt{p'(\rho)} = \sigma$ is constant in the isothermal case. It turns out that for different reasons, the equation of state p proportional to ρ is correct in the relativistic setting for pure radiation, and is taken as correct during the radiation phase of the Big Bang, after inflation, up to several hundred thousand years after the Big Bang, c.f. [**?**].

A second important case is the so called *isentropic* or γ *law* gas in which the equation of state is given by

$$p(v) = \frac{\sigma^2}{v^\gamma}, \tag{27}$$

where $\gamma > 1$. In this case the sound speed in (24), (25) should be taken to be

$$c(v) = \sqrt{-p'(v)} = \frac{\sigma\sqrt{\gamma}}{v^{\frac{\gamma+1}{2}}}.$$

More generally, motivated by the isothermal and isentropic equations of state, we define the so-called p-system as the system of equations

$$\begin{aligned} v_t - u_x &= 0 \\ u_t + p(v)_x &= 0 \end{aligned}$$

where $p(v)$ is a general function satisfying the conditions $p(v) > 0$, $p'(v) < 0$, $p''(v) > 0$. These are the salient features of p required for the theory to be essentially the same, and include, the physical cases (**??**) and (**??**).

• Our purpose now is to identify the left and right going waves associated with the nonlinear wave equation (24), (25), and to describe the nonlinear interaction of such waves. Recall that we have identified the left and right going waves $f(x+ct)$ and $g(x-ct)$ of the wave equation using the *principle of superposition*—that sums and multiples of solutions remain solutions. This method required that the sound speed c be constant so that the wave equation would be homogeneous and linear, the conditions under which the principle of superposition is valid. For the nonlinear wave equation, this method is doomed to failure because $c = c(v)$ depends on the unknown solution, the equations are nonlinear, and the principle of superposition fails for nonlinear equations. We first construct the left and right going waves of the linear wave equation by constructing the Riemann invariants for its equivalent 2×2 system (24), (25). We will then extend this method to uncover the left and right going *elementary waves* of the nonlinear wave equation, and use it to describe the nonlinear interaction of elementary waves.

By this route we discover a new sort of *nonlinear* superposition principle that describes the interaction of elementary *nonlinear* right and left going waves.

• To start, consider the *linear* wave equation

$$v_t - u_\xi = 0, \tag{28}$$

$$u_t - c^2 v_x = 0, \tag{29}$$

under the assumption that $c = const$. We first define the characteristic curves, that is, the trajectories $(x(t), t)$ of the left and right moving sound waves.

Definition 1. *The* 1*-characteristic curves, or back characteristics, are the curves* $(x(t), t)$ *moving in direction of negative* x *at characteristic (sound) speed*

$$\frac{dx}{dt} = -c. \tag{30}$$

The 2*-characteristic curves, or forward characteristics, are the curves* $(x(t), t)$ *moving in direction of positive* x *at characteristic (sound) speed*

$$\frac{dx}{dt} = c. \tag{31}$$

Now we have seen that a solution w is constant along 1- and 2- characteristics for pure left and right moving waves $f(x + ct)$ and $g(x - ct)$, but not for the superposition of two such waves. The following theorem identifies individual functions $r(u, v)$ and $s(u, v)$ called *Reimann invariants*, such that r is constant along 1-characteristics and s is constant along 2-characteristics for any solution (v, u) of (26), (27):

Theorem 2. *The function*

$$r(u, v) = u + cv \tag{32}$$

is constant along 1*-characteristics curves, and*

$$s(u, v) = u - cv \tag{33}$$

is constant along 2*-characteristic curves.*

Proof: For (30), use the notation

$$r(x,t) \equiv r(u(x,t), v(x,t)) = u(x,t) + cv(x,t)$$

and differentiate $r(x(t), t)$ along a 1-characteristic $(x(t), t)$ using $dx/dt = -c$:

$$\begin{aligned} \frac{d}{dt} r(x(t), t)) &= r_x \frac{dx}{dt} + r_t = -cr_x + r_t \\ &= -c(u + cv)_x + (u + cv)_t \\ &= c(v_t - u_x) + (u_t - c^2 u_x) = 0, \end{aligned}$$

the last two parentheses being zero by the equations (26), (27) that (v, u) are assumed to satisfy. By similar reasoning, s is constant along 2-characteristics. □

The Riemann invariants r and s tell us how to uncover the left and right going waves. Consider first the right going 2-waves. To construct them, restrict to solutions in which r is everywhere constant,

$$r(x, t) = r_0 = constant.$$

Under this assumption, we can create a localized 2-wave from initial data $s_0(x)$ localized between $x_1 \le x \le x_2$, (c.f. Figure 2),

$$s_0(x) = \begin{cases} s_L, & x < x_1 \\ s_0(x), & x_1 \le x \le x_2, \\ s_R, & x_2 < x. \end{cases} \tag{34}$$

We can continue the solution from initial values at $t = 0$ by taking s to be constant along the 2-characteristics of speed c emanating from any initial point x at $t = 0$. In particular, $r = r_0 = constant$ is consistent with the equations because all derivatives of the constant state are zero. We need only assume the initial data $s_0(x)$ is continuous to ensure the procedure produces a continuous solution moving to the right for all time. Such a simple 2-wave moving to the right is diagrammed in Figure 2.

Consider next the left going 1-waves. To construct them, restrict to solutions in which s is everywhere constant,

$$s(x,t) = s_0 = \mathit{constant}.$$

Under this assumption, we can create a localized 1-wave from initial data $r_0(x)$ localized between $x_3 \leq x \leq x_4$:

$$r_0(x) = \begin{cases} r_L, & x < x_3 \\ r_0(x), & x_3 \leq x \leq x_4, \\ r_R, & x_4 < x. \end{cases} \tag{35}$$

Again, we can continue the solution from its initial values by taking r to be constant along the 1-characteristics of speed $\dot{x} = -c$ emanating from initial points x at $t = 0$. In particular, $r = r_0 = \mathit{constant}$ is consistent with the equations because all derivatives of the constant state are zero. We need only assume the initial data $r_0(x)$ is continuous to ensure the procedure produces a continuous solution moving to the left for all time. A simple 1-wave moving to the left is diagrammed in Figure 1.

Using the Riemann invariants, we can understand how the left and right moving simple waves interact and separate. For this consider the initial data obtained by concatenating the simple wave data (32), (33), as diagrammed in Figure 3. Since r is constant along 1-characteristics and s is constant along 2-characteristics, the initial waves are separated because of the background constant state s_R on the initial 1-wave and r_L on the initial 2-wave. The waves then interact in a diamond shaped region where both r and s are changing in time, and finally the waves separate, the outgoing waves of the interaction propagating against the background constant states s_L and r_R of the the opposite Riemann invariants. (See Figure 3.)

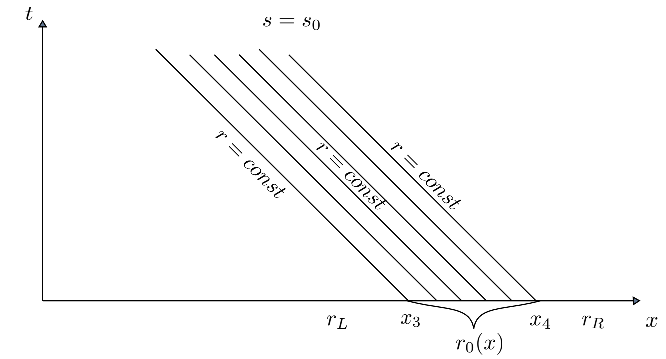

Figure 1: A Pure Linear 1-Wave

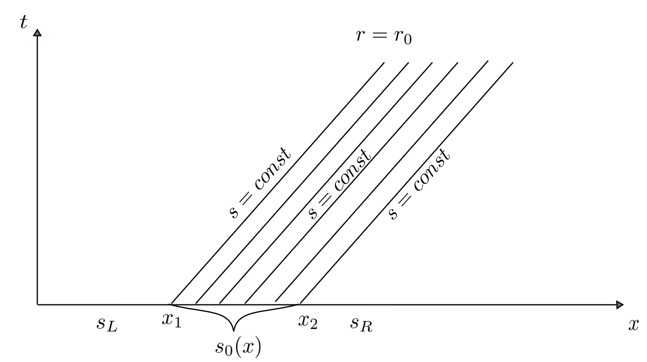

Figure 2: A Pure Linear 2-Wave

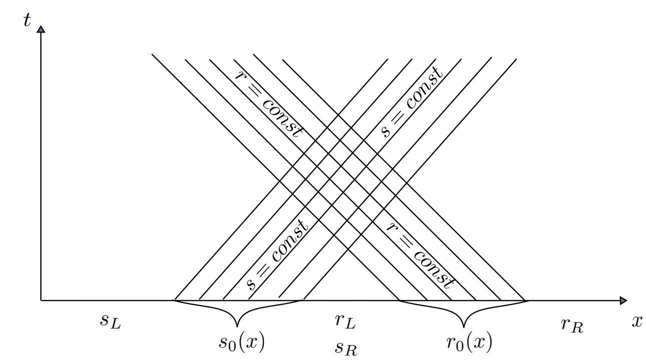

Figure 3: Interaction of Linear Waves

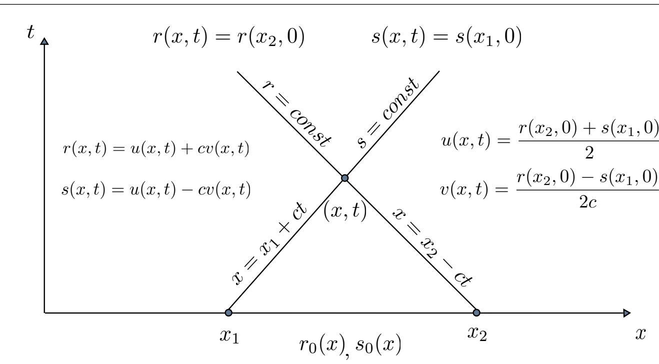

Figure 4: Formula for Solution when $c = Constant$.

Finally, we can use the Riemann invariants to derive a formula for the general solution of the initial value problem for the wave equation, c.f. Figure 4. We will see that no such formula exists for the nonlinear wave equation. For this, consider general initial data $r_0(x)$ and $s_0(x)$. Based on these given initial functions, we find the value of the solution $v(x,t)$, $u(x,t)$ at an arbitrary point (x,t). So let the two characteristic passing through (x,t) be the curve $x = x_1 + ct$, and the 1-characteristic the curve $x = x_2 - ct$. Then since r is constant along 1-characteristics and s is constant along 2-characteristics, it follows that

$$\begin{aligned} r(x,t) &= u(x,t) + cv(x,t) = r(x_2,0), \\ s(x,t) &= u(x,t) - cv(x,t) = s(x_1,0), \end{aligned}$$

so solving for $u(x,t)$ and $v(x,t)$ leads to a formula for the solution (v,u) in terms of the initial data:

$$\begin{aligned} v(x,t) &= \frac{r(x_2,0) - s(x_1,0)}{2c}, \\ u(x,t) &= \frac{r(x_2,0) + s(x_1,0)}{2}. \end{aligned}$$

• We now look to identify the left and right going waves for the *nonlinear* wave equation

$$v_t - u_\xi = 0, \tag{36}$$

$$u_t - c(v)^2 v_x = 0, \tag{37}$$

nonlinear because the sound speed $c = c(v)$ depends on the unknown solution v.

Following the procedure that worked in the linear case, define the characteristic curves again as the trajectories $(x(t),t)$ of the left and right moving sound waves.

Definition 3. *The* 1*-characteristic curves, or back characteristics, are the curves* $(x(t),t)$ *moving in direction of* *negative x at characteristic (sound) speed*

$$\frac{dx}{dt} = -c(v(x,t)). \tag{38}$$

The 2-characteristic curves, or forward characteristics, are the curves $(x(t), t)$ moving in direction of positive x at characteristic (sound) speed

$$\frac{dx}{dt} = c(v(x,t)). \tag{39}$$

But now because in the nonlinear case the sound speed $c(v)$ is not constant, the ODE's can be solved only when $v(x,t)$ is known, that is, only after a solution (v, u) of (34), (35) is given. Said differently, in the nonlinear case the characteristic curves depend on the solution—as they should because sound speeds really do depend on the density. So assume a solution, and hence $v(x,t)$ is given, and the 1-and 2-characteristic curves are defined. The following theorem identifies the Riemann invariants $r(u,v)$ and $s(u,v)$ satisfying the condition that r is constant along 1-characteristics and s is constant along 2-characteristics for any solution (v,u) of (26), (27).

Theorem 4. *Let $h(v)$ be an anti-derivative of $c(v)$,*

$$h'(v) = c(v).$$

Then for any solution $v(x,t), u(x,t)$ of (38), (39), the function

$$r(u,v) = u + h(v) \tag{40}$$

is constant along 1-characteristics curves, and

$$s(u,v) = u - h(v) \tag{41}$$

is constant along 2-characteristic curves.

Proof: For (38), use the notation

$$r(x,t) \equiv r(u(x,t), v(x,t)) = u(x,t) + h(v(x,t)),$$

and differentiate $r(x(t),t)$ along a 1-characteristic $(x(t),t)$ using $dx/dt = -c(v)$:

$$\begin{aligned}\frac{d}{dt}r(x(t),t)) &= r_x\frac{dx}{dt} + r_t = -c(v)r_x + r_t\\ &= -c(v)(u+h(v))_x + (u+h(v))_t\\ &= -c(v)(u_x + h'(v)v_x) + (u_t + h'(v)v_t)\\ &= c(v)(v_t - u_x) + (u_t - c(v)^2 v_x) = 0,\end{aligned}$$

where we have used $h'(v) = c(v)$ and that the last two parentheses are zero by the equations (34), (35). By similar reasoning, s is constant along 2-characteristics. $\Box$

The Riemann invariants r and s enable us to unravel the left and right going waves for the nonlinear wave equation. Consider first the right going 2-waves. To construct them, restrict again to solutions in which r is everywhere constant,

$$r(x,t) = r_0 = constant.$$

Under this assumption, we can again create a localized 2-wave from initial data $s_0(x)$ localized between $x_1 \le x \le x_2$:

$$s_0(x) = \begin{cases} s_L, & x < x_1\\ s_0(x), & x_1 \le x \le x_2,\\ s_R, & x_2 < x.\end{cases} \tag{42}$$

We can continue the solution from initial values at $t=0$ by taking s to be constant along the 2-characteristics of speed $c(v)$ emanating from any initial point x at $t=0$. In particular, $r = r_0 = constant$ is again consistent with the equations because all derivatives of the constant state are still zero. We need only assume the initial data $s_0(x)$ is continuous to ensure the procedure produces a continuous solution moving to the right for all time. A simple nonlinear 2-wave moving to the right is diagrammed in Figure 6.

However, remarkably, even though the speed $\dot{x}(t) = c(v)$ of the characteristic curves $(x(t), t)$ depends on the solution $v(x,t)$, along the simple 2-wave, the speed of each individual characteristic *is* constant, (albeit a different constant for each characteristic), because both r and s and hence both u and v are constant along characteristics. That is, $r = r_0$ fixes r, and s is constant along each 2-characteristic by our theorem. The main point is then that the simple 2-wave is propagating against a constant background value r_0 of the opposite Riemann invariant. As soon as the opposite Riemann invariant r starts changing along the 2-characteristic, the speeds of the 2-characteristic curves will change due to the nonlinearities, and so the characteristic curves will no longer be straight lines. This change of wave speed is indicated in Figure 7.

Consider next the left going 1-waves. To construct them, restrict to solutions in which s is everywhere constant,

$$s(x,t) = s_0 = constant.$$

Under this assumption, we can create a localized 1-wave from initial data $r_0(x)$ localized between $x_3 \leq x \leq x_4$:

$$r_0(x) = \begin{cases} r_L, & x < x_3 \\ r_0(x), & x_3 \leq x \leq x_4, \\ r_R, & x_4 < x. \end{cases} \tag{43}$$

Again, we can continue the solution from its initial values by taking r to be constant along the 1-characteristics of speed $\dot{x}(t) = -c(v)$ emanating from initial points x at $t = 0$. In particular, $r = r_0 = constant$ is consistent with the equations because all derivatives of the constant state are zero. We need only assume the initial data $r_0(x)$ is continuous to ensure the procedure produces a continuous solution moving to the left for all time. A simple nonlinear 1-wave moving to the left is diagrammed in Figure 5.

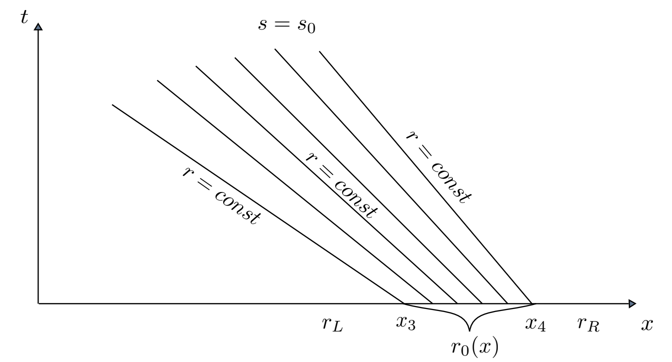

Figure 5: A Pure Nonlinear 1-Wave

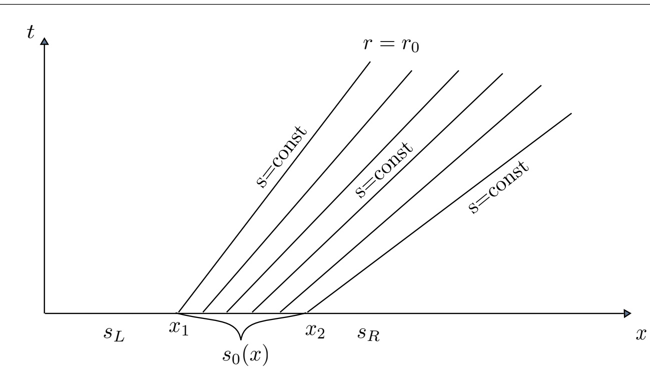

Figure 6: A Pure Nonlinear 2-Wave

Using the Riemann invariants, we can understand how the left and right moving nonlinear simple waves interact and separate. For this consider the initial data obtained by concatenating the simple wave data (40), (41), as diagrammed in Figure 7. Since r is constant along 1-characteristics and s is constant along 2-characteristics, the initial waves are separated because of the background constant state s_R on the initial 1-wave and r_L on the initial 2-wave. Before interaction, the both characteristics propagate with a constant speed and hence are straight lines, because both Riemann invariants r and s, and hence u and v, and hence $c(v)$, are constant along each characteristic. The speeds of different characteristics will differ because the constant value of v and hence $c(v)$ changes from characteristic to characteristic. The waves interact in a diamond shaped region where both r and s, as well as the speed of the characteristics along which they are constant, are changing in time. Then finally the waves separate from the interaction, and in the final state the outgoing waves of the interaction propagating against the background constant states s_L and r_R of the the opposite Riemann invariants. After interaction, r, s, u, v are all constant along each characteristic, so each characteristic is again a straight line in (x, t)-plane moving at a constant speed. But due to nonlinearities, the speeds of different characteristics are different according to the value of v, and hence of $c(v)$, along them, c.f. Figure 7.

Finally, it remains to see if we can use the Riemann invariants to derive a formula for the general solution of the initial value problem like we did for the linear wave equation. In fact, no such formula exists because we don't have the characteristic curves until we have the solution, and we can't get a formula for the solution in terms of the initial data like we did above without knowing the characteristic curves. But what we can do is get information as to how the solution depends on the initial data. For example, since we know r and s are constant along characteristics, the solution cannot generate any values of r and s that weren't already present in the initial data. Thus, for example, if there are constants $\underline{r} < \bar{r}$, $\underline{s} < \bar{s}$ such that the initial data satisfies

$$\underline{r} \leq r_0(x) \leq \bar{r}, \tag{44}$$

$$\underline{s} \leq s_0(x) \leq \bar{s}, \tag{45}$$

then we know the solution stays within these bounds in r and s for all time. If further, the value of $0 < \underline{c} \leq c(v) \leq \bar{c}$ for these initial values, then we have a bound on the speed of the sound waves for all time. By this reasoning, we can see clearly how the solution depends on the initial data.

For example, consider general initial data $r_0(x)$ and $s_0(x)$ within the bounds (42),(43). Based on these given initial functions, we find the value of the solution $v(x,t)$, $u(x,t)$ at an arbitrary point (x,t). So assuming two characteristics never cross before time t, (no shock formation before time t!), there must be unique 1 and 2-characteristic curves passing through the point (x,t). Since the speed of these characteristics are bounded away from zero and finite, these characteristic curves must emanate from points at time $t = 0$. So say the 1-characteristic through (x,t) emanates from $x = x_2$ and the 2-characteristic through (x,t) emanates from $x = x_1$ at time $t = 0$. Then since r is constant along 1-characteristics and s is constant along 2-characteristics, it follows again that

$$r(x,t) = u(x,t) + h(v(x,t)) = r(x_2, 0),$$
$$s(x,t) = u(x,t) - h(v(x,t)) = s(x_1, 0),$$

so solving for $u(x,t)$ and $v(x,t)$ leads to a formula for the solution (v,u) in terms of the initial data:

$$h(v(x,t)) = \tfrac{r(x_2,0)-s(x_1,0)}{2c}, \tag{46}$$

$$u(x,t) = \tfrac{r(x_2,0)+s(x_1,0)}{2}. \tag{47}$$

Since $h'(v) = c > \underline{c} > 0$, values of $h(v)$ uniquely determine values of v, (i.e., h has an inverse!), so we can use (44) to solve for $v(x,t)$ from knowledge of h.

Now (44) and (45) don't give us a formula for the solution because we need the solution to define the characteristic curves to continue the solution in the first place. But assuming there is a solution, (these ideas could be developed into an existence proof in a graduate class in PDE's), (44) and (45) tell us how the solution depends on initial values. In particular, the value of the solution at (x_0,t_0) cannot depend on values of the initial data further that $\bar{c}t_0$ away. That is, the solution at (x_0,t_0) depends only on values of the initial data $r_0(x), s_0(x)$ at $t=0$ for x within the range

$$x_0 - \bar{c}t_0 \le x \le x_0 + \bar{c}t_0. \tag{48}$$

In this sense we say that *information cannot propagate faster than speed $\bar{c}$.* More precisely, we make the following definition:

Definition 5. *The domain of dependence of a point (x_0,t_0) in a solution of a PDE is the set of all x such that values of the initial data at x can affect the solution at (x_0,t_0).*

Conclude: For our nonlinear wave equation, the domain of dependence of (x_0,t_0) is contained within the interval (46). The domain of dependence of a solution is very important in numerical computations of solutions. In particular, the famous CFL (Courant-Freidrichs-Levy) condition states that a numerical method will go unstable and will not converge to a solution if the domain of dependence of the numerical scheme does not contain the domain of dependence of the solution being simulated.

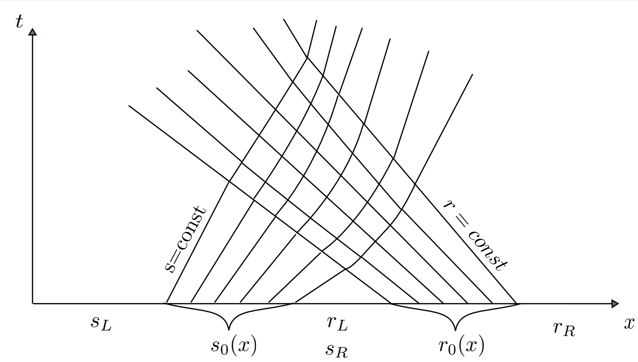

Figure 7: Interaction of Nonlinear Waves

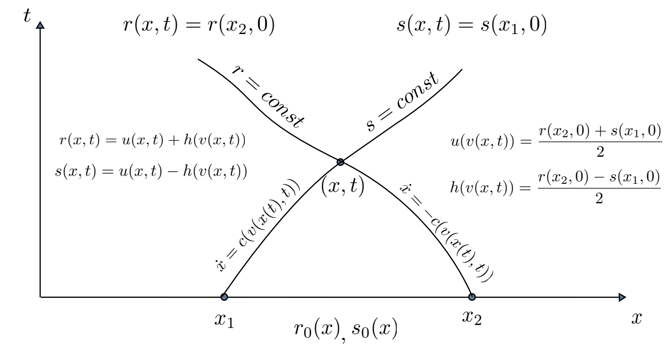

Figure 8: General Nonlinear Solution Allowing c Not Constant