8–Simple Waves and the Nonlinear Theory of Sound

MATH 22C

1. Introduction

In the last section we showed that the nonlinear wave equation admits nonlinear elementary waves that propagate to the left and to the right. But, because the equations are nonlinear, different solutions cannot be superimposed like solutions of the linear wave equation. That is, when the sound speed $c = c(v)$ depends on the solution v, general nonlinear solutions do not decompose exactly into a sum of left and right going waves as do solutions of the linear wave equation, when c is constant. We ended the last section by describing a sort of *nonlinear superposition* of left and right going waves for the nonlinear wave equation, based on Riemann invariants: functions r and s of the unknowns (u, v) that are constant along characteristic curves, the path of sound waves $\dot{x} = c(v)$ in the (x, t)-plane. By this we explained the propagation of left and right going nonlinear waves along the characteristics $ds/dt = \pm c$, when c depends on the solution.

In this section we describe a very general method for constructing the left and right going simple waves based on eigenvalues and eigenvectors that decompose the solution, something like the eigensolutions of linear constant coefficient ODE's. We apply the method to the linear and nonlinear wave equations and re-derive the simple left and right going wave from this eigenvalue point of view. This more general framework extends essentially unchanged to very general nonlinear systems of conservation laws. As a final payoff, we will find the simple waves for the compressible Euler equations as originally given in physical space,

$$\rho_t + (\rho u)_x = 0, \tag{1}$$

$$(\rho u)_t + (\rho u^2 + p(\rho))_x = 0, \tag{2}$$

and use them to calculate the speed of sound for the fully nonlinear equations. Recall that we got the speed of sound $c = \sqrt{p'(\rho)}$ for the *linearized* Euler equations, and we got the speed of sound $c = \sqrt{-p'(v)}$ for the p-system in Lagrangian coordinates, (that's the speed $d\xi/dt$ relative to a transformed spatial coordinate ξ), but we have not yet derived the speed of sound directly from (1) and (2). We end the section by giving sharp conditions for shock wave formation in simple waves, thereby demonstrating shock wave formation in the nonlinear wave equation and the compressible Euler equations of gas dynamics.

• Recall, then, that the wave equation $w_{tt} - c^2 w_{xx} = 0$ can be written as a first order system,

$$v_t - u_x = 0, \tag{3}$$

$$u_t - c^2 v_x = 0, \tag{4}$$

using $w_t = u$, $w_x = v$. This is valid for both the linear wave equation $c = const$, as well as the nonlinear wave equation $c = c(v)$, and so applies to the p-system of gas dynamics when $v = 1/\rho$ and u is the original velocity. To discover the simple waves by eigenvalue methods, write (3), (4) as a first order matrix system,

$$\begin{pmatrix} v \\ u \end{pmatrix}_t + \begin{pmatrix} 0 & -1 \\ -c^2 & 0 \end{pmatrix} \begin{pmatrix} v \\ u \end{pmatrix}_x = 0,$$

of general form

$$U_t + A(U)U_x = 0, \tag{5}$$

where

$$U = \begin{pmatrix} v \\ u \end{pmatrix},$$

and A is the 2×2 matrix

$$A(U) = \begin{pmatrix} 0 & -1 \\ -c^2 & 0 \end{pmatrix},$$

depending on U through the sound speed $c(v)$.

• The first order nonlinear PDE (5) has a structure comparable to a constant coefficient linear system of ODE's

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix},$$

of form

$$\dot{\mathbf{y}} = A\mathbf{y}$$

where

$$\mathbf{y} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

and for which a basis of solutions can be determined by the eigensolutions

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} e^{\lambda t}$$

where $\left(\lambda, R = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}\right)$ is any eigenpair satisfying

$$AR = \lambda R.$$

That is,

$$\dot{\mathbf{y}} = \lambda R e^{\lambda t} = A R e^{\lambda t} = A\mathbf{y}.$$

We now develop an analogous eigenvalue method for (5), the big difference being that our new methods will apply not to linear, but to fully *nonlinear* systems, and the systems will be PDE's, not ODE's.

2. Eigenfamilies of Simple Waves

• So consider a general first order nonlinear system of PDE's of the form (5),

$$U_t + A(U)U_x = 0,$$

and let (λ, R) be an eigenpair for A satisfying

$$AR = \lambda R,$$

where $R \neq 0$. Now since the 2×2 matrix $A = A(U)$ depends on the unknowns $U = (v, u)$, it follows that both λ and R must depend on U as well,

$$\lambda = \lambda(U) \in \mathcal{R}, \qquad R = R(U) = \begin{pmatrix} r_1(U) \\ r_2(U) \end{pmatrix},$$

so that

$$A(U)R(U) = \lambda(U)R(U),$$

holds for each U in some set $U \in \mathcal{U}$, the set of values of the unknowns U in which we look for solutions. Scalars λ and vectors R that depend on U are called *scalar* and *vector fields*, respectively.

Assume now that both $\lambda(U)$ and $R(U)$ are real valued, and depend smoothly on U, so that we can take derivatives of λ and R with respect to U as needed. Our goal is to make precise, and then verify, the following principle for constructing the simple wave solutions of system (5) associated with the λ-characteristic family:

The Simple-Wave Principle: *A λ-simple wave that solves (5) is constructed from states on an* **integral curve** *of the eigenvector $R(U)$ by asking that each state U on the integral curve propagate in the (x, t)-plane at its eigenspeed $\dot{x} = \lambda(U)$.*

In particular, the state U, and consequently the speed $\lambda(U)$, remains constant along the λ-characteristic $(x(t), t)$,

$$\dot{x} = \lambda(U).$$

Conclude that for a simple wave, the characteristic curve of speed $\dot{x} = \lambda(U)$ is a straight line of speed λ in the (x, t)-plane along which the solution is constant.

The *Simple-Wave Principle* provides a general version of the principle we already know for the Burgers equation $u_t + uu_x = 0$. That is, for Burgers equation, the eigenvalue is $\lambda = u$, the eigenvector is 1, the integral curve is the u-axis, and the principle says that states u propagate as constant along lines of speed $\lambda(u) = u$, which we already know is correct.

We first define an **integral curve** of $R(U)$.

Definition 1. *An* **integral curve** *of the vector field $R(U)$ is a curve $\mathbf{R}$ in the (u, v) plane with tangent vector R at each point; i.e., a curve whose parameterization $U(\xi) = (v(\xi), u(\xi))$ satisfies*

$$U'(\xi) = R(U(\xi)),$$

at every point. Indeed, we know a unique such curve always exist through each point U_0 because it is the unique solution of the initial value problem

$$\begin{aligned} U' &= R(U), \\ U(0) &= U_0, \end{aligned} \tag{6}$$

an autonomous ODE in the two unknowns $U = (v, u)$. Note that changing the parameterization of the integral curve merely changes the length of the eigenvector R, and hence eigenvectors are nonzero, but otherwise independent of length. So we can without loss of generality choose any parameterization, and in particular, if λ changes monotonically along the integral curve $\mathbf{R}$, then we can take $\xi = \lambda$ as the parameter. We now take $\xi = \lambda$ as the parameterization of the integral curve of $\mathbf{R}$, but this requires the assumption that λ change monotonically along R. We make this a definition, (first due to Peter Lax in about 1960).

Definition 2. *We say the eigenfield* (λ, R) *is* genuinely nonlinear *if* λ *changes monotonically along the integral curve* $\mathbf{R}$*, so that* $\mathbf{R}$ *can be parameterized as* $U(\lambda)$*. This is equivalent to requiring that the directional derivative of* λ *in the direction of* R *be everywhere nonzero,*

$$\nabla\lambda \cdot R \neq 0. \tag{7}$$

As we will see, the condition of *genuine nonlinearity* applies and is correct for the characteristic fields of the nonlinear wave equation and the p-system, as well as the original 2×2 compressible Euler equations, but it does not apply to the characteristic fields of the wave equation. (Nor does it apply to one of the characteristic fields of the compressible Euler equations that emerges when we incorporate the energy equation, not considered here.) To cover the case of these additional fields, we make the following definition, also due to Peter Lax:

Definition 3. *We say the eigenfield* (λ, R) *is* linearly degenerate *if* λ *is constant along the integral curve* $\mathbf{R}$*. This is the case when the directional derivative of* λ *in the direction of* R *is identically zero,*

$$\nabla\lambda \cdot R \equiv 0. \tag{8}$$

We now verify the *Simple-Wave Principle* for both cases (7) and (8).

- Assume first the case that an eigenfield (λ, R) of (5) is *genuinely nonlinear* in the sense of (7), so that $\lambda = \xi$ and

$U(\lambda)$ can be taken as the parameterization of the integral curve $\mathbf{R}$. According to the *Simple-Wave Principle*, if we make each state $U \in \mathbf{R}$ propagate along a straight line whose speed is the eigenvalue associated with that state,

$$\frac{dx}{dt} = \lambda(U),$$

then we have a simple solution of the nonlinear equations associated with the λ-eigenvalue. To see this, assume $U(\lambda)$ smoothly parameterizes the states on the integral curve $\mathbf{R}$ by it's eigenvalue $\lambda(U)$ at that state. We look for $\lambda(x,t)$ as a function of (x,t) such that $U(\lambda(x,t))$ solves (5). But according to the principle, the state $U(\lambda)$ should propagate as constant along straight lines of speed $\dot{x} = \lambda$, so $\lambda(x,t)$ should also propagate as constant along lines of speed λ. But this is just our condition that λ solve the Burgers equation! We can now put it all together.

Assume $\lambda(x,t)$ is a smooth solution of Burgers equation

$$\lambda_t + \lambda\lambda_x = 0.$$

We prove $U(\lambda(x,t))$ solves (5). But

$$\begin{aligned} \frac{\partial}{\partial t}U(\lambda(x,t)) &= U'(\lambda)\lambda_t, \\ \frac{\partial}{\partial x}U(\lambda(x,t)) &= U'(\lambda)\lambda_x, \end{aligned} \tag{9}$$

so

$$\begin{aligned} U_t + AU_x &= U'(\lambda)\lambda_t + AU'(\lambda)\lambda_x \\ &= U'(\lambda)\lambda_t + \lambda U'(\lambda)\lambda_x \\ &= U'(\lambda)\left\{\lambda_t + \lambda\lambda_x\right\} = 0, \end{aligned}$$

because $U'(\lambda) = R$ is the eigenvector tangent to $\mathbf{R}$ at $U(\lambda)$, and $\lambda(x,t)$ is assumed to solve Burgers equation $\lambda_t + \lambda\lambda_x = 0$.

Conclude that when (λ, R) is genuinely nonlinear so λ can be taken as a parameterization of the integral curve of R, a λ-simple wave is created by making states on the integral curve of an eigenvector propagate as constant along lines in the (x, t)-plane moving at the speed equal to the eigenvalue at that state–and asking the eigenvalue propagate as constant along lines of speed λ means $\lambda(x, t)$ solves Burgers equation.

• Consider now the second case when the eigen-field (λ, R) of (5) is *linearly degenerate* in the sense of (8), the case when λ is constant along the integral curve $U(\xi)$ of R. In this case assume $\xi(x, t)$ solves the linear transport equation

$$\xi_t + \lambda \xi_x = 0$$

expressing that ξ propagates as constant along lines of (constant) speed λ. In this case,

$$\begin{aligned} \frac{\partial}{\partial t} U(\xi(x,t)) &= U'(\xi)\xi_t, \\ \frac{\partial}{\partial x} U(\xi(x,t)) &= U'(\xi)\xi_x, \end{aligned} \tag{10}$$

so

$$\begin{aligned} U_t + AU_x &= U'(\xi)\xi_t + AU'(\xi)\xi_x \\ &= U'(\xi)\xi_t + \lambda U'(\xi)\xi_x \\ &= U'(\xi)\{\xi_t + \lambda \xi_x\} = 0, \end{aligned}$$

because $U'(\xi) = R$ is the eigenvector tangent to $\mathbf{R}$ at $U(\xi)$, and $\xi(x, t)$ solves the transport equation $\xi_t + \lambda \xi_x = 0$.

Conclude: Stated simply: A λ-simple wave is created by making states on the integral curve of an eigenvector propagate as constant along lines in the (x, t)-plane moving at the speed equal to the eigenvalue at that state. The Simple-Wave Principle applies to *genuinely nonlinear* and *linearly degenerate* characteristic fields, and in fact can be extended to all cases in between. All we required was that the eigenvalue λ be real valued. In fact, we never had to assume the matrix was 2×2 either. The Simple-Wave Principle is valid for any $n \times n$ system of form (5) with n unknowns $U = (u_1, ..., u_n)$ an $n \times n$ matrix A. All we needed was that $\lambda(U)$ be real valued and $A(U)R(U) = \lambda(U)R(U)$ for each U, and from this it follows that the Simple-Wave Principle applies.

3. Examples of Simple Waves

We now construct the families of simple waves for the linear and nonlinear wave equations, and then for the 2×2 compressible Euler equations expressed in the original Eulerian coordinates. The latter gives us the first calculation of the speed of sound for the fully nonlinear equations of gas dynamics.

Consider first the linear and nonlinear wave equation written as the first order system

$$\begin{pmatrix} v \\ u \end{pmatrix}_t + \begin{pmatrix} 0 & -1 \\ -c^2 & 0 \end{pmatrix} \begin{pmatrix} v \\ u \end{pmatrix}_x = 0,$$

of the form

$$U_t + AU_x = 0.$$

This is linear when $c = const$ and nonlinear when $c = c(v)$. In either case, to find the eigenvalues of A take

$$det \begin{pmatrix} -\lambda & -1 \\ -c^2 & -\lambda \end{pmatrix} = \lambda^2 - c^2 = 0,$$

so as we expected, whether linear or nonlinear, the eigenvalues are the wave speeds

$$\lambda_{\pm} = \pm c.$$

To find the associated eigenvectors, set $r_1 = 1$ (eigenvectors can be rescaled to any length), and solve

$$\begin{pmatrix} \mp c & -1 \\ -c^2 & \mp c \end{pmatrix} \begin{pmatrix} 1 \\ r_2 \end{pmatrix} = 0,$$

which gives

$$r_2 = \mp c.$$

Thus the two eigenfamilies

$$(-c, R_-), \quad (c, R_+)$$

are

$$\lambda_- \equiv \lambda_1 = -c, \quad R_- \equiv R_1 = \begin{pmatrix} 1 \\ c \end{pmatrix},$$

and

$$\lambda_+ \equiv \lambda_2 = +c, \quad R_+ \equiv R_2 = \begin{pmatrix} 1 \\ -c \end{pmatrix}.$$

The difference between the linear wave equation and the nonlinear wave equation is whether c is constant (linear) or $c = c(v)$ depends on v (nonlinear). In either case, we have:

Lemma 4. *The integral curves of the eigenvectors $R_1 \equiv R_1(U)$ are the curves in the (v, u)-plane along which the opposite Riemann invariant $s(u, v) = const$, and the integral curves of the eigenvectors $R_2 \equiv R_2(U)$ are the curves in the (v, u)-plane along which the opposite Riemann invariant $r(u, v) = const$, where (as in Section 7,)*

$$r(v, u) = u + h(v) \qquad (1 - \text{Riemann invariant}) \tag{11}$$

$$s(v, u) = u - h(v) \qquad (2 - \text{Riemann invariant}) \tag{12}$$

where

$$h'(v) = c(v). \tag{13}$$

Proof: Consider first the linear wave equation, $c = constant$, $\lambda_\pm = \pm c = const.$ and $R_\pm = (1, \mp c) = const.$ For the integral curves of $R_1 = (1, c)$ to be the curves $s(u, v) \equiv$

$u - cv = const$, we need only verify R_1 is orthogonal to ∇s in the (v,u)-plane. But (recall $\nabla s = (\partial s/\partial v, \partial s/\partial u)$ because $U = (v,u)$),

$$\begin{aligned} \nabla s \cdot R_1 &= \nabla(u - cv) \cdot (1, c) \\ &= (-c, 1) \cdot (1, c) = 0. \end{aligned}$$

Similarly

$$\begin{aligned} \nabla r \cdot R_2 &= \nabla(u + cv) \cdot (1, -c) \\ &= (c, 1) \cdot (1, -c) = 0. \end{aligned}$$

It follows by the Simple-Wave Principle that the states in a left going 1-simple wave lie at $s = const.$ and propagate at speed $\lambda_1 = -c$, while the states on a right going 2-simple wave lie at $r = const$ and propagate at speed $\lambda_2 = +c$, confirming what we established in the last section by the Riemann invariants directly. In the linear case, the wave speeds and eigenvectors are constant, and the Riemann invariants are straight lines in the (v,u)-plane.

Consider next the *nonlinear* wave equation, so $c = c(v)$, $\lambda_\pm = \pm c(v)$ and $R_\pm = (1, \mp c)$ all depend on (v,u). For the integral curves of $R_1 = (1, c(v))$ to be the curves $s(u,v) \equiv u - h(v) = const$, we need only verify R_1 is orthogonal to ∇s in the (v,u)-plane. But

$$\begin{aligned} \nabla s \cdot R_1 &= \nabla(u - h(v)) \cdot (1, c(v)) \\ &= (-h'(v), 1) \cdot (1, c(v)) = 0, \end{aligned}$$

because $h'(v) = c(v)$. Similarly

$$\begin{aligned} \nabla r \cdot R_2 &= \nabla(u + h(v)) \cdot (1, -c(v)) \\ &= (h'(v), 1) \cdot (1, -c(v)) = 0. \end{aligned}$$

It follows by the Simple-Wave Principle that the states in a left going 1-simple wave lie at $s = const.$ and propagate at speed $\lambda_1 = -c$, while the states on a right going 2-simple wave lie at $r = const$ and propagate at speed $\lambda_2 =$

$+c$, again confirming the Riemann invariant analysis in the last section. In the nonlinear case, the wave speeds and eigenvectors depend on the solution, but like the linear case, the Riemann invariants define known curves in the (v, u)-plane which describe the simple wave solutions of the p-system.

Consider for example the case of the isothermal p-system

$$\begin{aligned} v_t - u_x &= 0, \\ u_t + p(v)_x &= 0, \end{aligned} \tag{14}$$

the case when

$$p = \sigma^2/v,$$

so

$$c(v) = \sqrt{-p'(v)} = \frac{\sigma}{v},$$

and

$$h(v) = \sigma \ln(v).$$

In this case, the Riemann invariants are given by

$$\begin{aligned} r(u, v) &= u + \sigma \ln(v), \\ s(u, v) &= u - \sigma \ln(v). \end{aligned}$$

The integral curves $\mathbf{R}_1$ and $\mathbf{R}_2$ for the case $p = \sigma^2 \rho$ are diagrammed in Figure 1. The arrows point in the direction of increasing λ. The 2- and 1-simple waves corresponding to the states in bold face in Figure 1, are diagrammed in Figures 2 and 3.

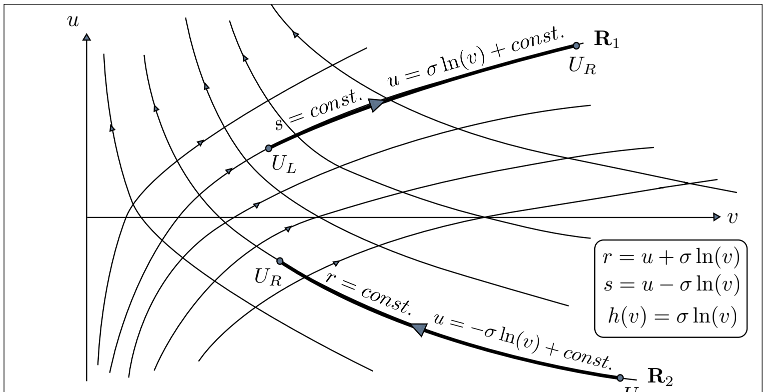

Figure 1: $\mathbf{R}_1$ and $\mathbf{R}_2$ for the p-system when $p = \sigma^2 \rho$.

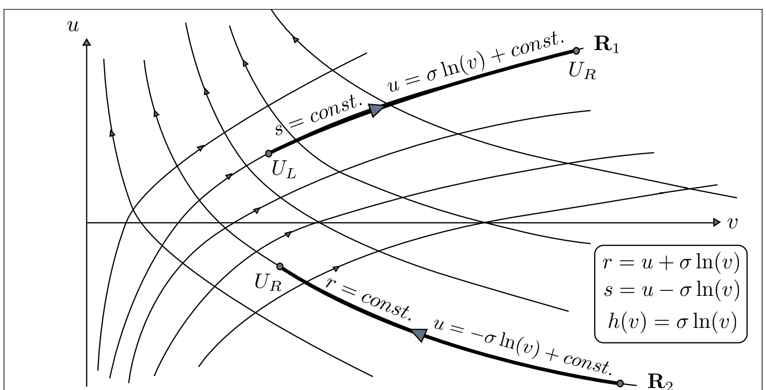

Figure 1: $\mathbf{R}_1$ and $\mathbf{R}_2$ for the p-system when $p = \sigma^2 \rho$.

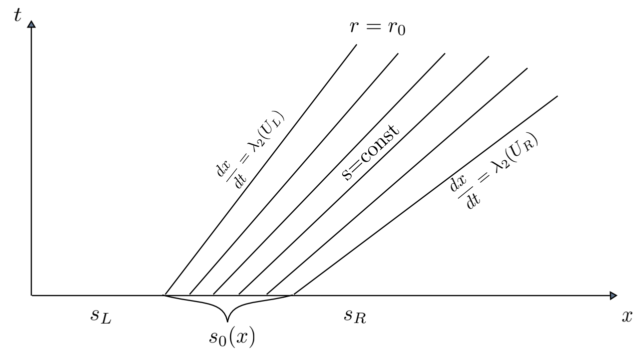

Figure 2: The 2-Simple Wave of Figure 1.

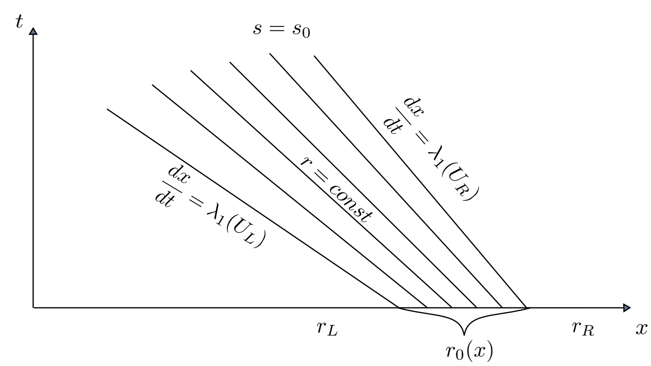

Figure 3: The 1-Simple Wave of Figure 1.

• As a final example, we compute the simple waves for the original compressible Euler equations in Eulerian coordinates (physical space) (1), (2),

$$\rho_t + (\rho u)_x = 0,$$
$$(\rho u)_t + (\rho u^2 + p(\rho))_x = 0.$$

The result provides at last the nonlinear speed of sound, and gives a rather complete description of the simple waves, that is, the non-interacting sound waves, that propagate down a 1-dimensional shock tube.

To start, write (1), (2) in matrix form (5) by setting $G = \rho u$, and get the equivalent system

$$\rho_t + G_x = 0, \tag{15}$$
$$G_t + \left(\frac{G^2}{\rho} + p(\rho)\right)_x = 0. \tag{16}$$

Differentiating the x-derivative in the second equation gives

$$\begin{aligned}\left(\frac{G^2}{\rho} + p(\rho)\right)_x &= \frac{2G}{\rho}G_x - \frac{G^2}{\rho^2}\rho_x + p'(\rho)\rho_x \\ &= (-u^2 + p'(\rho))\rho_x + 2uG_x,\end{aligned}$$

so that the matrix form reads

$$\begin{pmatrix} \rho \\ G \end{pmatrix}_t + \begin{pmatrix} 0 & 1 \\ -u^2 + p'(\rho) & 2u \end{pmatrix}\begin{pmatrix} \rho \\ G \end{pmatrix}_x = 0,$$

which is now of the desired form

$$U_t + AU_x = 0,$$

with

$$U = \begin{pmatrix} \rho \\ G \end{pmatrix},$$

and

$$A = \begin{pmatrix} 0 & 1 \\ -u^2 + p'(\rho) & 2u \end{pmatrix}.$$

To find the eigenvalues of A, solve

$$\begin{aligned} 0 = Det\,(A - \lambda I) &= Det \begin{pmatrix} -\lambda & 1 \\ -u^2 + p'(\rho) & 2u - \lambda \end{pmatrix} \\ &= \lambda^2 - 2u\lambda + u^2 - p'(\rho) = 0, \end{aligned}$$

yielding

$$\lambda_{\pm} = u \pm c,$$

where c is now the *Eulerian* sound speed

$$c = \sqrt{p'(\rho)}.$$

(Recall that the Lagrangian sound speed was $\sqrt{-p'(v)}$ which differs from this by a factor of ρ!) For the eigenvectors use

$$\begin{pmatrix} -\lambda & 1 \\ -u^2 + p'(\rho) & 2u - \lambda \end{pmatrix} \begin{pmatrix} 1 \\ r \end{pmatrix} = 0,$$

which gives[1]

$$r = \lambda.$$

Thus the two eigenfamilies

$$(\lambda_-, R_-), \quad (\lambda_+, R_+)$$

are

$$\lambda_- \equiv \lambda_1 = u - c, \quad R_- \equiv R_1 = \begin{pmatrix} 1 \\ u - c \end{pmatrix},$$

and

$$\lambda_+ \equiv \lambda_2 = u + c, \quad R_+ \equiv R_2 = \begin{pmatrix} 1 \\ u + c \end{pmatrix}.$$

Keep in mind that the components of vectors R_1 and R_2, while expressed in terms of ρ and u, provide values for directions in the (ρ, G)-plane, not the (ρ, u)-plane, where $G = \rho u$. To finish the description of the simple waves for the shock tube problem it remains only to describe the integral curves $\mathbf{R}_1$ and $\mathbf{R}_2$. Taking the cue from the p-system, we show

[1]Note $2u - \lambda_{\pm} = u \mp c = \lambda_{\mp}$, and $-u^2 + p'(\rho) = -(\lambda_-)(\lambda_+)$.

Lemma 5. *The integral curves of the eigenvectors R_1 are the curves $\mathbf{R}_1$ in the (ρ, G)-plane along which the opposite 2-Riemann invariant $s(\rho, G) = const$, and the integral curves $\mathbf{R}_2$ of the eigenvectors R_2 are the curves in the (ρ, G)-plane along which the opposite 1-Riemann invariant $r(\rho, G) = const$, where*

$$r(\rho, G) = \frac{G}{\rho} + h(1/\rho) = u + h(v), \tag{17}$$

$$s(\rho, G) = \frac{G}{\rho} - h(1/\rho) = u - h(v), \tag{18}$$

so that r and s are the Riemann invariants of the p-system expressed as functions of G and ρ because $v = 1/\rho$.

Proof: *As before, in order for the integral curves of $R_1 = (1, u + c(\rho))$ to be the curves $s(\rho, G) \equiv G/\rho - h(1/\rho) = const$, we need only verify R_1 is orthogonal to ∇s. To start note that*

$$\frac{dh}{d\rho} = \frac{dh}{dv}\frac{dv}{d\rho} = -\frac{c(\rho)}{\rho} = -\frac{\sqrt{p'(\rho)}}{\rho}. \tag{19}$$

Thus

$$\nabla s = \left(\frac{\partial s}{\partial \rho}, \frac{\partial s}{\partial G}\right) = \left(-\frac{G}{\rho^2} + \frac{c}{\rho}, \frac{1}{\rho}\right) = \frac{1}{\rho}\left(-u + c, 1\right)$$

in the (ρ, G)-plane, because $\bar{h}'(\rho) = -c(\rho)/\rho$. Using this we find

$$\nabla s \cdot R_1 \;=\; \frac{1}{\rho}(-u + c, 1) \cdot (1, u - c) = 0,$$

as claimed. Similarly,

$$\nabla r = \left(\frac{\partial r}{\partial \rho}, \frac{\partial r}{\partial G}\right) = \left(-\frac{G}{\rho^2} - c, \frac{1}{\rho}\right) = \frac{1}{\rho}\left(-u - c, 1\right),$$

so

$$\nabla r \cdot R_2 \;=\; \frac{1}{\rho}(-u-c,1)\cdot(1,u+c) = 0,$$

as claimed again, and the lemma is verified. □

• **Conclude:** It follows by the Simple-Wave Principle that the states in a left going 1-simple wave lie at $s = const.$ and propagate at speed $\lambda_1 = u - c$, while the states on a right going 2-simple wave lie at $r = const.$ and propagate at speed $\lambda_2 = u + c$, where $r = u + h(1/\rho)$ and $s = u - h(1/\rho)$. This is consistent with what we found for the p-system, but now x is physical distance along the shock tube, and c is the sound speed relative to the shock tube in real physical space, (the Eulerian sound speed). Thus, one of the big payoff's here is that we have found the true and exact speed of sound for sound wave propagation down a shock tube, (something Newton tried and failed to do!). Namely, the sound waves propagate at eigenspeeds $\lambda_{\pm} = u \pm c$, so the speed of sound, which is the speed over and above the velocity u of the gase, is

$$\text{Speed of Sound} = \sqrt{p'(\rho)}.$$

So again, in this nonlinear case, the wave speeds and eigenvectors depend on the solution, but the Riemann invariants are known curves in the (ρ, u)-plane which describe the simple wave solutions of the compressible Euler equations in physical space.

• As an explicit example, consider the case of isothermal gas dynamics, $p = \sigma^2\rho$. In this case the sound speed $c = \sqrt{p'(\rho)} = \sigma$ is constant, $h(1/\rho) = -\sigma \ln \rho$, and the simple wave solutions of the compressible Euler equations

$$\rho_t + (\rho u)_x = 0, \tag{20}$$

$$(\rho u)_t + (\rho u^2 + \sigma^2 \rho)_x = 0, \tag{21}$$

are completely described by the Riemann invariants

$$r(\rho, u) = u - \sigma \ln \rho,$$
$$s(\rho, u) = u + \sigma \ln \rho.$$

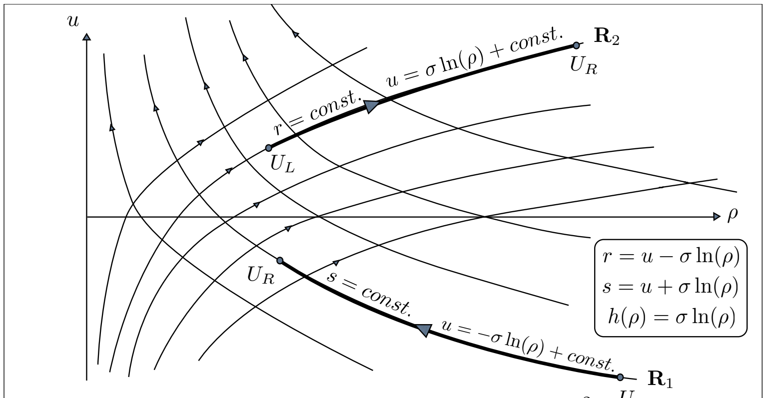

Figure 4: $\mathbf{R}_1$ and $\mathbf{R}_2$ for Compressible Euler when $p = \sigma^2 \rho$.

Figure 4 gives a complete description of the simple waves that propagate down a shock tube as modeled by the nonlinear compressible Euler equations. One can see that Figure 4 could have been obtained from Figure 1 by simply replacing the v-axis by the ρ-axis, $\rho = 1/v$. This is no surprise for we know the p-system describes the same physical states as the compressible Euler equation, only the spatial coordinate is different. The simple waves diagrammed in Figures 2 and 3 will be qualitatively the same as for the p-system, but the sound speeds at which constant values of s and r propagate on 1-simple waves and 2-simple waves, respectively, will be the Eulerian eigenvalues $\lambda_1 = u - c$ and $\lambda_2 = u + c$.

• **Characteristics of Euler:** Although the simple waves for the compressible Euler equations (20), (21) are completely described by Figure 4, a general solution involves the interaction of nonlinear waves. For interacting solutions, it remains to complete the picture of the compressible Euler equations by defining the characteristics curves along which sound waves propagate in general, and to prove that s is constant along 1-characteristics and r is constant along 2-characteristics as we saw in Section 7 for the p-system. Thus the value of a solution $U(x,t) = (\rho(x,t), u(x,t))$ of (20), (21) is determined from the initial data by the s-value it receives from the 1-characteristic and the r-value it receives from the 2-characteristic that passes through (x,t), as in Figure 5. Keep in mind that the picture is complete for simple waves, but for interacting solutions, the picture is incomplete in the sense that the speeds of the characteristics depend on the solution, and the description of the solution depends on the characteristics. But the method of characteristics and Riemann invariants gives us an accurate qualitative picture of nonlinear sound wave propagation as described by the compressible Euler equations. This nonlinear theory of wave propagation extends and refines the classical theory of linear wave propagation as described by the linear wave equation $\rho_{tt} - c^2\rho_{xx} = 0$, $c = \sqrt{p'(\rho_0)}$, the equation you get when you linearize the compressible Euler equations about a state of constant density ρ_0. We finish with the following theorem which completes the picture diagrammed in Figure 5 of nonlinear sound wave propagation down a shock tube, as modeled by the compressible Euler equations of gas dynamics.

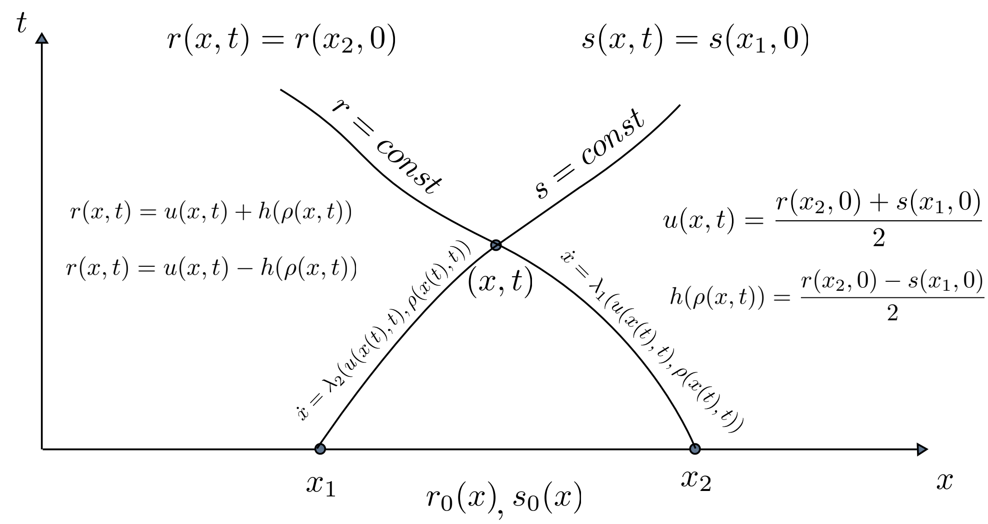

Figure 5: Nonlinear Sound Wave Propagation in Compressible Euler.

Theorem 6. *For a general solution* $U(x,t) = (\rho(x,t), u(x,t))$ *of the compressible Euler equations (20), (21), the* 1*-Riemann invariant* $r = u + h(1/\rho)$ *is constant along* 1*-characteristic curves* $(x(t), t)$ *satisfying*

$$\dot{x} = \lambda_1 = u - c,$$

and the 2*-Riemann invariant* $s = u - h(1/\rho)$ *is constant along* 2*-characteristic curves* $(x(t), t)$ *satisfying*

$$\dot{x} = \lambda_2 = u + c.$$

Proof: It suffices to show that for a given solution of (20), (21),

$$\frac{d}{dt} r(x(t), t) = 0,$$

for 1-characteristic curves , and

$$\frac{d}{dt}s(x(t),t) = 0,$$

for 2-characteristic curves $(x(t), t)$. For example,

$$\dot{x} = u - c,$$

along a 1-characteristic, so

$$\begin{aligned}
\frac{d}{dt}r(x(t),t) &= r_t + r_x\dot{x} = r_t + (u-c)r_x = r_t + ur_x - cr_x \\
&= u_t + h(1/\rho)_t + uu_x + uh(1/\rho)_x - cu_x - ch(1/\rho)_x \\
&= \{u_t + uu_x - ch(1/\rho)_x\}_1 \qquad (22) \\
&\qquad -\frac{c}{\rho}\{\rho_t + u\rho_x + u_x\rho\}_2 .
\end{aligned}$$

(Note for the last term, $-\frac{c}{\rho}u_x\rho = -cu_x$.) To finish we show that $\{\cdot\}_1$ and $\{\cdot\}_2$ vanish by equations (21) and (20), respectively. For $\{\cdot\}_2$ use

$$\{\cdot\}_2 = \{\rho_t + (u\rho)_x\}_2 = 0$$

by (20). For $\{\cdot\}_1$ use (19) to write

$$-ch(1/\rho)_x = \frac{c^2}{\rho}\rho_x = \frac{p(\rho)_x}{\rho},$$

so

$$\{\cdot\}_1 = \left\{u_t + uu_x + \frac{p(\rho)_x}{\rho}\right\}_1 = 0,$$

by the transport equation for u, c.f. equation (12) of Section 7. That is, starting from (21) we have

$$\begin{aligned}
0 &= (\rho u)_t + (\rho u^2 + p)_x \\
&= u(\rho_t + (\rho u)_x) + \rho\left(u_t + uu_x + \frac{p(\rho)_x}{\rho}\right) = \rho\{\cdot\}_1 ,
\end{aligned}$$

because the first parenthesis is zero by (20). We thereby conclude that $\frac{d}{dt}r(x(t),t) = 0$ along 1-characteristics curves, so r is constant along 1-characteristics, as claimed. The proof that s is constant along 2-characteristics is similar. $\Box$

4. Shock Wave Formation in Simple Waves

We have described a method for constructing the nonlinear version of the right and left going waves of the linear wave equation. Such waves are called *simple* waves or *non-interacting* waves because a general solution consists of the nonlinear interaction of simple waves from different families. Simple waves are constructed from eigenvalues and eigenvectors that decompose the matrix $A(U)$ in a first order nonlinear system of form

$$U_t + A(U)U_x = 0. \tag{23}$$

Although we explicitly solved for the two families of simple waves in the 2×2 p-system and compressible Euler equations, the method applies to any system of form (24) with n unknowns $U = (u_1, ..., u_n)$ and $A(U)$ an $n \times n$ matrix. The conclusion is that if $(\lambda(U), R(U))$ is an eigen-field satisfying

$$A(U)R(U) = \lambda(U)R(U),$$

(say for all $U \in \mathbf{U} \subset \mathcal{R}^2$), and $U(\xi)$ is a parameterization of an integral curve $\mathbf{R}$ of $R(U)$, so $U(\xi)$ solves the autonomous ODE

$$U'(\xi) = R(U(\xi)), \quad \underline{\xi} \leq \xi \leq \bar{\xi},$$

then we obtain simple (non-interacting) solutions $U(x,t)$ of (24) called λ-simple waves, by asking that states $U(\xi)$ on $\mathbf{R}$ propagate at speed $\lambda(U(\xi))$ in the xt-plane. We called this the *Simple-Wave Principle*. If further the eigen-family is genuinely nonlinear,

$$\nabla\lambda \cdot R \neq 0,$$

so that λ is monotone along the integral curve $\mathbf{R}$ and can therefore be taken as the parameter $\xi = \lambda$, $U = U(\lambda)$, then the λ-simple waves are solutions $U(\lambda(x,t))$ where $\lambda(x,t)$ is any solution of Burgers equation

$$\lambda_t + \lambda\lambda_x = 0.$$

This is easy to see because Burgers equation expresses that $\lambda(x,t)$ should be constant along lines of speed λ, and hence when λ solves $\lambda_t + \lambda\lambda_x = 0$, states $U(\lambda(x,t))$ must be constant along lines of speed λ as well, so the *Simple-Wave Principle* holds. We then found that the nonlinear wave equation in the form of the p-system, as well as the compressible Euler equations in physical space, are both 2×2 systems with two distinct eigen-families (λ_1, R_1) and (λ_2, R_2), both of which are genuinely nonlinear, the 1-integral curves are $s = \mathit{const}$, the 2-integral curves are $r = \mathit{const}$, and r is constant along 1-characteristics, s is constant along 2-characteristics. These two systems represent equivalent formulations of wave propagation down a shock tube.

We now end this section and begin the study of shock waves by giving a sharp estimate for the time at which two characteristics must intersect in the Burgers equation. The result thus characterizes shock wave formation in λ-simple waves, and in particular establishes that shock waves can form in any $n \times n$ system of form (24) that admits a genuinely nonlinear family of simple waves. Shock waves form in Burgers equation whenever the solution is *compressive*; i.e, whenever two characteristics approach each other. Since solutions of Burgers equation $u_t + uu_x = 0$ are constant along characteristics, a solution can be continued only if it is allowed to become a *discontinuous* function beyond the time when the first two characteristics intersect. A solution that suffers a jump discontinuity across a curve in the (x,t)-plane is called a shock wave. In the next section we describe the shock waves solutions of the p-system of gas dynamics. The theory of shock waves complements the theory of simple waves through the resolution of the so-called *Riemann problem*. The Riemann problem gives the complete description of wave propagation down a shock tube.

• Recall the nonlinear Burgers equation

$$u_t + uu_x = 0. \tag{24}$$

Since this can be written

$$\nabla_{\overrightarrow{(u,1)}}\, u(x,t) = 0,$$

the gradient of a solution vanishes in direction $\overrightarrow{(u,1)}$ in the (x,t) plane, so solutions are constant along lines of speed $dx/dt = u$. Thus a solution starting from initial data $u(x,0) = u_0(x)$ with $u_0(x_1) = u_1$ will take the constant value $u(x,t) = u_1$ along the line of speed $dx/dt = u_1$ emanating from $x = x_1$ at $t = 0$, given in the (x,t)-plane by

$$x = u_1 t + x_1.$$

This characteristic will intersect a second characteristic

$$x = u_2 t + x_2,$$

emanating from $x_2 > x_1$, at time $t = T$, when both have the same x value; i.e., when

$$u_1 T + x_1 = u_2 T + x_2,$$

or

$$T = -\frac{x_2 - x_1}{u_2 - u_1}.$$

Since $x_2 - x_1 > 0$, this will happen at $T > 0$ whenever $u_2 < u_1$. This implies that a shock wave will form if, initially, a value of u on the left is larger than a value of u on the right, a condition expressing that two characteristics are *compressive*. The sharp result is the following theorem which gives the precise *first* time of shock wave formation in terms of the initial data $u_0(x)$.

Theorem 7. *A smooth solution $u(x,t)$ of the initial value problem*

$$u_t + uu_x = 0, \tag{25}$$

$$u(x,0) = u_0(x), \tag{26}$$

cannot be continued as a smooth solution beyond time

$$T = \frac{1}{Max\,\{-u_0'(x)\}}, \tag{27}$$

where the Max is taken over all values of x where $u'(x) \leq 0$. (In particular, if $u'(x) \geq 0$ for all $x \in \mathcal{R}$, then (27) imposes no restriction.)

The theorem is a direct consequence of the following lemma:

Lemma 8. *If smooth initial data $u_0(x)$ has a negative derivative $u_0'(x_0) = u_0' < 0$ at $x = x_0$, then the solution $u(x,t)$ of (25) will suffer an infinite derivative along the characteristic emanating from $x = x_0$ before time $-1/u_0'$. That is,*

$$\lim_{t \to T} u_x(x(t),t) = -\infty$$

along the characteristic

$$x(t) = u_0\, t + x_0, \tag{28}$$

at some time $T > 0$,

$$T \leq \frac{1}{-u_0'}.$$

Here $u_0 \equiv u_0(x_0)$.

Thus (27) simply identifies the characteristic with the most negative derivative at $t = 0$, thereby identifying the shortest time to shock formation.

Proof of Lemma: To get an equation for u_x along the characteristic $(x(t), t)$ given in (28), differentiate the solution $u_x(x(t), t)$ with respect to t along the characteristic (28) to obtain

$$\frac{du_x}{dt} \equiv \frac{d}{dt} u_x(u_0 t + x_0, t) = u_{xx} u_0 + u_{xt}. \tag{29}$$

But since $u(x, t)$ is assumed to satisfy the Burgers equation, we can differentiate Burgers with respect to x to get another equation satisfied by u_x, namely

$$u_{xt} + u u_{xx} + u_x^2 = 0.$$

In particular, this holds all along the characteristic curve (28), so along (28) we have

$$u_0 u_{xx} + u_{xt} = -u_x^2,$$

because $u = u_0$ on (28). Using this in (30) then gives

$$\frac{du_x}{dt} = -u_x^2, \tag{30}$$

and ODE for u_x along the curve (28). But (30) is a Ricotti equation with the solution

$$u_x(x(t), t) = \frac{1}{\frac{1}{u_0'} + t}.$$

Since $u_0' < 0$, it follows that

$$lim_{t \to T}\, u_x(x(t), t) = -\infty,$$

where

$$T = \frac{1}{-u_0'}.$$

Note however that this argument has assumed the solution exists along the characteristic all the way up until time T. (A shock wave created by intersection with another characteristic may have destroyed it before this time!) But this does not flaw the proof because we only claim the solution cannot exist *beyond* time T. The structure of the argument, then, is to assume the solution exists up until time T, and then prove it cannot go beyond. This completes the proof of the Lemma. □

5. General Hyperbolic First Order Systems

An $n \times n$ first order system of the form

$$U_t + A(U)U_x = 0,$$

with $U = (u_1, ..., u_n)$ and $A(U)$ an $n \times n$ matrix whose entries depend on U, is said to be *strictly hyperbolic* if at each U the matrix admits n real and distinct eigenvalues

$$\lambda_1(U) < \cdots < \lambda_n(U),$$

a condition that implies that the corresponding eigenvectors $\{R_-(U), ..., R_n(U)\}$ form a basis at each point U. We say the system is *non-degenerate* if each characteristic field (λ_k, R_k) is either *genuinely nonlinear* or *linearly degenerate*. In this case, there exists n-families of simple waves obtained by the Simple-Wave Principle, and for a given solution, the k-characteristic curves are the curve $(x(t), t)$ in the (x, t)-plane that propagate at speed λ_k,

$$\dot{x} = \lambda_k(U).$$

These are the speeds at which information propagates, and because all λ_k are finite, the equations exhibit finite speed of propagation of information. Such equations are called *hyperbolic* because they act something like the wave equation $u_{tt} - c^2 u_{xx} = 0$, which looks something like $t^2 - c^2 x^2 = 1$, which is an hyperbola. (More generally, $u_{tt} + c^2 u_{xx} = 0$ is called *elliptic* and $u_t - c^2 u_{xx} = 0$ is called *parabolic*. This terminology is somewhat justified by looking at the solutions of form $e^{kx - \omega t}$ which solve each equation. The constraints are $\omega^2 - k^2 = 0$ for the wave equation, $\omega - k^2$ for the heat equation and $\omega^2 + k^2 = 0$ for Laplace's equation.) We have developed the the general setting for shock wave theory sufficient to contain all of the classical conservation laws of fluid mechanics. It was put forth by researchers at the Courant Institute in New York City in the decades after World War II.