MIDTERM EXAM I—SOLUTIONS
Math 25
Temple-F06

Write solutions on the paper provided. Put your name on this exam sheet, and staple it to the front of your finished exam. Do Not Write On This Exam Sheet.

Problem 1. (20pts) Use the field axioms for the real numbers to prove that if $a \in \mathbf{R}$, then $-a = (-1)a$. (Give a field axiom reason for every step. Prove any lemma you use.)

Solution:

$$a \cdot 0 = a(-1 + -1) = a(-1) + a \cdot 1 = (-1)a + a. \tag{1}$$

But $a \cdot 0 = a \cdot (0 + 0) = a \cdot 0 + a \cdot 0$, so adding $-(a \cdot 0)$ to both sides gives $0 = a \cdot 0$. Putting this into the LHS of (1) gives $0 = (-1)a + a$, and thus adding $-a$ to both sides of this gives

$$-a = (-1)a.$$

For full credit you need to give field axiom reasons.
Problem 2. (20pts) Give the precise mathematical definitions for the following:

(a) A sequence of real numbers. *A sequence is a function* $s : \mathbf{N} \to \mathbf{R}$.

(b) $\lim_{n\to\infty} s_n = s_0$. $\forall \epsilon > 0\ \exists N \in \mathbf{N}\ st\ \forall n > N,\ |s_n - s_0| < \epsilon.$

(c) $\lim_{n\to\infty} s_n = +\infty$ $\forall M > 0\ \exists N \in \mathbf{N}\ st\ \forall n > N, s_n > M.$

Problem 3. (20pts) Assume that $\lim_{n\to\infty} s_n = s_0$ and $\lim_{n\to\infty} t_n = t_0$ where s_0 and t_0 are finite real numbers. Give a careful proof that $lim_{n\to\infty}(s_n + t_n) = (s_0 + t_0)$.

Solution: Fix $\epsilon > 0$. We find N such that $n > N$ implies that $|(s_n + t_n) - (s_0 + t_0)| < \epsilon$. But since $s_n \to s_0$, there exists N_1 such that $n > N_1$ implies $|s_n - s_0| < \epsilon/2$, and since $t_n \to t_0$, there exists N_2 such that $n > N_2$ implies $|t_n - t_0| < \epsilon/2$. Choosing $N = Max\{N_1, N_2\}$, and using the triangle inequality we have that for $n > N$,

$$|(s_n + t_n) - (s_0 + t_0)| \leq |s_n - s_0| + |t_n - t_0)| < \epsilon/2 + \epsilon/2 = \epsilon.$$

Problem 4. (20pts) Let r be a real number such that $|r| < 1$, and let s_n denote the sequence of partial sums $s_n = \sum_{k=1}^{n} r^k = 1 + r + r^2 + r^3 + \cdots + r^n$. Derive a formula for s_n that does not involve a summation, and use it to evaluate $\lim_{n\to\infty} s_n$.

Solution: $s_n(1 - r) = s_n - rs_n = 1 - r^{n+1}$. Solving for s_n gives $s_n = \frac{1-r^{n+1}}{1-r}$. From this we see that $\lim_{n\to\infty} s_n = \frac{1}{1-r}$.

Problem 5. (20pts) The completeness axiom states that every set of real numbers bounded from above has a least upper bound.

(a) Give the precise definition of an upper bound for a set $A \subset \mathbf{R}$.

A number $x \in \mathbf{R}$ is an upper bound for A if $\forall\ a \in A$ we have $a \leq x$.

(b) Give the precise definition of the least upper bound for a set $A \subset \mathbf{R}$.

A number $x \in \mathbf{R}$ is a least upper bound for A if x is an upper bound for A, and $x \leq y$ for any other upper bound y for A.

(c) Let s_n be a sequence of real numbers that is bounded from above, and assume that for every $n \in \mathbf{N}$, $s_{n+1} \geq s_n$. Use the completeness axiom to characterize the limit, and then give a careful proof that the sequence s_n converges to that limit. (You may use any theorem we have proven so far!)

This is just the monotone convergence theorem in Section 10.