MIDTERM EXAM II
Math 25
Temple-F06

Write solutions on the paper provided. Put your name on this exam sheet, and staple it to the front of your finished exam. Do Not Write On This Exam Sheet.

Problem 1. (a) (4pts) Give the precise definition of an upper bound for a set $A \subset \mathbf{R}$.

(b) (4pts) Give the precise definition of the least upper bound for a set $A \subset \mathbf{R}$.

(c) (4pts) State the Completeness Axiom regarding a $A \subset \mathbf{R}$.

(d) (8pts) Let s_n be a non-decreasing sequence of real numbers that is bounded from above. Give a careful proof that the sequence s_n converges.

Problem 2. Let s_n be a sequence of real numbers.

(a) (2pts) Give the precise definition for $s_n \to s_0$ in the case when $s_0 \in \mathbf{R}$, and when $s_0 = +\infty$.

(b) (4pts) Define what it means for s_n to be a Cauchy sequence.

(c) (4pts) Give the negation of the statement "s_n is *Cauchy*".

(d) (5pts) Prove directly: If s_n converges to a real number s_0, then s_n is Cauchy.

(e) (5pts) Prove directly: If $s_n \to +\infty$, then s_n is not Cauchy.

Problem 3. Let s_n be a sequence of real numbers.

(a) (6pts) Define $\underline{s}_N$ and $\bar{s}_N$, the approximate $\liminf$ and approximate $\limsup$ of s_n, respectively.

(b) (6pts) Define $\underline{s} = \liminf s_n$ and $\bar{s} = \limsup s_n$ in terms of $\underline{s}_N$ and $\bar{s}_N$, respectively.

(c) (8pts) Use $\leq$ to give the correct inequalities that order the set $\{\underline{s}_N, \bar{s}_N, \underline{s}, \bar{s}\}$, and prove the right most inequality.

Problem 4. Let $s_n = \{1 + (-1)^n\} e^{1/n}$

(a) (10pts) Find the $\liminf s_n$ and $\limsup s_n$.

(b) (10pts) Define a subsequence that converges to $\limsup s_n$.

Problem 5. Let s_n be a sequence of real numbers.

(a) (10pts) Give an example of a sequence of real numbers s_n whose subsequential limit set is exactly the set $\{1/n : n \in \mathbf{N}\} \cup \{0\}$. (You may use a diagram to define your sequence.)

(b) (10pts) Prove that there is no sequence of real numbers whose subsequential limit set S is the set $\{1/n : n \in \mathbf{N}\}$. (You may use any result in the book.)

Math 25 Midterm Soln's

#1 (a) x is an upper bound for $A \subseteq \mathbb{R}$ if

$x \geq a \quad \forall a \in A$

(b) x is a LUB for A if $x \not> y$ $\forall$ upperbound y of A

(c) Every set bded from above has a LUB

(d) $S_{n+1} \geq S_n \quad \forall n$. Let $S_0 = \text{LUB}\{S_n\}$. We prove that $S_n \to S_0$. Fix $\varepsilon > 0$. We find N st $n \geq N \Rightarrow |S_n - S_0| < \varepsilon$. But S_0 the $\text{LUB}\{S_n\}$ implies $S_0 \geq S_n \; \forall n$, and $\exists N$ st $S_N > S_0 - \varepsilon$. Thus for $n > N$ we have $S_0 - \varepsilon < S_N \leq S_n \leq S_0$

so $|S_n - S_0| < \varepsilon$ ✓

#2 (a) $S_n \to S_0 \in \mathbb{R}$ if $\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ st

$\forall n > N \ |S_n - S_0| < \varepsilon$

$S_n \to +\infty$ if $\forall M > 0 \ \exists N \in \mathbb{N}$ st

$\forall n > N \quad S_n > M$.

(b) S_n is Cauchy if $\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$

st $\forall m, n > N$, $|S_n - S_m| < \varepsilon$

(c) $\neg(S_n \text{ is Cauchy}) \equiv$

$\equiv \neg(\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \text{ st } \forall m, n > N \ |S_n - S_m| < \varepsilon)$

$\equiv \exists \varepsilon > 0 \ \forall N \in \mathbb{N} \ \exists m, n > N \ |S_n - S_m| \geq \varepsilon$

d) Assume $S_n \to S_0 \in \mathbb{R}$. We show S_n is Cauchy. Fix $\varepsilon > 0$. We find N st $m,n > N \Rightarrow |S_n - S_m| < \varepsilon$. Choose N st $n > N \Rightarrow |S_n - S_0| < \frac{\varepsilon}{2}$. Then $m,n > N \Rightarrow$

$$|S_n - S_m| \leq |S_n - S_0 + S_0 - S_m| \leq |S_n - S_0| + |S_m - S_0|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \leq \varepsilon \checkmark$$

e) Assume $S_n \to +\infty$. Choose $\varepsilon = 1$. We prove that $\forall N \in \mathbb{N}\ \exists m,n > N$ st $|S_n - S_m| \geq \varepsilon$. Fix N. Choose any $n > N$. Then since $S_n \to +\infty$, setting $M = S_n + 1$ we know $\exists m > N$ st $S_m > M = S_n + 1$. Thus $|S_n - S_m| > 1$, proving S_n not Cauchy.

(4)

#3 (a) $\overline{S}_N = \text{Sup}\{S_n : n > N\}$

$\underline{S}_N = \text{Inf}\{S_n : n > N\}$ (6)

(b) $\overline{S} = \lim_{N\to\infty} \overline{S}_N$

$\underline{S} = \lim_{N\to\infty} \underline{S}_N$ (6)

(c) $\underline{S}_N \le \underline{S} \le \overline{S} \le \overline{S}_N$ (4)

Prove: $\overline{S} \le \overline{S}_N$

$\overline{S}_N = \text{Sup}\{S_n : n > N\} \ge \text{Sup}\{S_n : n > M\} = \overline{S}_M$

if $M \ge N$. $\therefore$ $\overline{S}_N$ is non-increasing sequ

$\Rightarrow$ $\overline{S}_N \ge \overline{S}$ its limit. (4)

#4 $s_n = (1+(-1)^n)e^{1/n}$

(a) $\liminf s_n = 0$, $\limsup s_n = 2$

because $e^{1/n} \longrightarrow 1$

(b) $s_{n_k} \longrightarrow 2$ for $n_k = 2k$

#5 (a) Example: Define s_n as follows –

$1, 1, \frac{1}{2}, 1, \frac{1}{2}, \frac{1}{3}, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$

$s_1\ s_2\ s_3\ s_4 \ldots$

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(other possible examples exist)

(b) $A = \{\frac{1}{n} : n \in \mathbb{N}\}$ cannot be S' for any s_n because it is not closed – i.e., $\frac{1}{n} \longrightarrow 0$ but $0 \notin A$.

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