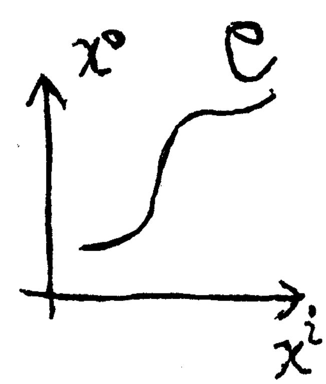
Introduction/Special Relativity

Recall: Einsteins Theory of Gravity takes the assumption that the gravitational field is given by a symmetric, non-degenerate bilinear form g defined on spacetime. A coordinate system $\underset{\sim}{x} \equiv (x^0, x^1, x^2, x^3)$ given on spacetime determines the components $g_{ij}(\underset{\sim}{x})$, a symmetric non-deg matrix at each $\underset{\sim}{x}$. This determines the differential $ds \equiv$ arclength

$$ds^2 = \pm g_{ij}\, dx^i dx^j.$$

- Given a curve $\underset{\sim}{x}(\xi)$ on spacetime, the g-length of the curve is given by

$$ds = \sqrt{\pm g_{ij}\, dx^i dx^j} = \sqrt{\pm g_{ij}\, \dot{x}^i \dot{x}^j}\, d\xi$$



since

$$dx^i = \dot{x}^i d\xi.$$

$$\Rightarrow \qquad \Delta s = \int_{\xi_1}^{\xi_2} \sqrt{\pm g_{ij}\, \dot{x}^i \dot{x}^j}\, d\xi$$

Assumption 1: $ds = c\,d\tau$ where $d\tau$ is the proper time change (aging time) for an observer traversing $\mathcal{C}$.

E.g., if $[x^i]$ = meters, $[t]$ = seconds, $[x^0 = ct]$ meters

By taking $ds = c\,d\tau$, $x^0 = ct$, x^0 and ds have units of meters $\Rightarrow$ space & time have dim. of length.

Assumption 2: (timelike) Paths of minimal or critical length $\equiv$ geodesics of g are the free fall paths

Assumption 3: Non-rotating frames are $\parallel$-transported by Connection for g along free-fall paths. (Diff Geom)

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Assumption of Special Relativity: $\exists$ a global coordinate system $\underset{\sim}{x}$ such that

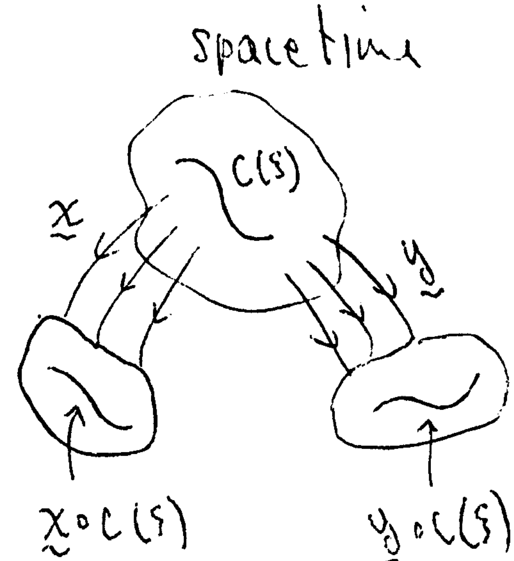
$$g_{ij} = \begin{bmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix} \equiv \eta_{ij}$$

everywhere. $\Rightarrow$ "spacetime is flat"

$\underset{\sim}{x}$ $\equiv$ Lorentz frame is orthonormal frame for g.

- Q: how do g_{ij} transform under a change of coordinates $y^i = y^i(\underset{\sim}{x})$?

Ans: Let $C(\xi)$ be a curve in spacetime. In



$\underset{\sim}{x}$-coords: $x^i(\xi) \equiv x^i \circ C(\xi)$

$\underset{\sim}{y}$-coords: $y^\alpha(\xi) = y^\alpha \circ C(\xi)$

We want:

$$\int_{\xi_1}^{\xi_2} \sqrt{\pm g_{ij}\, \dot{x}^i \dot{x}^j}\, d\xi = \int_{\xi_1}^{\xi_2} \sqrt{\pm \bar{g}\, \dot{y}\, \dot{y}}\, d\xi$$

so it suffices to make

$$g_{ij}\dot{x}^i \dot{x}^j = \bar{g}_{\alpha\beta}\dot{y}^\alpha \dot{y}^\beta.$$

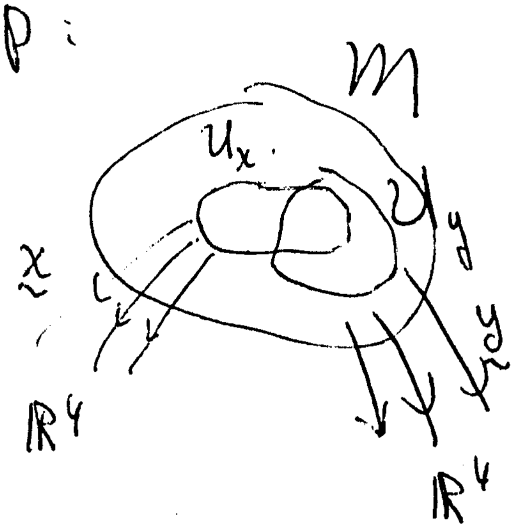
But $y^\alpha = y^\alpha(\underset{\sim}{x}) \Rightarrow \boxed{\dot{y}^\alpha = \frac{\partial y^\alpha}{\partial x^i}\dot{x}^i}$

$$\Rightarrow g_{ij}\dot{x}^i\dot{x}^j = \bar{g}_{\alpha\beta}\dot{y}^\alpha\dot{y}^\beta = \bar{g}_{\alpha\beta}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\dot{x}^i\dot{x}^j$$

$$\Rightarrow \boxed{g_{ij} = \bar{g}_{\alpha\beta}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}}$$

• **Spacetime Manifold:** Precisely, we assume Spacetime is a 4-d Manifold of Events which can be covered locally with coordinate charts $\underset{\sim}{x} : \mathcal{U} \longrightarrow \mathbb{R}^4$ st charts are smooth on the overlap:

$\underset{\sim}{x} \circ \underset{\sim}{y}^{-1}$ smooth, 1-1, onto, smooth inverse.



The metric g is given, & transforms by

$$g_{ij} = \bar{g}_{\alpha\beta} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j}$$

• Tangent Space TM^4: Suppose we wish to diff a function $f(p)$ along a curve $c(\xi) \subset M^4$. We can do this in a coord. System $\underset{\sim}{x}$:

$$\frac{d}{d\xi}\, \underset{\sim}{x} \circ$$

- Tangent Space: We'd like a tangent vector to a curve $c(\xi)$ to be "$X = \frac{dc}{d\xi}$", makes no sense. However, all we use "$\frac{dc}{d\xi}$" for is to differentiate functions along the curve

Let $f(p)$ denote scalar fn

$$f : \mathcal{M}^4 \longrightarrow \mathbb{R}$$

Then deriv. of f along c:

$$\frac{d}{d\xi} f(c(\xi)) \text{ is well defined.}$$

Let $f \circ \underset{\sim}{x}^{-1} : \mathbb{R}^4 \rightarrow \mathbb{R} \equiv$ "the fn f written in $\underset{\sim}{x}$-coord's"

$f \circ \underset{\sim}{y}^{-1} : \mathbb{R}^4 \rightarrow \mathbb{R} \equiv$ "the fn f written in $\underset{\sim}{y}$-coords"

Then

$$\frac{df}{d\xi}(c(\xi)) \equiv \frac{d}{d\xi}(f \circ \underset{\sim}{x}^{-1})(\overbrace{\underset{\sim}{x} \circ c(\xi)}^{\text{curve in x-coords}}) = \frac{\partial f}{\partial x^i} \dot{x}^i \equiv \dot{x}^i \frac{\partial}{\partial x^i} f$$

$$= \frac{d}{d\xi}(f \circ \underset{\sim}{y}^{-1})(\underset{\sim}{y} \circ c(\xi)) = \frac{\partial f}{\partial y^\alpha} \dot{y}^\alpha \equiv \dot{y}^\alpha \frac{\partial}{\partial y^\alpha}$$

<u>Defn</u>: The tangent vector to curve $\mathcal{C}$ is the differential operator $X = \dot{x}^i \frac{\partial}{\partial x^i} = \dot{y}^\alpha \frac{\partial}{\partial y^\alpha}$

where
$$\dot{x}^i \frac{\partial y^\alpha}{\partial x^i} = \dot{y}^\alpha .$$

This implies that (FIP)
$$\frac{\partial}{\partial x^i} = \frac{\partial y^\alpha}{\partial x^i}\frac{\partial}{\partial y^\alpha} ,$$ ← An equality at each pt p

which give the transformation laws for vectors. We think of $\left\{ \frac{\partial}{\partial x^0}, \dots, \frac{\partial}{\partial x^3} \right\}$ as the $\underset{\sim}{x}$-coord basis for the tangent space $T_p M^4$ at pt p.

<u>Thm</u>: $T_p M^4$ is a vector space of dimension 4 (FIP)

- <u>Defn</u>: if $X = a^i \frac{\partial}{\partial x^i}$, $Y = b^i \frac{\partial}{\partial x^i}$, then
$$\langle X, Y \rangle = g_{ij} a^i b^j \quad \text{so that}$$
$$|X| = |g_{ij} a^i b^j|^{1/2}$$

"vector tangent to curve $\mathcal{C}(\xi)$"
$$\Delta s = \int_{\xi_1}^{\xi_2} |g_{ij} \dot{x}^i \dot{x}^j|^{1/2} d\xi = \int_{\xi_1}^{\xi_2} \left| \frac{dc}{d\xi} \right| d\xi$$

☒ <u>SPECIAL</u> <u>RELATIVITY</u>: Assume $g_{ij} = \eta_{ij} = \mathrm{diag}(-1,1,1,1)$ in $\underset{\sim}{x}$-coordinates. E.g., if we assume

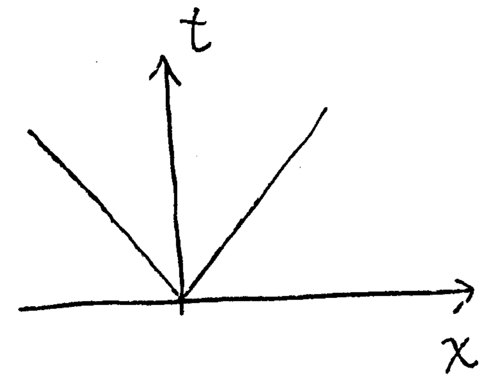
$$ds^2 = \eta_{ij}\,dx^i dx^j = -d(x^0)^2 + d(x^1)^2 + \cdots + d(x^3)^2$$

gives the metric in <u>meters</u>, then $x^0 = ct$, t in seconds, $s = c\tau$, τ proper time in sec's.

$\Rightarrow$ proper time change in second for observer traversing $c(s)$ is

$$\Delta\tau = \frac{1}{c}\int_{\xi_1}^{\xi_2}\sqrt{-\{-(\dot{x}^0)^2+(\dot{x}^1)^2+\cdots+(\dot{x}^3)^2\}}\,d\xi$$

<u>Defn</u>: a vector $X = a^i\frac{\partial}{\partial x^i}$ is



(1) timelike if $\langle X, X\rangle < 0$

(2) lightlike if $\langle X, X\rangle = 0$

(3) spacelike if $\langle X, X\rangle > 0$.

~~<u>Assumption</u>: The tangent vector to the "world line" or curve associated with any particle satisfies $\frac{dc}{d\xi}$ is timelike~~

"All particles move with speed $< c$"

<u>Assumption</u> ①: All particles move with speed $< c$ in the $\underset{\sim}{x}$-coordinates

<u>Precisely</u>: $\underset{\sim}{x}(t)$ the world line of a particle

$\Rightarrow \dot{x}^i \frac{\partial}{\partial x^i}$ is timelike. FIP

<u>Assumption</u> ②: The tangent vector to light rays is Lightlike. $\Leftrightarrow$ "light rays travel w. speed c"

~~For each O-N frame can identify vectors with points in space~~

We can use the O-N frame $\underset{\sim}{x}$ to identify vectors in $T_{\underset{\sim}{0}}M^4$ with points in the space: "identify components with pts in the spacetime"

$$X \equiv X^i \frac{\partial}{\partial x^i}\Big|_{\underset{\sim}{0}} \longleftrightarrow P = \Phi(X): x^i(P) = X^i$$

Under this identification, we can interpret

$$ds(X) \equiv \sqrt{\left| -(X^0)^2 + \cdots + (X^3)^2 \right|},$$

as follows

This is the exponential map FIP

Assumption (5)

X timelike: $ds(X) = c\,d\tau$ is the proper time change between events $P_0 = \Phi(0)$ and $P_1 = \Phi(X)$ as measured by observer moving with velocity vector X; Ie, if

$$\underset{\sim}{x} \circ c(\xi) \equiv \underset{\sim}{x}(\xi) \text{ satisfies } \dot{\underset{\sim}{x}}(\xi) = X, \quad \underset{\sim}{x}(0) = 0$$

then $x^i(\xi) = X^i \xi \Rightarrow$

$$c\Delta\tau = \int_0^1 \sqrt{1 - (X^0)^2 + (X^1)^2 + \cdots + (X^3)^2}\, d\xi = ds(X)$$

Note: if $d\xi \equiv ds$, (arclength param. of $c(\xi)$, then

$$ds = |X|\, ds \Rightarrow |X| = 1.$$

[i.e., $ds = g_{ij}\dot{x}^i \dot{x}^j d\xi \Rightarrow \langle X, X\rangle = 1$ iff $ds = d\xi$]

in which case we call X the 4-vel. of observer.

~~X spacelike~~;

X spacelike: $ds(X)$ is the length in meters of a rod as measured by observer moving in a frame in which P_0 and P_1 occur at same time. E.g., in $\underset{\sim}{x}$-frame, $X^0 = 0 \Rightarrow ds(X) = \sqrt{(X^1)^2 + (X^2)^2 + (X^3)^2}$ is pos def metric giving Euclidean lengths.

- Property of Flat Space: ① You can identify vectors in T_pM with T_qM by:

$$g_{ij} = \eta_{ij} \text{ in coords } \underset{\sim}{x}^i$$

$$\Leftrightarrow \quad X_p = a^i \frac{\partial}{\partial x^i}\Big|_p \longleftrightarrow X_q = a^i \frac{\partial}{\partial x^i}\Big|_q$$

"vectors with same components at different pts are said to be ∥-translations"

② You can identify T_0M with M by:

$$X = a^i \frac{\partial}{\partial x^i}\Big|_{\underset{\sim}{x}=0} \longleftrightarrow x^i = a^i \equiv \underset{\sim}{x} \text{ coords of a pt in } M$$

Define: $X \in T_pM$, $I(X) = q \in M : \underset{\sim}{x}^i(q) - \underset{\sim}{x}^i(p) = X^i$

<u>Example</u> : 1-d $\qquad g_{ij} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

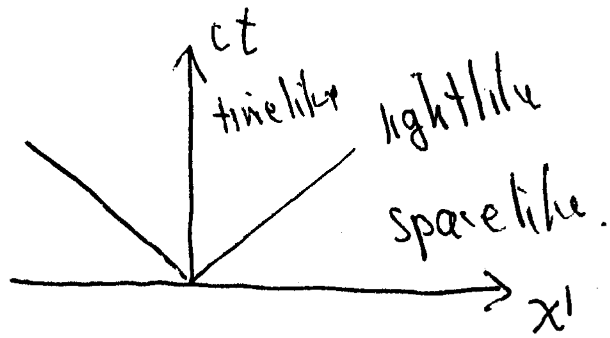
<u>Check</u>: $\left\{ \frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1} \right\}$ form an o-N basis

$$\left\langle \frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^0} \right\rangle = g_{ij} \, e_0^i \, e_0^j = -1$$

$$\left\langle \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^1} \right\rangle = +1$$

$$\left\langle \frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1} \right\rangle = 0$$

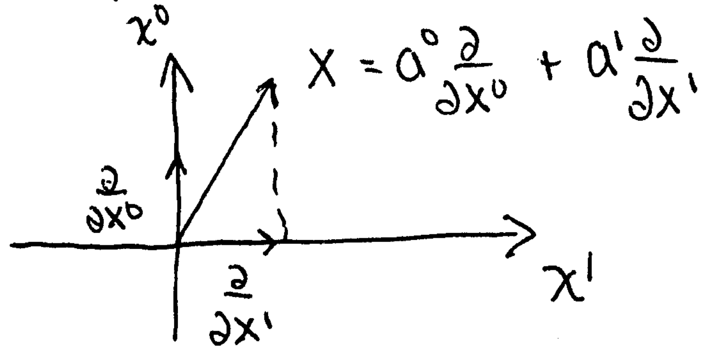
<u>Check</u>: $\frac{\partial}{\partial x^0} \pm \frac{\partial}{\partial x^1}$ lightlike $\quad [1, \pm 1] \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \pm 1 \end{bmatrix} = 0$



Q: What are the <u>other</u> o-N frames, and how are they related to $\underline{x}$-coordinates?

<u>Ans</u>: Let X be vector $X = a^0 \frac{\partial}{\partial x^0} + a^1 \frac{\partial}{\partial x^1}$

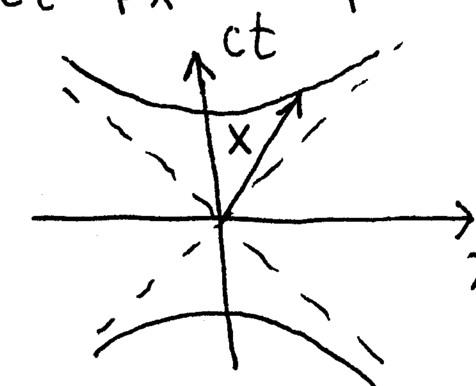
Since metric everywhere the same, we can identify components with pts in the space:



I.e. set $a^0 \equiv ct$, $a^1 \equiv x$, $X = ct \frac{\partial}{\partial x^0} + x \frac{\partial}{\partial x^1}$

Let X be timelike, unit length

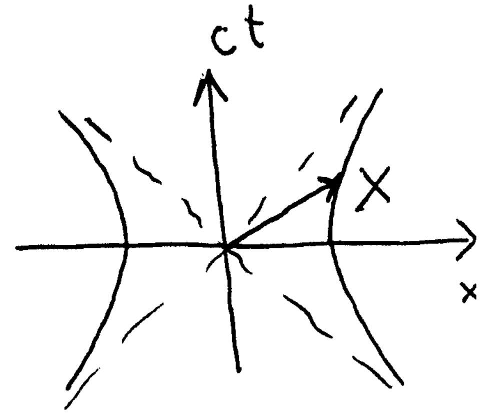
$$\langle X, X \rangle = (ct, x) \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \end{bmatrix} = -ct^2 + x^2 = -1$$



$\Rightarrow X$ lies on unit hyperbola$_1$

Let X be spacelike, unit length

$$\langle X, X \rangle = -ct^2 + x^2 = 1$$



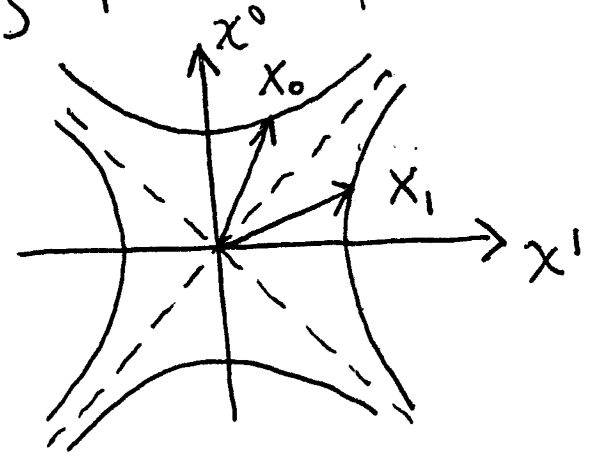
$\Rightarrow X$ lies on unit hyperbola$_2$

Thus: if $\langle X_0, X_1 \rangle = 0$, $\langle X_0, X_0 \rangle = -1$, $\langle X_1, X_1 \rangle = 1$

$$= \langle X_0, X_1 \rangle = (ct_0, x_0) \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} ct_1 \\ x_1 \end{bmatrix} = (-ct_0, x_0) \cdot (ct_1, x_1)$$

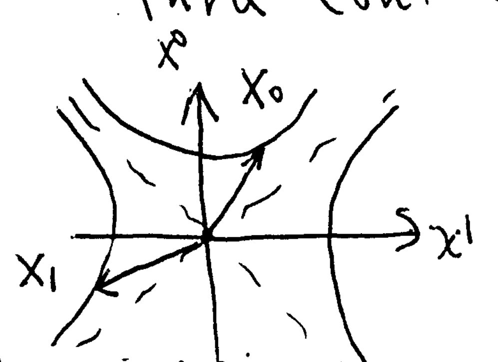
$\Rightarrow$ "$(ct_1, x_1) \parallel \pm (x_0, ct_0)$ $\Rightarrow$ X_1 is the reflection of X_0 in line $x = \pm ct$

If we assume X_0 timelike, positive direction and $\{X_0, X_1\}$ positively oriented, then

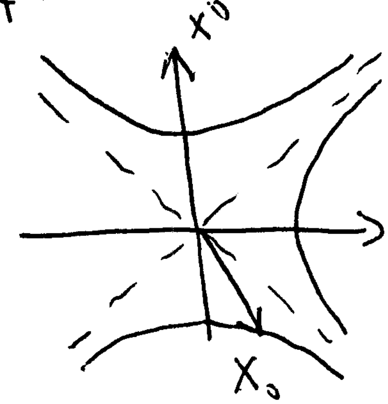


<u>Note</u> ① cannot get to neg oriented frame thru cont transformations

② Cannot get to time inverted or space inverted thru cont sequ. of tran's.



space inversion



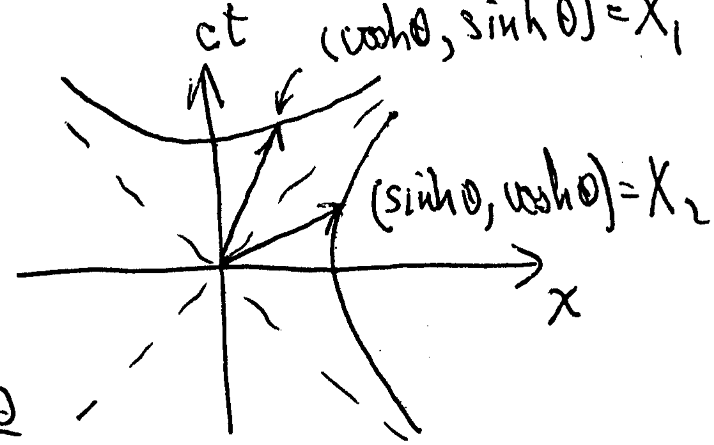
time inversion

Note: $\cosh^2\theta - \sinh^2\theta = 1$

$\Rightarrow ct = \cosh\theta,\ x = \sinh\theta$ lies on unit hyperbola. Our notation

(a, b) $\quad a = x^0$-coord

$\quad\quad\quad b = x^1$-coord



$$\Rightarrow X_0 = \cosh\theta \frac{\partial}{\partial x^0} + \sinh\theta \frac{\partial}{\partial x^1}$$

$$X_1 = \sinh\theta \frac{\partial}{\partial x^0} + \cosh\theta \frac{\partial}{\partial x^1}$$

gives all pos oriented, time oriented, O.N. frames, $-\infty < \theta < \infty$.

Note: ① $\forall$ X_0 with $|X_0| = 1$, you can complete it to an O.N. frame. If further X_0 is pos-time directed, then you can complete it uniquely to frame (X_0, X_1) with X_1 pos. space directed.

② All vectors can be completed to O.N. frame except lightlike vectors, which are $\perp$ to themselves.

(3) For any X, $|X| \neq 0$, you can define the orthogonal projection onto X:

$$\mathrm{Proj}_X Y = \frac{\langle X, Y \rangle}{\langle X, X \rangle} X \qquad \text{(FIP)}$$

Keep $\pm$ sign from metric

Using this, you can define the Gramm-Schmidt process to construct an O-N basis in $\mathbb{R}^4$ from any 4 linearly indept non-light like vectors.

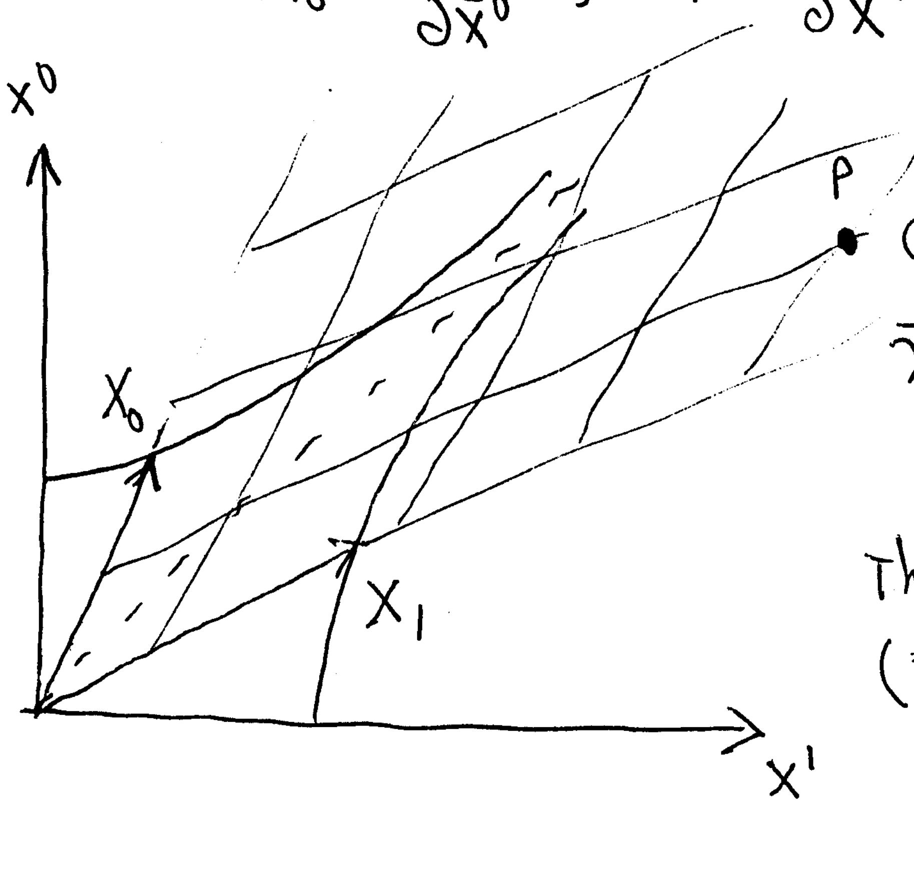
- <u>Lorentz Transformations</u>: Given $g_{ij} = \eta_{ij}$ in $\underset{\sim}{x} \equiv (x^0, x^1)$ coordinates. Construct another O-N frame for each

$$X_0 = \cosh\theta \frac{\partial}{\partial x^0} + \sinh\theta \frac{\partial}{\partial x^1}$$

$$X_1 = \sinh\theta \frac{\partial}{\partial x^0} + \cosh\theta \frac{\partial}{\partial x^1}$$

I.e., translate these vectors to each point of spacetime ("‖-translation in a flat spacetime") and choose $\{X_0, X_1\}$ to be the coord. basis vectors for a new coord system $(\bar{x}^0, \bar{x}^1)$ on spacetime as follows: We need

$$X_0 = \frac{\partial}{\partial \bar{x}^0}, \quad X_1 = \frac{\partial}{\partial \bar{x}^1}$$



$aX_0 + bX_1$

$\bar{x}^0(p) = a, \ \bar{x}^1(p) = b$

This defines the $(\bar{x}^0, \bar{x}^1)$ coord system.

Clearly, $\frac{\partial}{\partial \bar{x}^0} = X_0$, $\frac{\partial}{\partial \bar{x}^1} = X_1$ FIP, and

thus $\bar{g}_{ij} = \eta_{ij}$ because $\{X_0, X_1\}$ is an O-N basis at each point of spacetime.

Q: How is the (x^0, x^1) coord. system related to the $(\bar{x}^0, \bar{x}^1)$ coord system?

Ans: $x^0 \frac{\partial}{\partial x^0} + x^1 \frac{\partial}{\partial x^1} = \bar{x}^0 \frac{\partial}{\partial \bar{x}^0} + \bar{x}^1 \frac{\partial}{\partial \bar{x}^1}$

$\Leftrightarrow$ "coord's name same point p"

But:
$$\frac{\partial}{\partial \bar{x}^0} = \cosh\theta \frac{\partial}{\partial x^0} + \sinh\theta \frac{\partial}{\partial x^1}$$
$$\frac{\partial}{\partial \bar{x}^1} = \sinh\theta \frac{\partial}{\partial x^0} + \cosh\theta \frac{\partial}{\partial x^1}$$

$$\Rightarrow x^0 \frac{\partial}{\partial x^0} + x^1 \frac{\partial}{\partial x^1} = (\bar{x}^0 \cosh\theta + \bar{x}^1 \sinh\theta)\frac{\partial}{\partial x^0} + (\bar{x}^0 \sinh\theta + \bar{x}^1 \cosh\theta)\frac{\partial}{\partial x^1}$$

$$\Leftrightarrow \begin{bmatrix} x^0 \\ x^1 \end{bmatrix} = \begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix} \begin{bmatrix} \bar{x}^0 \\ \bar{x}^1 \end{bmatrix}$$

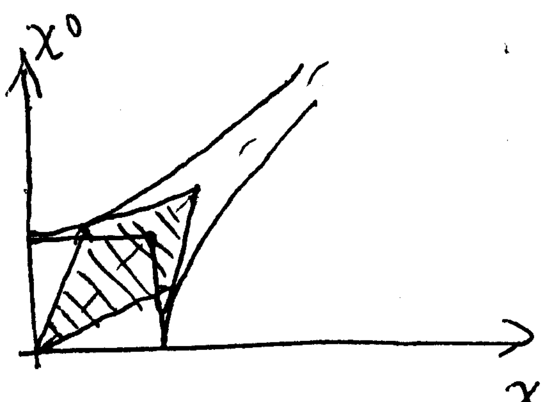
<u>Theorem</u>: The positively oriented, time oriented, homogeneous Lorentz transformations are given by $\underset{\sim}{x} = L(\theta)\underset{\sim}{\bar{x}}$, where $L(\theta)$ is given by

$$\begin{bmatrix} x^0 \\ x^1 \end{bmatrix} = L(\theta)\begin{bmatrix} \bar{x}^0 \\ \bar{x}^1 \end{bmatrix} = \begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix}\begin{bmatrix} \bar{x}^0 \\ \bar{x}^1 \end{bmatrix}$$

for $-\infty < \theta < \infty$.

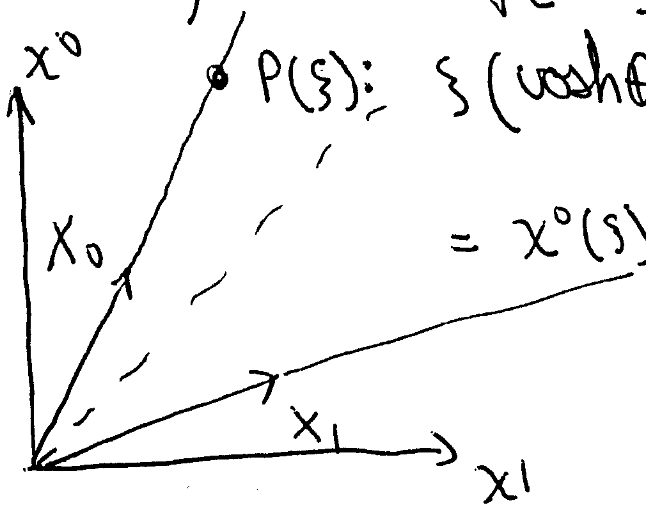
<u>Note</u> 1 : $\det L(\theta) = \cosh^2\theta - \sinh^2\theta = 1 \Rightarrow$ Lorentz transformations preserve the coordinate volume:

I.e., Vol of ▨ = 1.



<u>Note</u> 2 : $L(\theta)^{-1} = L(-\theta) = \begin{bmatrix} \cosh\theta & -\sinh\theta \\ -\sinh\theta & \cosh\theta \end{bmatrix}$

<u>Note</u> ② we can rewrite in terms of the velocity of the $\bar{x}$-frame as observed in the x-frame by writing $\begin{Bmatrix}\cosh\theta \\ \sinh\theta\end{Bmatrix}$ as a fn of v:



$$P(\xi) : \xi\left(\cosh\theta \frac{\partial}{\partial x^0} + \sinh\theta \frac{\partial}{\partial x^1}\right)$$
$$= x^0(\xi)\frac{\partial}{\partial x^0} + x^1(\xi)\frac{\partial}{\partial x^1}$$

<u>Conclude</u>: $P(\xi)$ parameterizes the $\bar{x}^0$-axis

$$\Leftrightarrow \quad \frac{dx^1}{dx^0} = \frac{dx^1/d\xi}{dx^0/d\xi} = \frac{\sinh\theta}{\cosh\theta} \qquad \begin{cases} x^0(\xi) = \xi\cosh\theta \\ x^1(\xi) = \xi\sinh\theta \end{cases}$$

$$\therefore \quad \frac{dx^1}{dx^0} = \frac{1}{c}\frac{dx^1}{dt} = \frac{1}{c}v = \frac{\sinh\theta}{\cosh\theta} = \tanh\theta$$

$$1 - \tanh^2\theta = \operatorname{sech}^2\theta = \frac{1}{\cosh^2\theta} \Rightarrow \cosh^2\theta = \frac{1}{1-\left(\frac{v}{c}\right)^2}$$

$$-1 + \tanh^{-2}\theta = \operatorname{csch}^2\theta = \frac{1}{\sinh^2\theta} \Rightarrow \sinh^2\theta = \frac{\left(\frac{v}{c}\right)^2}{1-\left(\frac{v}{c}\right)^2}$$

$\Rightarrow$ Lorentz Transformation

$$\begin{bmatrix} x^0 \\ x^1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{1-(v/c)^2}} & \frac{v/c}{\sqrt{1-(v/c)^2}} \\ \frac{v/c}{\sqrt{1-(v/c)^2}} & \frac{1}{\sqrt{1-(v/c)^2}} \end{bmatrix} \begin{bmatrix} \bar{x}^0 \\ \bar{x}^1 \end{bmatrix}$$

gives L. trans, where the bar frame moves with vel. v rel to unbarred frame.

Here, $\sqrt{1-\left(\frac{v}{c}\right)^2} = 1 - \frac{1}{2}\left(\frac{v}{c}\right)^2 + O\left(\frac{v}{c}\right)^4$

$$\frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} = \frac{1}{1+\{\ \}} = 1 - \{\ \} + O(\{\ \})^2$$

$$= 1 + \frac{1}{2}\left(\frac{v}{c}\right)^2 + O\left(\frac{v}{c}\right.$$

$$\Rightarrow \quad \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} = 1 + \frac{1}{2}\left(\frac{v}{c}\right)^2 + O\left(\frac{v}{c}\right)^4$$

$$\Rightarrow \quad L(\theta) = \mathrm{Id} + O\left(\frac{v}{c}\right)$$

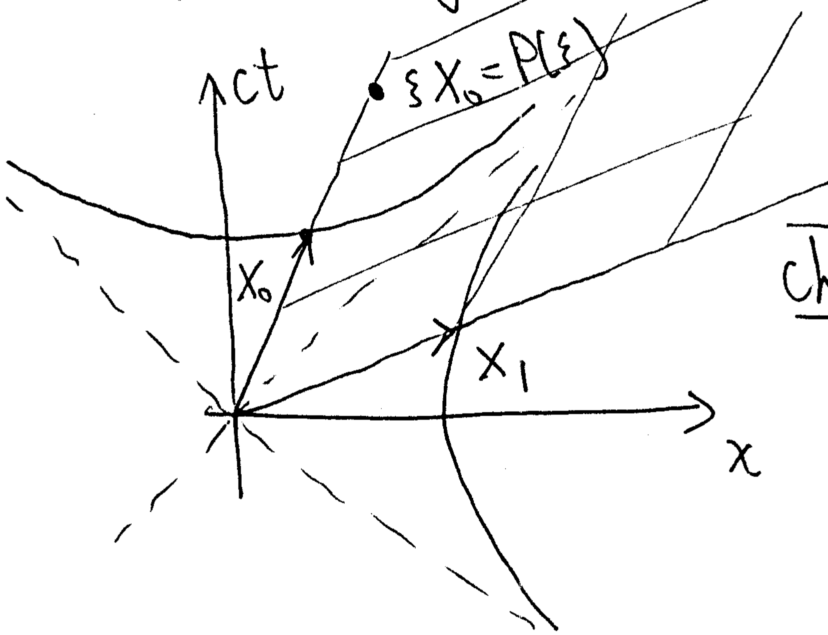
- Time Dialation: An observer fixed in the unbarred frame (say at origin) moves along a curve $x^1 = 0$, $x^0 = \xi$. His "aging time" betw t_1 and t_2 is given by

$$c\Delta\tau = \int_{ct_1}^{ct_2} \sqrt{-\eta_{ij}\dot{x}^i\dot{x}^j}\, d\xi = \int_{ct_1}^{ct_2} d\xi = c\Delta t$$

Conclude: Proper time & coordinate time agree for an observer fixed in L-frame:

By symmetry, an observer fixed on $\bar{x}^0$-coord axis ages according to the change in his $\bar{t}$-coordinate

But: starting clocks at $t = \bar{t} = 0$, the time change for observer fixed in $\bar{\underset{\sim}{x}}$-coordinate between 0 and $P(\xi) = \xi X_0$ is $\xi = \bar{t}c$ because X_0 represents a unit $\bar{t}$ time change.



Really $P(\xi) = \bar{\Phi}(\xi X_0)$

Check: $c\Delta \bar{t} = \int_0^{\xi_0} \| \xi X_0 \| \, d\xi = \int_0^{\xi_0} \xi \, d\xi = \xi_0$

But in $\underset{\sim}{x}$-coords, $\xi X_0 = (\xi \cosh\theta, \xi \sinh\theta)$

$\Rightarrow ct = \xi \cosh\theta = c\bar{t} \cosh\theta$

$$\therefore \quad \bar{t} = \frac{1}{\cosh\theta} t = \sqrt{1 - \tanh^2\theta}\, t = \sqrt{1 - \left(\frac{v}{c}\right)^2}\, t$$

$1 - \left(\frac{v}{c}\right)^2 = 1 - \tanh^2\theta = \operatorname{sech}^2\theta$

"Moving clocks appear to run slowly

Conclude: $\bar{t} = \sqrt{1-\left(\frac{v}{c}\right)^2}\; t$

thus $\Delta\bar{t} = \sqrt{1-\left(\frac{v}{c}\right)^2}\; \Delta t < \Delta t$

"Moving observers carry clocks that appear to move slow rel to coord clocks"

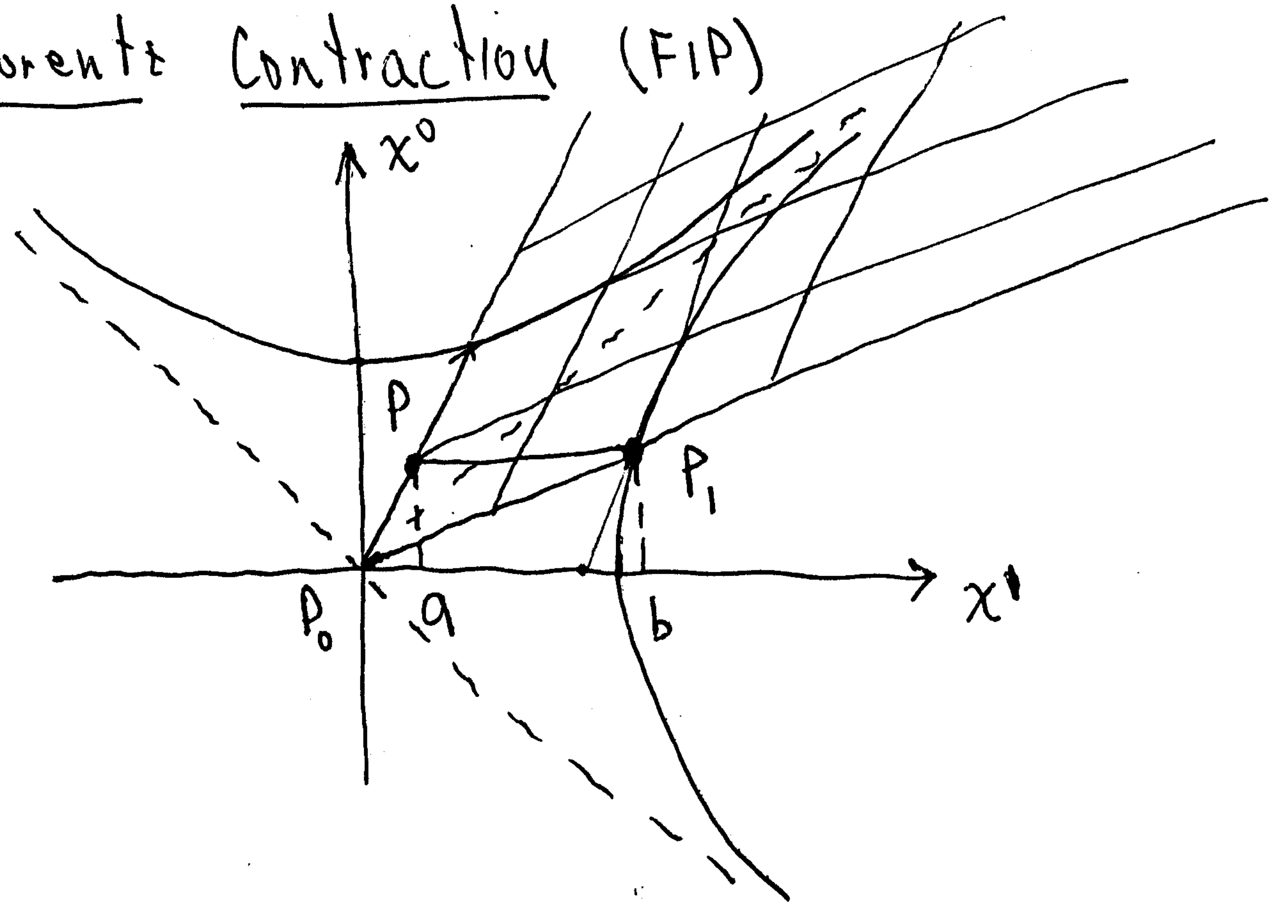
<u>Twin Paradox</u>: Any observer that moves & returns will age less than fixed observer.

Q: Explain lack of symmetry?

<u>Homework</u>: Show $L(\theta)L(\bar{\theta}) = L(\theta+\bar{\theta})$, and use this to show that if the $\bar{\underset{\sim}{x}}$-frame moves with vel v relative to $\underset{\sim}{x}$-frame, and the $\bar{\bar{\underset{\sim}{x}}}$-frame moves with velocity $\bar{v}$ rel to the barred frame, then the following velocity transformation law holds:

$$\bar{\bar{v}} = \frac{v+\bar{v}}{1+\frac{v\bar{v}}{c^2}}$$

Lorentz Contraction (FIP)



A meter stick in the barred frame has $P_0 = 0$ and P_1 s t $X_1 = \frac{\partial}{\partial \bar{x}^1}$, $P_1 = \Phi(X_1)$, X_1 the spacelike unit vector in the barred o-n frame.

Thus $X_1 = \sinh\theta \frac{\partial}{\partial x^0} + \cosh\theta \frac{\partial}{\partial x^1} \Rightarrow P_1 = (\sinh\theta, \cosh\theta)$

$$\underset{\sim}{x}(P_1) = \underset{\sim}{x} \circ \Phi(X_1) = (\sinh\theta, \cosh\theta).$$

Let $L_{\underset{\sim}{x}}$ denote the length of meter stick as measured in the $\underset{\sim}{x}$-frame, (unbarred).

In this frame, the length $L_{\underset{\sim}{x}}$ will be change in x'-coord. across the bar as measured at the same time x^0. Thus,

$$L_{\underset{\sim}{x}} = b - a$$

where $b = \cosh\theta$, and $a = x'$-coord of the point P on the $\bar{x}^0$-coord axis at the time $x^0 = \sinh\theta$. But $\bar{x}^0$-axis in $\underset{\sim}{x}$-coords is

$$\underset{\sim}{x}(\xi) = \xi(\cosh\theta, \sinh\theta),$$

so $\quad x^0(\xi) = \xi\cosh\theta = \sinh\theta \Rightarrow$

$$\xi = \tanh\theta$$

$\Rightarrow$
$$a = \xi\sinh\theta = \tanh\theta\sinh\theta$$

$\Rightarrow$
$$L_{\underset{\sim}{x}} = b - a = \cosh\theta - \tanh\theta\sinh\theta$$

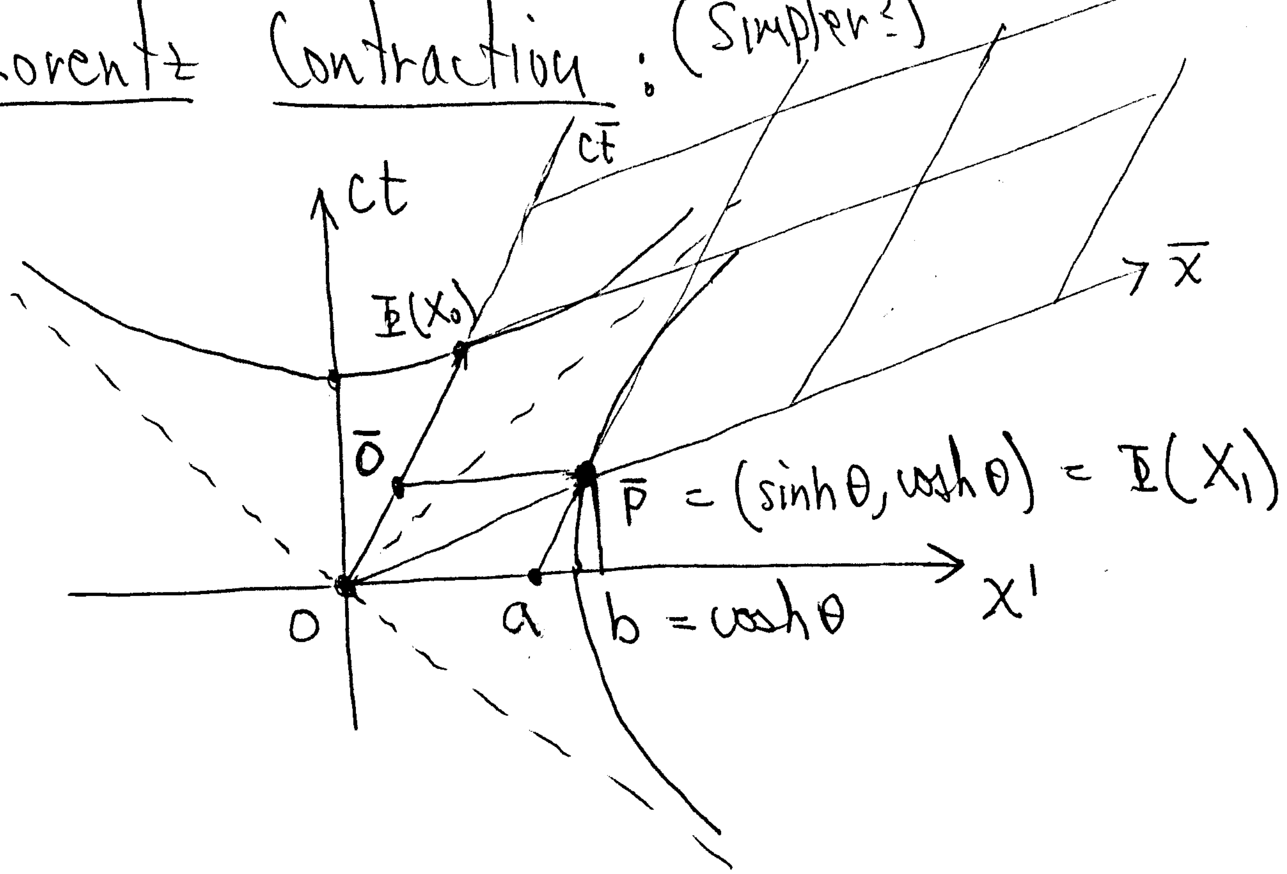
$$= \cosh\theta - \frac{\sinh^2\theta}{\cosh\theta} = \cosh\theta - \frac{\cosh^2\theta - 1}{\cosh\theta}$$

$$= \operatorname{sech}\theta = \sqrt{1-\left(\frac{v}{c}\right)^2}$$

"moving rods appear contracted by a factor $\sqrt{1-\left(\frac{v}{c}\right)^2}$"

$$L_{\underset{\sim}{x}} = \sqrt{1-\left(\frac{v}{c}\right)^2}\, L_{\underset{\sim}{\bar{x}}}$$

Lorentz Contraction: (Simpler?)



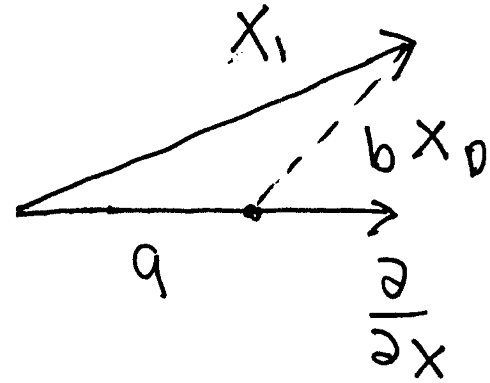
- The barred observer measures a unit bar fixed in his frame at $t=0$ as $\|X_1\| = 1$. The unbarred observer sees bar moving at speed v and measures bar at $t=0$ as equal to a

But $a = b - (b-a) = \cosh\theta - (b-a)$ and $(b-a)$ is the $\underset{\sim}{x}$-coord of $\bar{O} = \xi(\cosh\theta, \sinh\theta)$ where $\xi = \tanh\theta$ so $\bar{O}$ and $\bar{P}$ have same x^0-coordinate.

Thus $a = \cosh\theta - \cosh\theta\tanh\theta = \cosh\theta\,\mathrm{sech}^2\theta$

$= \mathrm{sech}\,\theta = \sqrt{1 - \left(\frac{v}{c}\right)^2}$ ✓

- <u>Note</u>: For an invariant formula for the Lorentz contraction factor, note:



$$a\frac{\partial}{\partial x} + bX_0 = X_1 \equiv \text{vector along moving rod}$$

or better:

$$a\frac{\partial}{\partial x} + bX_1^{\perp} = X_1 \quad \left(X_1^{\perp} \perp X_1 \text{ wrt } g\right)$$

Thus $\left\langle a\frac{\partial}{\partial x} + bX_1^{\perp}, X_1 \right\rangle = \langle X_1, X_1 \rangle$

$$a\left\langle \frac{\partial}{\partial x}, X_1 \right\rangle = \langle X_1, X_1 \rangle$$

$$\frac{1}{a} = \frac{\left\langle X_1, \frac{\partial}{\partial x} \right\rangle}{\langle X_1, X_1 \rangle} = \left\| \mathrm{Proj}_{X_1} \frac{\partial}{\partial x} \right\|$$

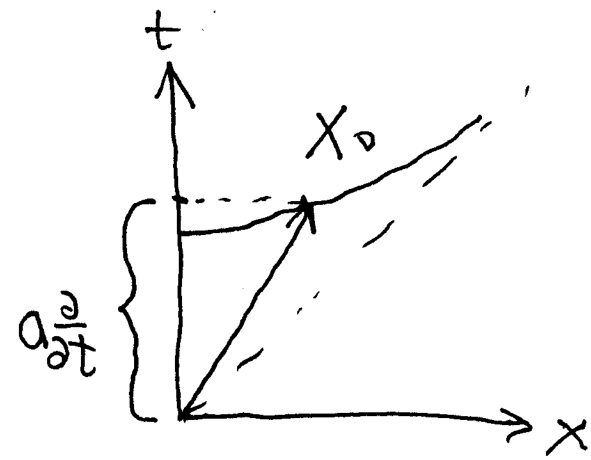
Conclude: A moving rod will be measured as $\frac{1}{a}$ times longer in its rest frame than in the moving frame, where

$$\boxed{\frac{1}{a} = \left\| \mathrm{Proj}_{X_1} \frac{\partial}{\partial x'} \right\|} \iff a = \frac{1}{\left\| \mathrm{Proj}_{X_1} \frac{\partial}{\partial x} \right\|}$$

contraction factor when measured in $\frac{\partial}{\partial x}$

- Similarly,

$a\frac{\partial}{\partial t} + b\frac{\partial}{\partial x} = X_0$ timelike



$$\left\langle a\frac{\partial}{\partial t}, \frac{\partial}{\partial t} \right\rangle = \left\langle X_0, \frac{\partial}{\partial t} \right\rangle$$

$$\boxed{a = \left\langle \frac{\partial}{\partial t}, X_0 \right\rangle}$$

= "time measured by $\frac{\partial}{\partial t}$ observer for a clock moving along X_0"