17.6) Homework assignment 17.5b gives us the following:

Proposition 1 *Every polynomial is continuous on* $\mathbb{R}$.

Suppose $f(x) = \frac{p(x)}{q(x)}$ on the domain $\{x \in \mathbb{R} q(x) \neq 0\}$ where $p(x)$ and $q(x)$ are polynomials. By Proposition 1, both $p(x)$ and $q(x)$ are continuous on $\mathbb{R}$. As long as $q(x) \neq 0$, the ratio $\frac{p(x)}{q(x)}$ is continuous (Theorem 17.4). But all x's where $q(x) = 0$ are not in the domain of $f(x)$. Hence, $f(x)$ is continuous on dom(f).

17.8) (a) If $f(x) \leq g(x)$ for a given x, then $f(x) - g(x) \leq 0$. So we have

$$\min(f, g)(x) = \frac{1}{2}(f(x) + g(x)) - \frac{1}{2}(-(f(x) - g(x))) = f(x)$$

and we obtain the minimum function output $f(x)$.

If $f(x) \geq g(x)$ for a given x, then $f(x) - g(x) \geq 0$. So we have

$$\min(f, g)(x) = \frac{1}{2}(f(x) + g(x)) - \frac{1}{2}(f(x) - g(x)) = g(x)$$

and we obtain the minimum function output $g(x)$.

Hence, $\min(f, g)$ function can be correctly defined as

$$\min(f, g) := \frac{1}{2}(f + g) - \frac{1}{2}|f - g|$$

(b) A quick calculation using formula in Section 17 Example 5 shows

$$-\max(-f, -g) = -\left(\frac{1}{2}(-f - g) - \frac{1}{2}| - f + g|\right) = \frac{1}{2}(f + g) - \frac{1}{2}|f - g| = \min(f, g)$$

(c) Looking at the formula obtained in (a), suppose f and g are continuous. Then $f + g$ and $f - g$ are continuous because of the addition and subtraction laws of continuity (Theorem 17.4), respectively. The function $|f - g|$ is continuous because the absolute value function preserves continuity (Theorem 17.3). Also, the functions $\frac{1}{2}(f + g)$ and $\frac{1}{2}|f - g|$ are continuous through the scaler multiplication law (Theorem 17.3). Finally, $\min(f, g)$ is continuous by another application of the addition law of continuity.

17.10) (a) The given function is

$$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

Construct the sequence $\{x_n\}$ by $x_n = \frac{1}{n}$ $\forall n \in \mathbb{N}$. For this sequence we have

$$\{x_n\} \to 0 \quad \text{and} \quad f(x_n) = 1 \ \ \forall n$$

Consequently, $\lim_{n\to\infty} f(x_n) = 1 \neq f(0) = 0$. Hence, f is not continuous at $x = 0$.

(b) The given function is

$$g(x) = \begin{cases} 0 & \text{if } x = 0 \\ \sin(\frac{1}{x}) & \text{if } x \neq 0 \end{cases}$$

Construct the sequence $\{x_n\}$ by

$$x_n = \frac{1}{\frac{\pi}{2} + 2\pi n} \quad \forall n \in \mathbb{N}$$

For this sequence we have

$$\{x_n\} \to 0 \quad \text{and} \quad g(x_n) = 1 \ \ \forall n$$

Consequently, $\lim\limits_{n\to\infty} g(x_n) = 1 \neq g(0) = 0$. Hence, g is not continuous at $x = 0$.

(d) The given function $P(x)$ with domain $[0, \infty)$ is

$$P(x) = \begin{cases} 15 & \text{if } 0 \leq x < 1 \\ 15 + 13n & \text{if } n \leq x < n + 1 \end{cases}$$

Choose $x_0 \in \mathbb{N}$, and let $m = x_0 - 1$. Construct the sequence $\{x_n\}$ by $x_n = x_0 - \frac{1}{n}$ $\forall n \in \mathbb{N}$. For this sequence we have

$$\{x_n\} \to x_0 \quad \text{and} \quad P(x_n) = 15 + 13m \ \ \forall n$$

since $m \leq x_n < m + 1$.

Consequently,

$$\lim_{n\to\infty} P(x_n) = 15 + 13m \neq P(x_0) = 15 + 13(m + 1)$$

since $m + 1 \leq x_0 < m + 2$. Hence, P is not continuous at x_0. Since x_0 was an arbitrary positive integer, $P(x)$ is discontinuous on the positive integers.

17.12)(b) Homework assignment 17.12a gives us the following:

Proposition 2 *Let f be a continuous function with domain (a, b). If $f(r) = 0$ for each rational number r in (a, b), then $f(x) = 0$ for all $x \in (a, b)$.*

Suppose f and g are continuous on (a, b) and $f(x) = g(x) \ \ \forall x \in \mathbb{Q}$. Consider the difference function $(f - g)(x)$ on (a, b), which is continuous by the subtraction law of continuity (Theorem 17.4). By our assumption, $(f - g)(x) = f(x) - g(x) = 0 \ \ \forall x \in \mathbb{Q}$. By the above Proposition, $(f - g)(x) = 0 \ \ \forall x \in (a, b)$. This implies $f(x) - g(x) = 0 \ \ \forall x \in (a, b)$. Thus, $f(x) = g(x) \ \ \forall x \in (a, b)$.

17.13)(b) First we show h is continuous at $x = 0$. Suppose $\{x_n\}$ is any sequence converging to 0. If x_n is rational, then $h(x_n) = x_n$. If x_n is irrational, then $h(x_n) = 0$. Either way, $|h(x_n)| \leq |x_n|$.

Now, we will show that $\{h(x_n)\}$ converges to $h(0)$. Let $\epsilon > 0$ be given. Since $\{x_n\}$ converges to 0, for this ϵ, there exists an N such that whenever $n \geq N$ we have $|x_n| < \epsilon$. But then for this same N, we have that $|h(x_n)| \leq |x_n| < \epsilon$ whenever $n \geq N$. Since $\epsilon > 0$ was arbitrary, we conclude that $h(x_n) \to h(0)$, i.e. h is continuous at $x = 0$.

Before showing that h is discontinuous at any $x \neq 0$, we state and prove the reverse triangle inequality: for any $a, b \in \mathbb{R}$ we have

$$\Big| |a| - |b| \Big| \leq |a - b|$$

To prove this, one uses the regular triangle inequality, which says that for any $x, y \in \mathbb{R}$ we have

$$|x + y| \le |x| + |y|$$

Let $x = a - b$, $y = b$ in the triangle inequality. Then

$$|a| \le |a - b| + |b|$$

Subtracting $|b|$ from both sides of the equation, we see that

$$|a| - |b| \le |a - b| \tag{1}$$

Now let $x = b - a$, $y = a$ in the triangle inequality. Then

$$|b| \le |b - a| + |a| = |a - b| + |a|$$

Subtracting $|a|$ from both sides of this equation gives

$$|b| - |a| \le |a - b| \tag{2}$$

Combining (1) and (2) yields the reverse triangle inequality.

Now, we will prove that h is discontinuous at every nonzero x using the reverse triangle inequality. Suppose $x \neq 0$. Then there exists some $\epsilon > 0$ such that $|x| > 2\epsilon$. Fix this ϵ. (A side note: we use 2ϵ rather than ϵ to make the end result neater, but the process is entirely the same either way, up to dividing all ϵ terms in the proof by 2.) We now break the situation up into two separate cases.

First, suppose $x \in \mathbb{R} \setminus \mathbb{Q}$. Then there exists a sequence $\{x_n\} \subset \mathbb{Q}$ such that $\{x_n\} \to x$. This means that for our particular ϵ, there exists some N such that whenever $n \ge N$ we have $|x_n - x| < \epsilon$. Fix this N. Since $\{x_n\} \subset \mathbb{Q}$, $h(x_n) = x_n$ for all n. Since $x \in \mathbb{R} \setminus \mathbb{Q}$, $h(x) = 0$. Thus

$$|h(x_n) - h(x)| = |x_n| = |x - (x - x_n)|$$

By applying the reverse triangle inequality to $a = x$, $b = x - x_n$, we see that

$$|h(x_n) - h(x)| \ge \Big| |x| - |x - x_n| \Big| = \Big| |x| - |x_n - x| \Big|$$

For our fixed N, we have $|x_n - x| < \epsilon$ whenever $n \ge N$, so that

$$|h(x_n) - h(x)| > \Big| 2\epsilon - \epsilon \Big| = \epsilon$$

whenever $n \ge N$. The fact that this above inequality holds for a particular ϵ and for any $n \ge N$ means that $h(x_n)$ cannot converge to $h(x)$, i.e. h is not continuous at x.

The second case can be proved without resorting to the reverse triangle inequality. Suppose $x \in \mathbb{Q}$ and let $\{x_n\} \subset \mathbb{R} \setminus \mathbb{Q}$ be a sequence converging to x. Since $x \neq 0$, we will continue to operate under the assumption that $|x| > 2\epsilon$. In this case, we have $h(x_n) = 0$ for all n, while $h(x) = x$. Thus

$$|h(x_n) - h(x)| = |0 - x| = |x| > 2\epsilon$$

for our particular choice of ϵ and for any $n \in \mathbb{N}$. Thus $h(x_n)$ does not converge to $h(x)$, i.e. h is not continuous at x.

Since we have shown h is discontinuous at any nonzero $x \in \mathbb{Q}$ as well as any nonzero $x \in \mathbb{R} \setminus \mathbb{Q}$, we conclude that h is discontinuous at any nonzero $x \in \mathbb{R}$. (Thanks to Evan Smothers)