29.2) Let $f(x) = \cos x$ which is continuous and differentiable on $\mathbb{R}$ from known facts. Consider $x, y \in \mathbb{R}$. By the Mean Value Theorem, there exists a c between x and y such that

$$\frac{f(x) - f(y)}{x - y} = f'(c) \Leftrightarrow \frac{\cos x - \cos y}{x - y} = -\sin c$$

Taking the absolute values of both sides, we obtain

$$\frac{|\cos x - \cos y|}{|x - y|} = |\sin c| \leq 1.$$

Rearranging, gives us the final result of

$$|\cos x - \cos y| \leq |x - y|.$$

Since $x, y \in \mathbb{R}$ were arbitrary, this inequality holds for all $x, y \in \mathbb{R}$, proving the claim.

28.8) Let f be differentiable on (a, b).

(ii) Suppose $f'(x) < 0 \ \forall x \in (a, b)$. Consider x_1 and x_2 with $a < x_1 < x_2 < b$. Since f is differentiable on (a, b), it is continuous and differentiable on $[x_1, x_2]$ by Theorem 28.2. By the Mean Value Theorem, there exists a $c \in (x_1, x_2)$ such that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) < 0,$$

where the inequality comes from the assumption. Since $x_2 - x_1 > 0$, we have

$$f(x_2) - f(x_1) < 0 \Rightarrow f(x_2) < f(x_1).$$

Thus, f is strictly decreasing.

(iii) Suppose $f'(x) \geq 0 \ \forall x \in (a, b)$. Consider x_1 and x_2 with $a < x_1 < x_2 < b$. Since f is differentiable on (a, b), it is continuous and differentiable on $[x_1, x_2]$ by Theorem 28.2. By the Mean Value Theorem, there exists a $c \in (x_1, x_2)$ such that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) \geq 0,$$

where the inequality comes from the assumption. Since $x_2 - x_1 > 0$, we have

$$f(x_2) - f(x_1) \geq 0 \Rightarrow f(x_1) \leq f(x_2).$$

Thus, f is increasing.

(iv) Suppose $f'(x) \leq 0 \ \forall x \in (a, b)$. Consider x_1 and x_2 with $a < x_1 < x_2 < b$. Since f is differentiable on (a, b), it is continuous and differentiable on $[x_1, x_2]$ by Theorem 28.2. By the Mean Value Theorem, there exists a $c \in (x_1, x_2)$ such that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) \leq 0,$$

where the inequality comes from the assumption. Since $x_2 - x_1 > 0$, we have

$$f(x_2) - f(x_1) \leq 0 \Rightarrow f(x_2) \leq f(x_1).$$

Thus, f is decreasing.

28.14) Suppose f is differentiable on $\mathbb{R}$, $1 \leq f'(x) \leq 2\ \forall x \in \mathbb{R}$, and $f(0) = 0$. For $x = 0$, the inequality $x \leq f(x) \leq 2x$ hold trivially.

Let $x > 0$. Since f is differentiable on $\mathbb{R}$, it is continuous on $\mathbb{R}$ by Theorem 28.2. By the Mean Value Theorem, there exists a $c \in (0, x)$ such that

$$\frac{f(x) - f(0)}{x - 0} = f'(c) \Leftrightarrow f'(c) = \frac{f(x)}{x}$$

Since $1 \leq f'(x) \leq 2\ \forall x \in \mathbb{R}$, we have

$$1 \leq \frac{f(x)}{x} \leq 2 \Leftrightarrow x \leq f(x) \leq 2x$$

Since $x > 0$ was arbitrary, the inequality holds for all $x > 0$. Combining this with the $x = 0$ case, we obtain

$$x \leq f(x) \leq 2x\ \forall x \geq 0,$$

proving the claim.

28.18) Let f be differentiable on $\mathbb{R}$ with $a := \sup\{|f'(x)| : x \in \mathbb{R}\} < 1$. Choose $s_0 \in \mathbb{R}$ and define sequence $\{s_n\}$ by $s_n = f(s_{x-1})$ for $n \geq 1$. Since f is differentiable on $\mathbb{R}$, it is continuous on $\mathbb{R}$ by Theorem 28.2. Consider $n \in \mathbb{N}$. By the Mean Value Theorem, there exists a c between s_n and s_{n-1} such that

$$\frac{f(s_n) - f(s_{n-1})}{s_n - s_{n-1}} = f'(c) \Leftrightarrow \frac{s_{n+1} - s_n}{s_n - s_{n-1}} = f'(c) \Leftrightarrow \frac{|s_{n+1} - s_n|}{|s_n - s_{n-1}|} = |f'(c)| \leq a,$$

by the assumption. Rearranging this inequality and the fact that $n \in \mathbb{N}$ was arbitrary, gives us

$$|s_{n+1} - s_n| \leq a|s_n - s_{n-1}| \text{ for } n \geq 1.$$

Notice by repeated use of the above inequality, we obtain

$$|s_n - s_{n-1}| \leq a|s_{n-1} - s_{n-2}| \leq a|s_{n-2} - s_{n-3}| \leq ... \leq a^{n-1}|s_1 - s_0|\ \forall n \in \mathbb{N}$$

Consider $m, n \in \mathbb{N}$ where without loss of generality $n > m$, with the above inequality, we have

$$|s_n - s_m| = |s_n - s_{n-1} + s_{n+1} - s_{n-2} + ... + s_{m+1} - s_m|$$

$$\leq |s_n - s_{n-1}| + |s_{n+1} - s_{n-2}| + ... + |s_{m+1} - s_m|$$

$$\leq a^{n-1}|s_1 - s_0| + a^{n-2}|s_1 - s_0| + ... + a^m|s_1 - s_0|$$

$$\leq a^m \left(\sum_{k=0}^{n-m-1} a^k\right) |s_1 - s_0| \leq a^m \left(\sum_{k=0}^{\infty} a^k\right) |s_1 - s_0| = \frac{a^m}{1-a}|s_1 - s_0|,$$

since its a geometric series with $a < 1$. Then, we have the following

$$|s_n - s_m| \leq \frac{a^m}{1-a}|s_1 - s_0|. \tag{1}$$

Now we are going to prove that $\{s_n\}$ is a Cauchy sequence. Let $\epsilon > 0$ be given. From (1), we have the following

$$|s_n - s_m| < \epsilon \quad \text{if} \quad |s_n - s_m| \leq \frac{a^m}{1-a}|s_1 - s_0|. < \epsilon$$

for $m, n \in \mathbb{N}$ where $n > m$. But

$$\frac{a^m}{1-a}|s_1 - s_0| < \epsilon \text{ if and only if } m > \log_a\left(\frac{(1-a)\epsilon}{|s_1 - s_0|}\right).$$

Choose

$$N = \log_a\left(\frac{(1-a)\epsilon}{|s_1 - s_0|}\right).$$

If $m, n > N$ with $n > m$, then

$$|s_n - s_m| < \epsilon$$

Thus, the sequence $\{s_n\}$ is Cauchy. Since $\mathbb{R}$ is complete, the sequence $\{s_n\}$ converges.