31.2) Let $f(x) = \sinh x = \frac{1}{2}(e^x - e^{-x})$. Then we have

$$f'(x) = \frac{1}{2}(e^x + e^{-x})$$

$$f''(x) = \frac{1}{2}(e^x - e^{-x})$$

$$f'''(x) = \frac{1}{2}(e^x + e^{-x})$$

$$f^{(4)}(x) = \frac{1}{2}(e^x - e^{-x})$$

In general, the nth derivative becomes

$$f^{(n)}(x) = \begin{cases} \frac{1}{2}(e^x - e^{-x}) & \text{if n is even} \\ \frac{1}{2}(e^x + e^{-x}) & \text{if n is odd} \end{cases}$$

Evaluation at $x = 0$, gives us

$$f^{(n)}(0) = \begin{cases} 1 & \text{if n is even} \\ 0 & \text{if n is odd} \end{cases}$$

Thus, the Taylor Series for $\cosh x$ is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + ... = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}.$$

For any bounded interval $(-M, M) \subseteq \mathbb{R}$, all the derivatives of f are bounded (by e^M), a direct application of Corollary 31.4 proves that

$$\sinh x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} \ \forall x \in \mathbb{R}$$

31.4) Use the following smooth function at a template

$$f(x) = \begin{cases} e^{-\frac{1}{x}} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

where $f^{(n)}(0) = 0 \ \forall n \in \mathbb{N}$ (See Example 3 in book).

(a)

$$f_a(x) = \begin{cases} e^{-\frac{1}{x-a}} & \text{if } x > a \\ 0 & \text{if } x \leq a \end{cases}$$

(b)

$$f_b(x) = \begin{cases} e^{\frac{1}{x-b}} & \text{if } x < b \\ 0 & \text{if } x \geq a \end{cases}$$

(c)

$$h_{a,b}(x) = \begin{cases} e^{-\frac{1}{(x-a)(x-b)}} & \text{if } x \in (a, b) \\ 0 & \text{if } x \notin (a, b) \end{cases}$$

(d)

$$h^*_{a,b}(x) = \begin{cases} \frac{e^{-\frac{1}{x-a}}}{e^{-\frac{1}{x-a}}+e^{\frac{1}{x-b}}} & \text{if } x \in (a,b) \\ 0 & \text{if } x \notin (a,b) \end{cases}$$

31.6) Fix $x > 0$. Let M be the unique solution to

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} x^k + \frac{Mx^n}{n!}.$$

Define

$$F(t) = f(t) + \sum_{k=1}^{n-1} \frac{(x-t)^k}{k!} f^{(k)}(t) + M\frac{(x-t)^n}{n!}.$$

for $t \in [0, x]$.

(a) Taking the derivative (using the Product Rule)

$$F'(t) = f'(t) + \sum_{k=0}^{n-1} \left(-\frac{(x-t)^{k-1}}{(k-1)!} f^{(k)}(t) + \frac{(x-t)^k}{k!} f^{(k+1)}(t) \right) - M\frac{(x-t)^{n-1}}{(n-1)!}$$

$$= f'(t) - f'(t) + \frac{(x-t)^{n-1}}{(n-1)!} f^{(n)}(t) + M\frac{(x-t)^{n-1}}{(n-1)!} = \frac{(x-t)^{n-1}}{(n-1)!}[f^{(n)}(t) - M],$$

where the sum collapsed because it was telescoping. Then, F is differentiable on $[0, x]$ with

$$F'(t) = \frac{(x-t)^{n-1}}{(n-1)!}[f^{(n)}(t) - M],$$

(b) We have

$$F(x) = f(x) + 0 = f(x).$$

Also, we have

$$F(0) = f(0) + \sum_{k=1}^{n-1} \frac{x^k}{k!} f^{(k)}(0) + M\frac{x^n}{n!} = \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} x^k + \frac{Mx^n}{n!} = f(x),$$

by the definition of M. Thus, $F(0) = f(x) = F(x)$, proving the claim.

(c) Since F is differentiable and (thus) continuous on $[0, x]$ along with $F(x) = F(0)$, by Rolle Theorem, there exists $c \in (0, x)$ such that

$$F'(c) = 0 \Leftrightarrow \frac{(x-c)^{n-1}}{(n-1)!}[f^{(n)}(c) - M] = 0 \Leftrightarrow f^{(n)}(c) = M,$$

proving Taylor's Error Formula (Theorem 31.3).