21.6) Suppose $f : S_1 \to S_2$ and $g : S_2 \to S_3$ are both continuous. Consider the function $g \circ f : S_1 \to S_3$. Let $U \subseteq S_3$ be open. Since g is continuous, $g^{-1}(U) \subseteq S_2$ is open by Theorem 21.3. Since f is continuous, $f^{-1}(g^{-1}(U)) \subseteq S_1$ is open by another application of Theorem 21.3. Then $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$ by the definition of composition and inverse. Thus, $(g \circ f)^{-1}(U) \subseteq S_1$ is open. Since U was arbitrary, this holds for all $U \subseteq S_3$. Therefore, $g \circ f$ is continuous by Theorem 21.3.

21.8) Suppose $f : S \to S^*$ is uniformly continuous and $\{s_n\} \subseteq S$ is a Cauchy sequence. Let $\epsilon > 0$ be given. Since f is uniformly continuous,

$$\exists \delta > 0 \text{ such that } \forall x, y \in S \text{ with } d(x,y) < \delta \Rightarrow d^*(f(x), f(y)) < \epsilon. \tag{1}$$

Since $\{s_n\}$ is Cauchy, for this δ

$$\exists N \text{ such that } \forall m, n > N \Rightarrow d(s_m, s_n) < \delta.$$

Then, for this N, we have

$$\forall m, n > N \Rightarrow d(s_m, s_n) < \delta \Rightarrow d^*(f(s_m), f(s_n)) < \epsilon,$$

by (1). Therefore, $\{f(s_n)\}$ is a Cauchy sequence as well.

21.10) (a) Let $f : (0,1) \to [0,1]$ be defined as

$$f(x) = \begin{cases} 0 & \text{if } 0 < x < \frac{1}{2} \\ 2x - \frac{1}{2} & \text{if } \frac{1}{2} \leq x \leq \frac{3}{4} \\ 1 & \text{if } \frac{3}{4} < x < 1 \end{cases}$$

(b) Let $g : (0,1) \to \mathbb{R}$ be defined as $g(x) = \tan(\pi x - \frac{\pi}{2})$.

(c) Let $h : [01] \cup [2,3] \to [0,1]$ be defined as $h(x) = -\frac{1}{2}x^2 + \frac{2}{3}x$

31.4) Use the following smooth function at a template

$$f(x) = \begin{cases} e^{-\frac{1}{x}} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

where $f^{(n)}(0) = 0 \ \forall n \in \mathbb{N}$ (See Example 3 in book).

(a)

$$f_a(x) = \begin{cases} e^{-\frac{1}{x-a}} & \text{if } x > a \\ 0 & \text{if } x \leq a \end{cases}$$

(b)

$$f_b(x) = \begin{cases} e^{\frac{1}{x-b}} & \text{if } x < b \\ 0 & \text{if } x \geq a \end{cases}$$

(c)

$$h_{a,b}(x) = \begin{cases} e^{-\frac{1}{(x-a)(x-b)}} & \text{if } x \in (a,b) \\ 0 & \text{if } x \notin (a,b) \end{cases}$$

(d)

$$h^*_{a,b}(x) = \begin{cases} \frac{e^{-\frac{1}{x-a}}}{e^{-\frac{1}{x-a}} + e^{\frac{1}{x-b}}} & \text{if } x \in (a,b) \\ 0 & \text{if } x \notin (a,b) \end{cases}$$

31.6) Fix $x > 0$. Let M be the unique solution to

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} x^k + \frac{Mx^n}{n!}.$$

Define

$$F(t) = f(t) + \sum_{k=1}^{n-1} \frac{(x-t)^k}{k!} f^{(k)}(t) + M\frac{(x-t)^n}{n!}.$$

for $t \in [0, x]$.

(a) Taking the derivative (using the Product Rule)

$$F'(t) = f'(t) + \sum_{k=0}^{n-1} \left(-\frac{(x-t)^{k-1}}{(k-1)!} f^{(k)}(t) + \frac{(x-t)^k}{k!} f^{(k+1)}(t) \right) - M\frac{(x-t)^{n-1}}{(n-1)!}$$

$$= f'(t) - f'(t) + \frac{(x-t)^{n-1}}{(n-1)!} f^{(n)}(t) + M\frac{(x-t)^{n-1}}{(n-1)!} = \frac{(x-t)^{n-1}}{(n-1)!}[f^{(n)}(t) - M],$$

where the sum collapsed because it was telescoping. Then, F is differentiable on $[0, x]$ with

$$F'(t) = \frac{(x-t)^{n-1}}{(n-1)!}[f^{(n)}(t) - M],$$

(b) We have

$$F(x) = f(x) + 0 = f(x).$$

Also, we have

$$F(0) = f(0) + \sum_{k=1}^{n-1} \frac{x^k}{k!} f^{(k)}(0) + M\frac{x^n}{n!} = \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} x^k + \frac{Mx^n}{n!} = f(x),$$

by the definition of M. Thus, $F(0) = f(x) = F(x)$, proving the claim.

(c) Since F is differentiable and (thus) continuous on $[0, x]$ along with $F(x) = F(0)$, by Rolle Theorem, there exists $c \in (0, x)$ such that

$$F'(c) = 0 \Leftrightarrow \frac{(x-c)^{n-1}}{(n-1)!}[f^{(n)}(c) - M] = 0 \Leftrightarrow f^{(n)}(c) = M,$$

proving Taylor's Error Formula (Theorem 31.3).