- 18.2) It would breakdown because the limit of subsequences x_0 and y_0 does not have to be in the interval (a, b). Since we would only assume continuity of f on (a, b), we would lose it at the endpoints a and b. This could cause the function to be unbounded and not contain it's minimum and/or maximum. For example, take $f(x) = \frac{1}{x}$ on (0, 1).
- 18.4) Suppose $S \subseteq \mathbb{R}$ and there exists a sequence $\{x_n\}$ in S that converges to $x_0 \notin S$. Consider the function $f(x) = \frac{1}{x x_0}$ on S. f is continuous on S since $x_0 \notin S$ by the fact $x x_0$ is a polynomial and the division law of continuity (Theorem 17.4). f is unbounded because of the division by zero that occurs within the sequence $f(x_n)$ (Theorem 9.10 and Exercise 9.10b in the case the sequence is negative).
- 18.6) We can rewrite the equation $x = \cos x$ as $x \cos x = 0$. Let $f(x) = x \cos x$ and m = 0. Notice that f(0) = -1 < 0 and $f(\frac{\pi}{2}) = \frac{\pi}{2} > 0$. Choose interval $(0, \frac{\pi}{2})$, so that $f(0) < m = 0 < f(\frac{\pi}{2})$. By the Intermediate Value Theorem, there exists at least one $x \in (0, \frac{\pi}{2})$ with f(x) = m = 0. With this x, we have

$$f(x) = 0 \Leftrightarrow x - \cos x = 0 \Leftrightarrow x = \cos x.$$

Hence, $x = \cos x$ for $x \in (0, \frac{\pi}{2})$.

- 18.8) Suppose f is a continuous function on \mathbb{R} and f(a)f(b) < 0 for some $a, b \in \mathbb{R}$. Since f(a)f(b) < 0, one of the two values f(a) or f(b) must be positive while the other is negative. Without loss of generality, assume f(a) < 0 and f(b) > 0. Let m = 0 and choose interval (a, b) since f(a) < m = 0 < f(b). By the Intermediate Value Theorem, there exists at least one $x \in (a, b)$ with f(x) = m = 0, proving the claim.
- 18.10) Suppose f is continuous on [0,2] and f(0) = f(2). Define g(x) = f(x+1) f(x) on [0,1]. Notice g(0) = f(2) f(1) and g(1) = f(1) f(0) = f(1) f(2) by assumption. So we have g(0) = -g(1). We consider two cases: a) g(0) = 0 or b) $g(0) \neq 0$.
- Case a) If g(0) = 0, this implies f(2) f(1) = 0 and consequently f(2) = f(1). So for x = 2 and y = 1, we have |x y| = 1 and f(x) = f(y), proving the claim.
- Case b) If $g(0) \neq 0$, without loss of generality we can assume g(0) < 0, which implies g(1) > 0. g is continuous since f is continuous and composition and differences preserve continuity. Let m = 0, and choose interval (0,1) since g(0) < m = 0 < g(1). By the Intermediate Value Theorem, there exists at least one $c \in (0,1)$ with g(c) = m = 0. With this c, we have

$$g(c) = 0 \Leftrightarrow f(c+1) - f(c) \Leftrightarrow f(c+1) = f(c).$$

So for x = c + 1 and y = c, we have |x - y| = 1 and f(x) = f(y), proving the claim.