19.2) (a) Let $\epsilon > 0$ be given. Notice

$$|f(x) - f(y)| = |3x + 11 - (3y + 11)| = 3|x - y|.$$

Then

$$|f(x) - f(y)| < \epsilon \Leftrightarrow 3|x - y| < \epsilon \Leftrightarrow |x - y| < \frac{\epsilon}{3}.$$

Choose $\delta = \frac{\epsilon}{3}$. Thus, if

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$$

and f is uniformly continuous on $\mathbb{R}$.

(b) Let $\epsilon > 0$ be given. Notice

$$|f(x) - f(y)| = |x^2 - y^2| = |x - y||x + y|.$$

So

$$|f(x) - f(y)| < \epsilon \Leftrightarrow |x - y||x + y| < \epsilon.$$

But $x, y \in [0, 3]$ which gives us the bound $|x + y| \leq 6$. Then

$$|x - y||x + y| \leq 6|x - y| < \epsilon \text{ if and only if } |x - y| < \frac{\epsilon}{6}.$$

Choose $\delta = \frac{\epsilon}{6}$. Thus, if

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$$

and f is uniformly continuous on $[0, 3]$.

19.4) (a) Suppose f is uniformly continuous on a bounded set S, but (for a contradiction) f is not bounded. So for any $n \in \mathbb{N}$ there exists an $x_n \in S$ where $|f(x_n)| > n$. In particular, we can use this fact to construct a sequence $\{x_n\}$ with $\lim_{n\to\infty} |f(x_n)| = \infty$. By the Bolzano-Weierstrass theorem, $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$. Since $\{x_{n_k}\}$ converges, it's a Cauchy sequence. By the uniform continuity of f, $\{f(x_{n_k})\}$ and consequently $\{|f(x_{n_k})|\}$ are both Cauchy sequences (Theorem 19.4). On $\mathbb{R}$, Cauchy sequences are convergent which means $\{|f(x_{n_k})|\}$ is bounded, but $\lim_{k\to\infty} |f(x_{n_k})| = \infty$. Contradiction. Hence, f is uniformly continuous.

(b) $f(x) = \frac{1}{x^2}$ is not bounded because of the division by zero at $x = 0$. By homework 19.4a), since interval $(0, 1)$ is a bounded set, f is not uniformly continuous on $(0, 1)$.

19.6) (a) $f'(x) = \frac{1}{2\sqrt{x}}$ is unbounded on $(0, 1]$ because of the division by zero that occurs at $x = 0$. We can build a (trivial) continuous extension of $f(x) = \sqrt{x}$ on $(0, 1]$ by $\tilde{f}(x) = \sqrt{x}$ on $[0, 1]$, which is continuous since x^p is continuous for $p = 1/2$ and $x \geq 0$. Since there exists a continuous extension of $f(x)$ on $(0, 1]$, f is uniformly continuous on $(0, 1]$ (Theorem 19.5).

(b) Notice that $|f'(x)| < \frac{1}{2}$ for all $x \in (1, \infty)$. Hence, f is differentiable with f' bounded on the interval $(1, \infty)$, which implies f is uniformly continuous on $[1, \infty)$ (Theorem 19.6).

19.8) Let $f(x) = \sin x$ which implies $f'(x) = \cos x$, so f is differentiable on $\mathbb{R}$. Let $x, y \in \mathbb{R}$. By the Mean Value Theorem, there exists $c \in \mathbb{R}$ such that

$$f'(c) = \frac{\sin x - \sin y}{x - y} \Rightarrow |f'(c)| = \frac{|\sin x - \sin y|}{|x - y|}.$$

But $|f'(x)| = |\cos x| \leq 1 \ \ \forall x \in \mathbb{R}$. Thus,

$$\frac{|\sin x - \sin y|}{|x - y|} \leq 1 \Rightarrow |\sin x - \sin y| \leq |x - y|$$

(b) Let $\epsilon > 0$ be given. Choose $\delta = \epsilon$. If

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| = |\sin x - \sin y| \leq |x - y| < \epsilon,$$

using inequality proved in homework 8a). Hence, f is uniformly continuous on $\mathbb{R}$.

18.10) (a) Yes. I observe

$$g(x) = \begin{cases} 0 & \text{if } x = 0 \\ x^2 \sin(\frac{1}{x}) & \text{if } x \neq 0 \end{cases}$$

is continuous on $\mathbb{R}$.

(b) g is uniformly continuous on any bounded subset $\mathbb{R}$ because g is continuous on any closed and bounded subset of $\mathbb{R}$. So for any bounded set $S \subseteq \mathbb{R}$, we can easily build a continuous extension on the closure of S (i.e. S^- in Definition 13.8).

(c) Yes. g is uniformly continuous on $\mathbb{R}$ because g' is bounded away from $x = 0$ (See the book solution for homework 19.9 for detailed discussion of this).