20.8) Consider the function $f(x) = x \sin \frac{1}{x}$.

For $\lim_{x\to\infty} f(x)$, we consider a sequence $x_n \subseteq (1, \infty)$ where $\{x_n\} \to \infty$. By definition,

$$\lim_{x\to\infty} f(x) := \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_n \sin \frac{1}{x_n} = \lim_{n\to\infty} \frac{\sin \frac{1}{x_n}}{\frac{1}{x_n}} = \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$$

from a well known trigonometric limit. Similarly, $\lim_{x\to-\infty} f(x) = 1$.

For $\lim_{x\to 0} f(x)$, we consider a sequence $x_n \subseteq (-1, 0) \cup (0, 1)$ where $\{x_n\} \to 0$. By definition,

$$\lim_{x\to 0} f(x) := \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_n \sin \frac{1}{x_n}$$

Notice we have the following bound

$$-x_n \leq \ x_n \sin \frac{1}{x_n} \leq x_n \quad \forall x_n$$

Since $\{x_n\} \to 0$ and consequently $\{-x_n\} \to 0$, we have

$$\lim_{x\to 0} f(x) = \lim_{n\to\infty} x_n \sin \frac{1}{x_n} = 0$$

by the Squeeze Theorem (Exercise 8.5). Moreover, since the two-sided limit exists, we have

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = \lim_{x\to 0} f(x) = 0$$

by Theorem 20.10.

20.16) Let $\lim_{x\to a^+} f_1(x) = L_1$ and $\lim_{x\to a^+} f_2(x) = L_2$.

(a) Suppose $f_1(x) \leq f_2(x)$ for all x in some interval (a, b). Consider the sequence $x_n \subseteq (a, b)$ where $\{x_n\} \to a$. Since $f_1(x_n) \leq f_2(x_n)$ $\forall n$, we have

$$\lim_{n\to\infty} f_1(x_n) \leq \lim_{n\to\infty} f_2(x_n)$$

By the definition of the limit, we obtain

$$L_1 = \lim_{x\to a^+} f_1(x) := \lim_{n\to\infty} f_1(x_n) \leq \lim_{n\to\infty} f_2(x_n) =: \lim_{x\to a^+} f_2(x) = L_2$$

Thus, $L_1 \leq L_2$, proving the claim.

(b) No. As a counterexample, consider the functions $f_1(x) = 0$ and $f_2(x) = x$ on $(0, 1)$.

20.17) Suppose $\lim_{x\to a^+} f_1(x) = \lim_{x\to a^+} f_3(x) = L$ and $f_1(x) \leq f_2(x) \leq f_3(x)$ $\forall x \in (a, b)$.

First, we want to show $\lim_{x\to a^+} f_2(x)$ exists and is finite. Consider the sequence $x_n \subseteq (a, b)$ where $\{x_n\} \to a$ along with the corresponding output sequence $f_2(x_n)$. Since by assumption $f_1(x_n) \leq f_2(x_n) \leq f_3(x_n)$ $\forall n$, the sequence $f_2(x_n)$ is bounded. By Bolzano-Weierstrass, there exists a convergent subsequence $f_2(x_{n_k})$ where we call the limit L_2. Thus, we have by definition

$$\lim_{x \to a^+} f_2(x) := \lim_{k \to \infty} f_2(x_{n_k}) = L_2.$$

Using the result in exercise 20.16a twice,

$$f_1(x) \leq f_2(x) \leq f_3(x) \quad \forall x \in (a, b) \Rightarrow L \leq L_2 \leq L.$$

Therefore, $L_2 = L$ or equivalently

$$\lim_{x \to a^+} f_2(x) = L.$$