24.2) For $x \in [0, \infty)$, let the sequence of functions $\{f_n\}$ be defined by $f_n(x) = \dfrac{x}{n}\ \forall n$.
(a) For $x = 0$, we have

$$f_n(0) = 0\ \forall n \Rightarrow \{f_n(0)\} \to 0.$$

For $x \in (0, \infty)$, we have

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x}{n} = 0.$$

Thus, we choose to define $f(x) = 0\ \forall x \in [0, \infty)$ so that $\{f_n\} \to f$ on $[0, \infty)$.

(b) For a given $n \in \mathbb{N}$, we have

$$\sup\{|f(x) - f_n(x)| : x \in [0, 1]\} = \sup\{\frac{x}{n} : x \in [0, 1]\} = \frac{1}{n}$$

Then

$$\lim_{n\to\infty} \sup\{|f(x) - f_n(x)| : x \in [0, 1]\} = \lim_{n\to\infty} \frac{1}{n} = 0$$

Thus, $\{f_n\} \rightrightarrows f$ on $[0, 1]$ by Proposition in Remark 24.4.

(c) For a given $n \in \mathbb{N}$, we have

$$\sup\{|f(x) - f_n(x)| : x \in [0, \infty)\} = \sup\{\frac{x}{n} : x \in [0, \infty)\} = \infty.$$

Then

$$\lim_{n\to\infty} \sup\{|f(x) - f_n(x)| : x \in [0, \infty)\} = \infty.$$

Thus, $\{f_n\}$ does not uniformly converge to f on $[0, \infty)$ by Proposition in Remark 24.4.

24.6) Let the sequence of functions $\{f_n\}$ be $f_n(x) = \left(x - \dfrac{1}{n}\right)^2$ be defined on $x \in [0, 1]$.
(a) Let $x \in [0, 1]$. Then

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \left(x - \frac{1}{n}\right)^2 = x^2$$

Thus, we choose to define $f(x) = x^2\ \forall x \in [0, 1]$ so that $\{f_n\} \to f$ on $x \in [0, 1]$.

(b) Yes. Let $\epsilon > 0$ be given and $x \in [0, 1]$. Then we have

$$|f_n(x) - f(x)| = \left|(x - \frac{1}{n})^2 - x^2\right| = \left|\frac{1}{n^2} - \frac{2x}{n}\right| \leq \frac{1}{n^2} - \frac{2}{n}$$

from the Triangle Inequality and $x \in [0, 1]$. Since $n^2 > n\ \forall n \in \mathbb{N}$, we have the following bound

$$|f_n(x) - f(x)| \leq \frac{1}{n^2} - \frac{2}{n} < \frac{3}{n}.$$

So

$$|f_n(x) - f(x)| < \epsilon \Leftrightarrow \frac{3}{n} < \epsilon \Leftrightarrow n > \frac{3}{\epsilon}$$

Choose $N = \dfrac{3}{\epsilon}$. Thus,

$$\forall n > N \Rightarrow |f_n(x) - f(x)| < \epsilon$$

Since $x \in [0,1]$ was arbitrary, it holds for all $x \in [0,1]$. Therefore, $\{f_n\} \rightrightarrows f$ on $[0,1]$ by definition.

24.8) Let the sequence of functions $\{f_n\}$ be $f_n(x) = \sum_{k=0}^{n} x^k$ be defined on $x \in [0,1]$.

(a) For $x = 1$, $f_n(1) = n \; \forall n$. Clearly,

$$\lim_{n\to\infty} f_n(1) = \infty,$$

so the limit does not exist. Therefore, the sequence $\{f_n\}$ does not converge pointwise on $[0,1]$.

(b) No. Since $\{f_n\}$ does not converge pointwise on $[0,1]$ (see part a), $\{f_n\}$ cannot converge uniformly $[0,1]$.

24.10) Suppose $\{f_n\} \rightrightarrows f$ and $\{g_n\} \rightrightarrows g$ on a set S. Consider the sequence of functions $\{f_n + g_n\}$ on S. Let $\epsilon > 0$ be given and $x \in S$. Notice

$$|(f_n + g_n)(x) - (f + g)(x)| = |f_n(x) - f(x) + g_n(x) - g(x)| \leq |f_n(x) - f(x)| + |g_n(x) - g(x)|$$

by the Triangle Inequality.

Consider the number $\frac{\epsilon}{2} > 0$. Since $\{f_n\} \rightrightarrows f$, there exists N_1 such that

$$\forall n > N_1 \Rightarrow |f_n(x) - f(x)| < \frac{\epsilon}{2}. \tag{1}$$

Also, since $\{g_n\} \rightrightarrows g$, there exists N_2 such that

$$\forall n > N_2 \Rightarrow |g_n(x) - g(x)| < \frac{\epsilon}{2}. \tag{2}$$

Choose $N = \max\{N_1, N_2\}$. Then

$$\forall n > N \Rightarrow |(f_n + g_n)(x) - (f + g)(x)| \leq |f_n(x) - f(x)| + |g_n(x) - g(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

by (1) and (2). Since $x \in S$ was arbitrary, it holds for all $x \in S$. Therefore, $\{f_n + g_n\} \rightrightarrows f + g$