26.2) (a) Start with the geometric series

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \text{ for } |x| < 1.$$

Taking the derivative to both sides, we obtain

$$\sum_{n=1}^{\infty} nx^{(n-1)} = \frac{1}{(1-x)^2} \text{ for } |x| < 1.$$

Then, we multiply both sides by x

$$\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2} \text{ for } |x| < 1$$

to obtain the desired result.

(b) Notice

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = \sum_{n=1}^{\infty} n \left(\frac{1}{2}\right)^n.$$

Using the formula derived in a for $x = \frac{1}{2}$, this evaluates to

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{\frac{1}{2}}{(1-\frac{1}{2})^2} = 2.$$

(c) Note

$$\sum_{n=1}^{\infty} \frac{n}{3^n} = \sum_{n=1}^{\infty} n \left(\frac{1}{3}\right)^n \text{ and } \sum_{n=1}^{\infty} \frac{(-1)^n n}{3^n} = \sum_{n=1}^{\infty} n \left(-\frac{1}{3}\right)^n.$$

Using the formula derived in a for $x = \frac{1}{3}$ and $x = -\frac{1}{3}$, this evaluates to

$$\sum_{n=1}^{\infty} \frac{n}{3^n} = \frac{\frac{1}{3}}{(1-\frac{1}{3})^2} = \frac{3}{4} \text{ and } \sum_{n=1}^{\infty} \frac{(-1)^n n}{3^n} = \frac{-\frac{1}{3}}{(1+\frac{1}{3})^2} = -\frac{3}{16}.$$

26.4) (a) Start with the power series for e^x

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \forall x \in \mathbb{R}.$$

Substitution of x by $(-x^2)$ gives us

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n} \quad \forall x \in \mathbb{R}.$$

(b) Let

$$F(x) = \int_0^x e^{-t^2}\, dt$$

Using the power series obtained in (a) and integrating term-by-term (Theorem 26.4), we arrive at the power series for $F(x)$

$$F(x) = \int_0^x e^{-t^2}\, dt = \int_0^x \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} t^{2n}\, dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{x^{2n+1}}{2n+1}$$

26.4) Let

$$s(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \text{ and } c(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}.$$

(a) Differentiating $s(x)$ term-by-term (Theorem 26.5), we have

$$s'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = c(x).$$

Notice we keep the sum starting at $n = 0$ since the 1st term of $s(x)$ is not a constant.

Differentiating $c(x)$ term-by-term (Theorem 26.5), we have

$$c'(x) = \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k-1)!} x^{2k-1} = -\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)!} x^{2k-1} = -\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = -s(x),$$

where we reindexed the sum with $n = k - 1$.

(b) We can implicitly differentiate to obtain

$$(s^2 + c^2)' = 2ss' + 2cc' = 2sc - 2cs = 0 \ \forall x \in \mathbb{R},$$

proving the claim.

(c) Applying the Fundamental Theorem of Calculus to the equation obtained in (a), we have

$$(s^2 + c^2)' = 0 \ \forall x \in \mathbb{R} \Rightarrow s^2 + c^2 = C \ \forall x \in \mathbb{R}$$

where C is a constant of integration. Since it holds for all $x \in \mathbb{R}$, we let $x = 0$, then

$$C = [s(0)]^2 + [c(0)]^2 = 0^2 + 1^2 = 1$$

Hence, $C = 1$ and we have

$$s^2 + c^2 = 1,$$

proving the claim.