28.2) (a) Let $f(x) = x^3$. By definition, we have

$$f'(2) := \lim_{x\to 2} \frac{f(x)-f(2)}{x-2} = \lim_{x\to 2} \frac{x^3-8}{x-2} = \lim_{x\to 2} \frac{(x-2)(x^2+2x+4)}{x-2} = \lim_{x\to 2} x^2+2x+4 = 12.$$

(b) Let $g(x) = x+2$. By definition, we have

$$g'(a) := \lim_{x\to a} \frac{g(x)-g(a)}{x-a} = \lim_{x\to a} \frac{x+2-(a-2)}{x-a} = \lim_{x\to a} \frac{x-a}{x-a} = \lim_{x\to a} 1 = 1.$$

(c) Let $f(x) = x^2 \cos x$. By definition, we have

$$f'(0) := \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{x^2\cos x}{x} = \lim_{x\to 0} x\cos x = 0.$$

(d) Let $r(x) = \dfrac{3x+4}{2x-1}$. By definition, we have

$$r'(1) := \lim_{x\to 1} \frac{r(x)-r(1)}{x-1} = \lim_{x\to 1} \frac{\frac{3x+4}{2x-1} - 7\left(\frac{2x-1}{2x-1}\right)}{x-1} = \lim_{x\to 1} \frac{3x+4-(14x-7)}{(x-1)(2x-1)}$$

$$= \lim_{x\to 1} \frac{-11(x-1)}{(x-1)(2x-1)} = \lim_{x\to 1} \frac{-11}{2x-1} = -11.$$

28.6) Let

$$f(x) = \begin{cases} x^2 \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

(a) Let $x \neq 0$. We have

$$f'(x) = 2x\sin\frac{1}{x} - \cos\frac{1}{x}$$

through the use of the Product Rule (Theorem 28.3) and the Chain Rule (Theorem 28.4). Since $x \neq 0$ was arbitrary, the derivative exists for all $a \neq 0$ and.

$$f'(a) = 2a\sin\frac{1}{a} - \cos\frac{1}{a}$$

(b) By definition, we have

$$f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{x^2\sin\frac{1}{x}}{x} = \lim_{x\to 0} x\sin\frac{1}{x},$$

since $x \neq 0$ in the limit. Notice we have the following bound

$$-x \leq \ x\sin\frac{1}{x} \leq x \ \ \forall x$$

Since $\{x\} \to 0$ and consequently $\{-x\} \to 0$, we have

$$f'(0) = \lim_{x \to 0} \ x \sin \frac{1}{x} = 0$$

by the Squeeze Theorem (Exercise 8.5). Thus, f is differentiable at $x = 0$ with $f'(0) = 0$.

(c) $f'(x)$ is discontinuous because $\cos \dfrac{1}{x}$ is discontinuous through oscillation. See Exercise 28.6b for a way to prove this.

28.6) Let

$$f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

(a) f is continuous at $x = 0$ by Exercise 17.9c.

(b) By definition, we have

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x \sin \frac{1}{x}}{x} = \lim_{x \to 0} \sin \frac{1}{x},$$

since $x \neq 0$ in the limit. This limit does not exist through oscillation. More specifically, let $g(x) = \sin \dfrac{1}{x}$ and construct the sequence $\{x_n\}$ by

$$x_n = \frac{1}{\frac{\pi}{2} + 2\pi n} \quad \forall n \in \mathbb{N}$$

For this sequence we have

$$\{x_n\} \to 0 \quad \text{and} \quad g(x_n) = 1 \ \ \forall n.$$

Consequently, $\lim_{n \to \infty} \ g(x_n) = 1$. Construct another sequence $\{y_n\}$ by

$$y_n = \frac{1}{2\pi n} \quad \forall n \in \mathbb{N}$$

For this sequence we have

$$\{y_n\} \to 0 \quad \text{and} \quad g(y_n) = 0 \ \ \forall n.$$

Consequently, $\lim_{n \to \infty} \ g(y_n) = 0$. So we have

$$\lim_{n \to \infty} \ g(x_n) = 1 \neq \lim_{n \to \infty} \ g(y_n) = 0,$$

where $\{x_n\} \to 0$ and $\{y_n\} \to 0$. So by definition of the limit,

$$\lim_{x \to 0} g(x) = \lim_{x \to 0} \sin \frac{1}{x}$$

does not exist. Thus, f is not differentiable at $x = 0$.

28.8) Let

$$f(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{I} := \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

(a) Let $x = 0$ and consider a sequence $\{x_n\} \to 0$. Break this sequence into two subsequences $\{x_{n_j}\} \subseteq \mathbb{I}$ and $\{x_{n_k}\} \subseteq \mathbb{Q}$ where $\{x_n\} = \{x_{n_j}\} \cup \{x_{n_k}\}$. There are 2 cases to consider: Case 1) $\{x_{n_k}\}$ is an infinite set and Case 2) $\{x_{n_j}\}$ is an infinite set.

Case 1) Assume $\{x_{n_k}\}$ is an infinite subsequence. Then $\{x_{n_k}\} \to 0$ since $\{x_n\} \to 0$ by Theorem 11.2. Because $x_{n_k} \in \mathbb{Q}\ \forall k$, we have

$$\lim_{k\to\infty} \ f(x_{n_k}) = \lim_{k\to\infty} \ (x_{n_k})^2 = 0.$$

Consequently,

$$\lim_{n\to\infty} \ f(x_n) = 0.$$

Case 2) Assume $\{x_{n_j}\}$ is an infinite subsequence. Then $\{x_{n_j}\} \to 0$ since $\{x_n\} \to 0$ by Theorem 11.2. Because $x_{n_j} \in \mathbb{I}\ \forall j$, we have

$$\lim_{j\to\infty} \ f(x_{n_k}) = \lim_{j\to\infty} \ 0 = 0.$$

Consequently,

$$\lim_{n\to\infty} \ f(x_n) = 0.$$

If both subsequences happen to be infinite, then from Case 1 and Case 2, we have

$$\lim_{n\to\infty} \ f(x_n) = 0.$$

Since $\{x_n\} \to 0$ was an arbitrary sequence, we have

$$\lim_{n\to\infty} \ f(x_n) = f(0) = 0$$

for all sequences converging to 0. Thus, f is continuous at $x = 0$ by definition.

(b) Let $x \neq 0$. Since $\mathbb{R} = \mathbb{Q} \cup \mathbb{I}$, there are two cases: Case 1) $x \in \mathbb{Q}$ and Case 2) $x \in \mathbb{I}$. We need the following

Proposition 1 *For all $x \in \mathbb{R}$ there exists a sequence of rational numbers which converges to x. Also, there exists a sequence of irrational numbers which converges to x.*

Case 1) Let $x \in \mathbb{Q}$ where $x \neq 0$. Consider a sequence of irrationals $\{x_n\} \subset \mathbb{I}$ where $\{x_n\} \to x$. Then

$$\lim_{n\to\infty} \ f(x_n) = \lim_{n\to\infty} \ 0 = 0 \neq f(x) = x^2.$$

Thus, f is not continuous at x by definition.

Case 2) Let $x \in \mathbb{I}$. Consider a sequence of rationals $\{x_n\} \subset \mathbb{Q}$ where $\{x_n\} \to x$. Then

$$\lim_{n\to\infty} \ f(x_n) = \lim_{n\to\infty} \ (x_n)^2 = x^2 \neq f(x) = 0.$$

Thus, f is not continuous at x by definition.

Since $x \neq 0$ was arbitrary, f is not continuous when $x \neq 0$.

(c) By the definition of the derivative and limits, we have

$$f'(0) := \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{f(x)}{x} := \lim_{n\to\infty} \frac{f(x_n)}{x_n},$$

where $\{x_n\} \to 0$. Like part a), we break this sequence into two $\{x_n\}$ subsequences $\{x_{n_j}\} \subseteq \mathbb{I}$ and $\{x_{n_k}\} \subseteq \mathbb{Q}$ where $\{x_n\} = \{x_{n_j}\} \cup \{x_{n_k}\}$.

For $\{x_{n_k}\} \subseteq \mathbb{Q}$, if this is an infinite subsequence, we have

$$\lim_{n\to\infty} \frac{f(x_n)}{x_n} = \lim_{k\to\infty} \frac{f(x_{n_k})}{x_{n_k}} = \lim_{k\to\infty} \frac{(x_{n_k})^2}{x_{n_k}} = \lim_{k\to\infty} x_{n_k} = 0,$$

since $\{x_{n_k}\} \to 0$.

For $\{x_{n_j}\} \subseteq \mathbb{I}$, if this is an infinite subsequence, we have

$$\lim_{n\to\infty} \frac{f(x_n)}{x_n} = \lim_{j\to\infty} \frac{f(x_{n_j})}{x_{n_j}} = \lim_{j\to\infty} \frac{0}{x_{n_j}} = 0.$$

For any sequence $\{x_n\}$ one (if not both) subsequences $\{x_{n_j}\} \subseteq \mathbb{I}$ and $\{x_{n_k}\} \subseteq \mathbb{Q}$ must be infinite. For each of the three options, the above analysis gives us

$$f'(0) = \lim_{n\to\infty} \frac{f(x_n)}{x_n} = 0.$$

Hence, f is differentiable at $x = 0$ with $f'(0) = 0$.

28.14) (a) Using the definition, we have

$$f'(a) := \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

Now let $h = x - a$ which is equivalent to $x = a + h$. So the limit can be transformed to

$$f'(a) := \lim_{x\to a} \frac{f(x) - f(a)}{x - a} = \lim_{x-a\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}.$$

Thus,

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}.$$

(b) From (a), we have

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}.$$

Substituting $h = -h$, we also have

$$f'(a) = \lim_{h\to 0} \frac{f(a-h) - f(a)}{-h} = \lim_{h\to 0} \frac{f(a) - f(a-h)}{h}.$$

So adding them together, we obtain

$$2f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} + \lim_{h\to 0} \frac{f(a) - f(a-h)}{h} = \lim_{h\to 0} \frac{f(a+h) - f(a-h)}{h}.$$

Thus, dividing by 2 gives us

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a-h)}{2h}.$$