This equation holds because $\max\{a,b\} = \frac{1}{2}(a+b) + \frac{1}{2}|a-b|$ is true for all $a, b \in \mathbb{R}$, a fact which is easily checked by considering the cases $a \geq b$ and $a < b$. By Theorem 17.4(i), $f+g$ and $f-g$ are continuous at x_0. Hence $|f-g|$ is continuous at x_0 by Theorem 17.3. Then $\frac{1}{2}(f+g)$ and $\frac{1}{2}|f-g|$ are continuous at x_0 by Theorem 17.3, and another application of Theorem 17.4(i) shows that $\max(f,g)$ is continuous at x_0. □

Exercises

17.1. Let $f(x) = \sqrt{4-x}$ for $x \leq 4$ and $g(x) = x^2$ for all $x \in \mathbb{R}$.

(a) Give the domains of $f+g$, fg, $f \circ g$ and $g \circ f$.

(b) Find the values $f \circ g(0)$, $g \circ f(0)$, $f \circ g(1)$, $g \circ f(1)$, $f \circ g(2)$ and $g \circ f(2)$.

(c) Are the functions $f \circ g$ and $g \circ f$ equal?

(d) Are $f \circ g(3)$ and $g \circ f(3)$ meaningful?

17.2. Let $f(x) = 4$ for $x \geq 0$, $f(x) = 0$ for $x < 0$, and $g(x) = x^2$ for all x. Thus $\text{dom}(f) = \text{dom}(g) = \mathbb{R}$.

(a) Determine the following functions: $f+g$, fg, $f \circ g$, $g \circ f$. Be sure to specify their domains.

(b) Which of the functions $f, g, f+g, fg, f \circ g, g \circ f$ is continuous?

17.3. Accept on faith that the following familiar functions are continuous on their domains: $\sin x$, $\cos x$, e^x, 2^x, $\log_e x$ for $x > 0$, x^p for $x > 0$ [p any real number]. Use these facts and theorems in this section to prove that the following functions are also continuous.

(a) $\log_e(1+\cos^4 x)$

(b) $[\sin^2 x + \cos^6 x]^\pi$

(c) 2^{x^2}

(d) 8^x

(e) $\tan x$ for $x \neq$ odd multiple of $\frac{\pi}{2}$

(f) $x \sin(\frac{1}{x})$ for $x \neq 0$

(g) $x^2 \sin(\frac{1}{x})$ for $x \neq 0$

(h) $\frac{1}{x}\sin(\frac{1}{x^2})$ for $x \neq 0$

17.4. Prove that the function $\sqrt{x}$ is continuous on its domain $[0, \infty)$. *Hint*: Apply Example 5 in §8.

17.5. **(a)** Prove that if $m \in \mathbb{N}$, then the function $f(x) = x^m$ is continuous on $\mathbb{R}$.

(b) Prove that every *polynomial function* $p(x) = a_0 + a_1x + \cdots + a_nx^n$ is continuous on $\mathbb{R}$.

17.6. A *rational function* is a function f of the form p/q where p and q are polynomial functions. The domain of f is $\{x \in \mathbb{R} : q(x) \neq 0\}$. Prove that every rational function is continuous. *Hint*: Use Exercise 17.5.

17.7. **(a)** Observe that if k is in $\mathbb{R}$, then the function $g(x) = kx$ is continuous by Exercise 17.5.

(b) Prove that $|x|$ is a continuous function on $\mathbb{R}$.

(c) Use (a) and (b) and Theorem 17.5 to give another proof of Theorem 17.3.

17.8. Let f and g be real-valued functions.

(a) Show that $\min(f, g) = \frac{1}{2}(f + g) - \frac{1}{2}|f - g|$.

(b) Show that $\min(f, g) = -\max(-f, -g)$.

(c) Use (a) or (b) to prove that if f and g are continuous at x_0 in $\mathbb{R}$, then $\min(f, g)$ is continuous at x_0.

17.9. Prove that each of the following functions is continuous at x_0 by verifying the ϵ-δ property of Theorem 17.2.

(a) $f(x) = x^2$, $x_0 = 2$;

(b) $f(x) = \sqrt{x}$, $x_0 = 0$;

(c) $f(x) = x\sin(\frac{1}{x})$ for $x \neq 0$ and $f(0) = 0$, $x_0 = 0$;

(d) $g(x) = x^3$, x_0 arbitrary.
Hint for (d): $x^3 - x_0^3 = (x - x_0)(x^2 + x_0x + x_0^2)$.

17.10. Prove that the following functions are discontinuous at the indicated points. You may use either Definition 17.1 or the ϵ-δ property in Theorem 17.2.

(a) $f(x) = 1$ for $x > 0$ and $f(x) = 0$ for $x \leq 0$, $x_0 = 0$;

(b) $g(x) = \sin(\frac{1}{x})$ for $x \neq 0$ and $g(0) = 0$, $x_0 = 0$;

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(c) $\operatorname{sgn}(x) = -1$ for $x < 0$, $\operatorname{sgn}(x) = 1$ for $x > 0$, and $\operatorname{sgn}(0) = 0$, $x_0 = 0$;

(d) $P(x) = 15$ for $0 \le x < 1$ and $P(x) = 15+13n$ for $n \le x < n+1$, x_0 a positive integer.

The function sgn is called the *signum function*; note that $\operatorname{sgn}(x) = \frac{x}{|x|}$ for $x \neq 0$. The definition of P, the postage-stamp function *circa* 1979, means P takes the value 15 on the interval $[0, 1)$, the value 28 on the interval $[1, 2)$, the value 41 on the interval $[2, 3)$, etc.

17.11. Let f be a real-valued function with $\operatorname{dom}(f) \subseteq \mathbb{R}$. Prove that f is continuous at x_0 if and only if, for every *monotonic* sequence (x_n) in $\operatorname{dom}(f)$ converging to x_0, we have $\lim f(x_n) = f(x_0)$. *Hint*: Don't forget Theorem 11.3.

17.12. **(a)** Let f be a continuous real-valued function with domain (a, b). Show that if $f(r) = 0$ for each rational number r in (a, b), then $f(x) = 0$ for all $x \in (a, b)$.

(b) Let f and g be continuous real-valued functions on (a, b) such that $f(r) = g(r)$ for each rational number r in (a, b). Prove that $f(x) = g(x)$ for all $x \in (a, b)$.

17.13. **(a)** Let $f(x) = 1$ for rational numbers x and $f(x) = 0$ for irrational numbers. Show that f is discontinuous at every x in $\mathbb{R}$.

(b) Let $h(x) = x$ for rational numbers x and $h(x) = 0$ for irrational numbers. Show that h is continuous at $x = 0$ and at no other point.

17.14. For each rational number x, write x as $\frac{p}{q}$ where p, q are integers with no common factors and $q > 0$, and then define $f(x) = \frac{1}{q}$. Also define $f(x) = 0$ for all $x \in \mathbb{R} \setminus \mathbb{Q}$. Thus $f(x) = 1$ for each integer, $f(\frac{1}{2}) = f(-\frac{1}{2}) = f(\frac{3}{2}) = \cdots = \frac{1}{2}$, etc. Show that f is continuous at each point of $\mathbb{R} \setminus \mathbb{Q}$ and discontinuous at each point of $\mathbb{Q}$.

17.15. Let f be a real-valued function whose domain is a subset of $\mathbb{R}$. Show that f is continuous at x_0 in $\operatorname{dom}(f)$ if and only if, for every sequence (x_n) in $\operatorname{dom}(f) \setminus \{x_0\}$ that converges to x_0, we have $\lim f(x_n) = f(x_0)$.

Now select any $a_0 < b_0$ in I and suppose, say, that $f(a_0) < f(b_0)$. We will show that f is strictly increasing on I. By (1) we have

$$\begin{array}{llll} f(x) < f(a_0) & \text{for} & x < a_0 & [\text{since } x < a_0 < b_0], \\ f(a_0) < f(x) < f(b_0) & \text{for} & a_0 < x < b_0, & \\ f(b_0) < f(x) & \text{for} & x > b_0 & [\text{since } a_0 < b_0 < x]. \end{array}$$

In particular,

$$f(x) < f(a_0) \quad \text{for all} \quad x < a_0, \tag{2}$$

$$f(a_0) < f(x) \quad \text{for all} \quad x > a_0. \tag{3}$$

Consider any $x_1 < x_2$ in I. If $x_1 \le a_0 \le x_2$, then $f(x_1) < f(x_2)$ by (2) and (3). If $x_1 < x_2 < a_0$, then $f(x_1) < f(a_0)$ by (2), so by (1) we have $f(x_1) < f(x_2)$. Finally, if $a_0 < x_1 < x_2$, then $f(a_0) < f(x_2)$, so that $f(x_1) < f(x_2)$. ■

Exercises

18.1. Let f be as in Theorem 18.1. Show that if the function $-f$ assumes its maximum at $x_0 \in [a, b]$, then f assumes its minimum at x_0.

18.2. Reread the proof of Theorem 18.1 with $[a, b]$ replaced by (a, b). Where does it break down? Discuss.

18.3. Use calculus to find the maximum and minimum of $f(x) = x^3 - 6x^2 + 9x + 1$ on $[0, 5)$.

18.4. Let $S \subseteq \mathbb{R}$ and suppose there exists a sequence (x_n) in S that converges to a number $x_0 \notin S$. Show that there exists an unbounded continuous function on S.

18.5. **(a)** Let f and g be continuous functions on $[a, b]$ such that $f(a) \ge g(a)$ and $f(b) \le g(b)$. Prove that $f(x_0) = g(x_0)$ for at least one x_0 in $[a, b]$.

(b) Show that Example 1 can be viewed as a special case of part (a).

18.6. Prove that $x = \cos x$ for some x in $(0, \frac{\pi}{2})$.

18.7. Prove that $x2^x = 1$ for some x in $(0, 1)$.

18.8. Suppose that f is a real-valued continuous function on $\mathbb{R}$ and that $f(a)f(b) < 0$ for some $a, b \in \mathbb{R}$. Prove that there exists x between a and b such that $f(x) = 0$.

18.9. Prove that a polynomial function f of odd degree has at least one real root. *Hint*: It may help to consider first the case of a cubic, i.e., $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ where $a_3 \neq 0$.

18.10. Suppose that f is continuous on $[0, 2]$ and that $f(0) = f(2)$. Prove that there exist x, y in $[0, 2]$ such that $|y - x| = 1$ and $f(x) = f(y)$. *Hint*: Consider $g(x) = f(x + 1) - f(x)$ on $[0, 1]$.

18.11. **(a)** Show that if f is strictly increasing on an interval I, then $-f$ is strictly decreasing on I.

(b) State and prove Theorems 18.4 and 18.5 for strictly decreasing functions.

18.12. Let $f(x) = \sin(\frac{1}{x})$ for $x \neq 0$ and let $f(0) = 0$.

(a) Observe that f is discontinuous at 0 by Exercise 17.10(b).

(b) Show that f has the intermediate value property on $\mathbb{R}$.

§19 Uniform Continuity

Let f be a real-valued function whose domain is a subset of $\mathbb{R}$. Theorem 17.2 tells us that f is continuous on a set $S \subseteq \text{dom}(f)$ if and only if

$$\begin{array}{l}\text{for each } x_0 \in S \text{ and } \epsilon > 0 \text{ there is } \delta > 0 \text{ so that} \\ x \in \text{dom}(f),\ |x - x_0| < \delta \text{ imply } |f(x) - f(x_0)| < \epsilon.\end{array} \qquad (*)$$

The choice of δ depends on $\epsilon > 0$ and on the point x_0 in S.

Example 1

We verify (*) for the function $f(x) = \frac{1}{x^2}$ on $(0, \infty)$. Let $x_0 > 0$ and $\epsilon > 0$. We need to show that $|f(x) - f(x_0)| < \epsilon$ for $|x - x_0|$ sufficiently small. Note that

$$f(x) - f(x_0) = \frac{1}{x^2} - \frac{1}{x_0^2} = \frac{x_0^2 - x^2}{x^2x_0^2} = \frac{(x_0 - x)(x_0 + x)}{x^2x_0^2}. \qquad (1)$$

Exercises

19.1. Which of the following continuous functions are uniformly continuous on the specified set? Justify your answers. Use any theorems you wish.

(a) $f(x) = x^{17}\sin x - e^x \cos 3x$ on $[0, \pi]$,

(b) $f(x) = x^3$ on $[0, 1]$,

(c) $f(x) = x^3$ on $(0, 1)$,

(d) $f(x) = x^3$ on $\mathbb{R}$,

(e) $f(x) = \frac{1}{x^3}$ on $(0, 1]$,

(f) $f(x) = \sin\frac{1}{x^2}$ on $(0, 1]$,

(g) $f(x) = x^2 \sin\frac{1}{x}$ on $(0, 1]$.

19.2. Prove that each of the following functions is uniformly continuous on the indicated set by directly verifying the ϵ-δ property in Definition 19.1.

(a) $f(x) = 3x + 11$ on $\mathbb{R}$,

(b) $f(x) = x^2$ on $[0, 3]$,

(c) $f(x) = \frac{1}{x}$ on $[\frac{1}{2}, \infty)$.

19.3. Repeat Exercise 19.2 for the following.

(a) $f(x) = \frac{x}{x+1}$ on $[0, 2]$,

(b) $f(x) = \frac{5x}{2x-1}$ on $[1, \infty)$.

19.4. **(a)** Prove that if f is uniformly continuous on a bounded set S, then f is a bounded function on S. *Hint*: Assume not. Use Theorems 11.5 and 19.4.

(b) Use (a) to give yet another proof that $\frac{1}{x^2}$ is not uniformly continuous on $(0, 1)$.

19.5. Which of the following continuous functions is uniformly continuous on the specified set? Justify your answers, using appropriate theorems or Exercise 19.4(a).

(a) $\tan x$ on $[0, \frac{\pi}{4}]$,

(b) $\tan x$ on $[0, \frac{\pi}{2})$,

(c) $\frac{1}{x}\sin^2 x$ on $(0, \pi]$,

(d) $\frac{1}{x-3}$ on $(0, 3)$,

(e) $\frac{1}{x-3}$ on $(3, \infty)$,

(f) $\frac{1}{x-3}$ on $(4, \infty)$.

19.6. (a) Let $f(x) = \sqrt{x}$ for $x \geq 0$. Show that f' is unbounded on $(0, 1]$ but that f is nevertheless uniformly continuous on $(0, 1]$. Compare with Theorem 19.6.

(b) Show that f is uniformly continuous on $[1, \infty)$.

19.7. (a) Let f be a continuous function on $[0, \infty)$. Prove that if f is uniformly continuous on $[k, \infty)$ for some k, then f is uniformly continuous on $[0, \infty)$.

(b) Use (a) and Exercise 19.6(b) to prove that $\sqrt{x}$ is uniformly continuous on $[0, \infty)$.

19.8. (a) Use the Mean Value theorem to prove that

$$|\sin x - \sin y| \leq |x - y|$$

for all x, y in $\mathbb{R}$; see the proof of Theorem 19.6.

(b) Show that $\sin x$ is uniformly continuous on $\mathbb{R}$.

19.9. Let $f(x) = x \sin(\frac{1}{x})$ for $x \neq 0$ and $f(0) = 0$.

(a) Observe that f is continuous on $\mathbb{R}$; see Exercises 17.3(f) and 17.9(c).

(b) Why is f uniformly continuous on any bounded subset of $\mathbb{R}$?

(c) Is f uniformly continuous on $\mathbb{R}$?

19.10. Repeat Exercise 19.9 for the function g where $g(x) = x^2 \sin(\frac{1}{x})$ for $x \neq 0$ and $g(0) = 0$.

19.11. Accept the fact that the function $\tilde{h}$ in Example 9 is continuous on $\mathbb{R}$; prove that it is uniformly continuous on $\mathbb{R}$.

§20 Limits of Functions

A function f is continuous at a point a provided the values $f(x)$ are near the value $f(a)$ for x near a [and $x \in \text{dom}(f)$]. See Definition 17.1

Exercises

20.1. Sketch the function $f(x) = \frac{x}{|x|}$. Determine, by inspection, the limits $\lim_{x\to\infty} f(x)$, $\lim_{x\to 0^+} f(x)$, $\lim_{x\to 0^-} f(x)$, $\lim_{x\to -\infty} f(x)$ and $\lim_{x\to 0} f(x)$ *when they exist*. Also indicate when they do not exist.

20.2. Repeat Exercise 20.1 for $f(x) = \frac{x^3}{|x|}$.

20.3. Repeat Exercise 20.1 for $f(x) = \frac{\sin x}{x}$. See Example 9 of §19.

20.4. Repeat Exercise 20.1 for $f(x) = x \sin \frac{1}{x}$.

20.5. Prove the limit assertions in Exercise 20.1.

20.6. Prove the limit assertions in Exercise 20.2.

20.7. Prove the limit assertions in Exercise 20.3.

20.8. Prove the limit assertions in Exercise 20.4.

20.9. Repeat Exercise 20.1 for $f(x) = \frac{1-x^2}{x}$.

20.10. Prove the limit assertions in Exercise 20.9.

20.11. Find the following limits.

(a) $\lim_{x\to a} \frac{x^2-a^2}{x-a}$ **(b)** $\lim_{x\to b} \frac{\sqrt{x}-\sqrt{b}}{x-b}$, $b > 0$

(c) $\lim_{x\to a} \frac{x^3-a^3}{x-a}$

Hint for (c): $x^3 - a^3 = (x-a)(x^2+ax+a^2)$.

20.12. **(a)** Sketch the function $f(x) = (x-1)^{-1}(x-2)^{-2}$.

(b) Determine $\lim_{x\to 2^+} f(x)$, $\lim_{x\to 2^-} f(x)$, $\lim_{x\to 1^+} f(x)$ and $\lim_{x\to 1^-} f(x)$.

(c) Determine $\lim_{x\to 2} f(x)$ and $\lim_{x\to 1} f(x)$ if they exist.

20.13. Prove that if $\lim_{x\to a} f(x) = 3$ and $\lim_{x\to a} g(x) = 2$, then

(a) $\lim_{x\to a}[3f(x) + g(x)^2] = 13$,

(b) $\lim_{x\to a} \frac{1}{g(x)} = \frac{1}{2}$,

(c) $\lim_{x\to a} \sqrt{3f(x) + 8g(x)} = 5$.

20.14. Prove that $\lim_{x\to 0^+} \frac{1}{x} = +\infty$ and $\lim_{x\to 0^-} \frac{1}{x} = -\infty$.

20.15. Prove $\lim_{x\to -\infty} f(x) = 0$ and $\lim_{x\to 2^+} f(x) = +\infty$ for the function f in Example 4.

20.16. Suppose that the limits $L_1 = \lim_{x\to a^+} f_1(x)$ and $L_2 = \lim_{x\to a^+} f_2(x)$ exist.

(a) Show that if $f_1(x) \leq f_2(x)$ for all x in some interval (a, b), then $L_1 \leq L_2$.

(b) Suppose that, in fact, $f_1(x) < f_2(x)$ for all x in some interval (a, b). Can you conclude that $L_1 < L_2$?

20.17. Show that if $\lim_{x\to a^+} f_1(x) = \lim_{x\to a^+} f_3(x) = L$ and if $f_1(x) \leq f_2(x) \leq f_3(x)$ for all x in some interval (a, b), then $\lim_{x\to a^+} f_2(x) = L$. *Warning*: This is not immediate from Exercise 20.16(a), because we are not assuming that $\lim_{x\to a^+} f_2(x)$ exists; this must be proved.

20.18. Let $f(x) = \frac{\sqrt{1+3x^2}-1}{x^2}$ for $x \neq 0$. Show that $\lim_{x\to 0} f(x)$ exists and determine its value. Justify all claims.

20.19. The limits defined in Definition 20.3 do not depend on the choice of the set S. As an example, consider $a < b_1 < b_2$ and suppose that f is defined on (a, b_2). Show that if the limit $\lim_{x\to a^S} f(x)$ exists for either $S = (a, b_1)$ or $S = (a, b_2)$, then the limit exists for the other choice of S and these limits are identical. Their common value is what we write as $\lim_{x\to a^+} f(x)$.

20.20. Let f_1 and f_2 be functions such that $\lim_{x\to a^S} f_1(x) = +\infty$ and such that the limit $L_2 = \lim_{x\to a^S} f_2(x)$ exists.

(a) Prove that $\lim_{x\to a^S}(f_1 + f_2)(x) = +\infty$ if $L_2 \neq -\infty$. *Hint*: Use Exercise 9.11.

(b) Prove that $\lim_{x\to a^S}(f_1 f_2)(x) = +\infty$ if $0 < L_2 \leq +\infty$. *Hint*: Use Theorem 9.9.

(c) Prove that $\lim_{x\to a^S}(f_1 f_2)(x) = -\infty$ if $-\infty \leq L_2 < 0$.

(d) What can you say about $\lim_{x\to a^S}(f_1 f_2)(x)$ if $L_2 = 0$?

§21 * More on Metric Spaces: Continuity

In this section and the next section we continue the introduction to metric space ideas initiated in §13. More thorough treatments appear in [25], [33] and [36]. In particular, for this brief introduction we avoid the technical and somewhat confusing matter of relative topologies that is not, and should not be, avoided in the more thorough treatments.

Exercises

21.1. Show that if the functions $f_1, f_2, \ldots, f_k$ in Proposition 21.2 are uniformly continuous, then so is γ.

21.2. Consider $f: S \to S^*$ where (S, d) and (S^*, d^*) are metric spaces. Show that f is continuous at $s_0 \in S$ if and only if

> for every open set U in S^* containing $f(s_0)$, there is an open set V in S containing s_0 such that $f(V) \subseteq U$.

21.3. Let (S, d) be a metric space and choose $s_0 \in S$. Show that $f(s) = d(s, s_0)$ defines a uniformly continuous real-valued function f on S.

21.4. Consider $f: S \to \mathbb{R}$ where (S, d) is a metric space. Show that the following are equivalent:
(i) f is continuous;
(ii) $f^{-1}((a, b))$ is open in S for all $a < b$;
(iii) $f^{-1}((a, b))$ is open in S for all rational $a < b$.

21.5. Let E be a noncompact subset of $\mathbb{R}^k$.

(a) Show that there is an unbounded continuous real-valued function on E. *Hint*: Either E is unbounded or else its closure E^- contains $\boldsymbol{x}_0 \notin E$. In the latter case, use $\frac{1}{g}$ where $g(\boldsymbol{x}) = d(\boldsymbol{x}, \boldsymbol{x}_0)$.

(b) Show that there is a bounded continuous real-valued function on E that does not assume its maximum on E.

21.6. For metric spaces (S_1, d_1), (S_2, d_2), (S_3, d_3), prove that if $f: S_1 \to S_2$ and $g: S_2 \to S_3$ are continuous, then $g \circ f$ is continuous from S_1 into S_3. *Hint*: It is somewhat easier to use Theorem 21.3 than to use the definition.

21.7. **(a)** Observe that if $E \subseteq S$ where (S, d) is a metric space, then (E, d) is also a metric space. In particular, if $E \subseteq \mathbb{R}$, then $d(a, b) = |a - b|$ for $a, b \in E$ defines a metric on E.

(b) For $\gamma: [a, b] \to \mathbb{R}^k$, give the definition of continuity of γ.

21.8. Let (S, d) and (S^*, d^*) be metric spaces. Show that if $f: S \to S^*$ is uniformly continuous, and if (s_n) is a Cauchy sequence in S, then $(f(s_n))$ is a Cauchy sequence in S^*.

21.9. We say a function f *maps* a set E onto a set F provided $f(E) = F$.

(a) Show that there is a continuous function mapping the unit square

$$\{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1\}$$

onto $[0, 1]$.

(b) Do you think there is a continuous function mapping $[0, 1]$ onto the unit square?

21.10. Show that there exist continuous functions

(a) mapping $(0, 1)$ onto $[0, 1]$,

(b) mapping $(0, 1)$ onto $\mathbb{R}$,

(c) mapping $[0, 1] \cup [2, 3]$ onto $[0, 1]$.

21.11. Show that there do not exist continuous functions

(a) mapping $[0, 1]$ onto $(0, 1)$,

(b) mapping $[0, 1]$ onto $\mathbb{R}$.

§22 * More on Metric Spaces: Connectedness

Consider a subset E of $\mathbb{R}$ that is not an interval. As noted in the proof of Corollary 18.3, the property

$$y_1, y_2 \in E \quad \text{and} \quad y_1 < y < y_2 \quad \text{imply} \quad y \in E$$

must fail. So there exist y_1, y_2, y in $\mathbb{R}$ such that

$$y_1 < y < y_2, \qquad y_1, y_2 \in E, \qquad y \notin E. \qquad (*)$$

The set E is not "connected" because y separates E into two pieces. Put another way, if we set $U_1 = (-\infty, y)$ and $U_2 = (y, \infty)$, then we obtain disjoint open sets such that

$$E \subseteq U_1 \cup U_2, \qquad E \cap U_1 \neq \varnothing, \qquad E \cap U_2 \neq \varnothing.$$

The last observation can be promoted to a useful general definition.

22.6 Definition.
Let S be a subset of $\mathbb{R}$. Let $C(S)$ be the set of all bounded continuous real-valued functions on S and, for $f, g \in C(S)$, let

$$d(f, g) = \sup\{|f(x) - g(x)| : x \in S\}.$$

With this definition, $C(S)$ becomes a metric space [Exercise 22.6]. Now note that a sequence (f_n) in this metric space converges to a point [function!] f provided $\lim_{n\to\infty} d(f_n, f) = 0$, that is

$$\lim_{n\to\infty} [\sup\{|f_n(x) - f(x)| : x \in S\}] = 0. \qquad (*)$$

Put another way, for each $\epsilon > 0$ there exists a number N such that

$$|f_n(x) - f(x)| < \epsilon \quad \text{for all} \quad x \in S \quad \text{and} \quad n > N.$$

We will study this important concept in the next chapter, but without using metric space terminology. See Definition 24.2 and Remark 24.4 where (*) is called *uniform convergence.*

A sequence (f_n) in $C(S)$ is a Cauchy sequence with respect to our metric exactly when it is *uniformly Cauchy* as defined in Definition 25.3. In our metric space terminology, Theorem 25.4 simply asserts that $C(S)$ is a *complete* metric space.

Exercises

22.1. Show that there do not exist continuous functions

(a) mapping $[0, 1]$ onto $[0, 1] \cup [2, 3]$,

(b) mapping $(0, 1)$ onto $\mathbb{Q}$.

22.2. Show that $\{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\}$ is a connected subset of $\mathbb{R}^2$.

22.3. Prove that if E is a connected subset of a metric space (S, d), then its closure E^- is also connected.

22.4. Consider the following subset of $\mathbb{R}^2$:

$$E = \left\{\left(x, \sin\frac{1}{x}\right) : x \in (0, 1]\right\};$$

E is simply the graph of $f(x) = \sin\frac{1}{x}$ along the interval $(0, 1]$.

(a) Sketch E and determine its closure E^-.

(b) Show that E^- is connected.

(c) Show that E^- is not path-connected.

22.5. Let E and F be connected sets in some metric space.

(a) Prove that if $E \cap F \neq \varnothing$, then $E \cup F$ is connected.

(b) Give an example to show that $E \cap F$ need not be connected. Incidentally, the empty set *is* connected.

22.6. **(a)** Show that $C(S)$ given in Definition 22.6 is a metric space.

(b) Why did we require the functions in $C(S)$ to be bounded when no such requirement appears in Definition 24.2?

22.7. Show that the metric space B in Exercise 13.3 can be regarded as $C(\mathbb{N})$.

22.8. Consider $C(S)$ for a subset S of $\mathbb{R}$. For a fixed s_0 in S, define $F(f) = f(s_0)$. Show that F is a uniformly continuous real-valued function on the metric space $C(S)$.

22.9. Consider $f, g \in C(S)$ where $S \subseteq \mathbb{R}$. Let $F(t) = tf + (1-t)g$. Show that F is a uniformly continuous function from $\mathbb{R}$ into $C(S)$.

22.10. Let f be a uniformly continuous function in $C(\mathbb{R})$. For each $x \in \mathbb{R}$, let f_x be the function defined by $f_x(y) = f(x+y)$. Let $F(x) = f_x$; show that F is uniformly continuous from $\mathbb{R}$ into $C(\mathbb{R})$.

22.11. Consider $C(S)$ where $S \subseteq \mathbb{R}$, and let $\mathcal{E}$ consist of all f in $C(S)$ such that $\sup\{|f(x)| : x \in S\} \leq 1$.

(a) Show that $\mathcal{E}$ is closed in $C(S)$.

(b) Show that $C(S)$ is connected.

(c) Show that $\mathcal{E}$ is connected.

22.12. Consider a subset $\mathcal{E}$ of $C(S)$, $S \subseteq \mathbb{R}$. A function f_0 in $\mathcal{E}$ is *interior* to $\mathcal{E}$ if there exists a finite subset F of S and an $\epsilon > 0$ such that

$$\{f \in C(S) : |f(x) - f_0(x)| < \epsilon \text{ for } x \in F\} \subseteq \mathcal{E}.$$

The set $\mathcal{E}$ is *open* if every function in $\mathcal{E}$ is interior to $\mathcal{E}$.

(a) Reread Discussion 13.7.

(b) Show that the family of open sets defined above forms a topology for $C(S)$. *Remarks.* This topology is different from the one given by the metric in Definition 22.6. In fact, this topology does not come from any metric at all! It is called the *topology of pointwise convergence* and can be used to study the convergence in Definition 24.1 just as the metric in Definition 22.6 can be used to study the convergence in Definition 24.2.

22.13. Show that a function $f: \mathbb{R} \to \mathbb{R}$ is continuous if and only if its graph $G = \{(x, f(x)) : x \in \mathbb{R}\}$ is connected and closed in $\mathbb{R}^2$. See C.E. Burgess's article, Continuous Functions and Connected Graphs, *American Mathematical Monthly*, vol. **97** (1990), pp. 337-339.

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§23

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Exercises

23.1. For each of the following power series, find the radius of convergence and determine the exact interval of convergence.

(a) $\sum n^2x^n$ **(b)** $\sum(\frac{x}{n})^n$

(c) $\sum(\frac{2^n}{n^2})x^n$ **(d)** $\sum(\frac{n^3}{3^n})x^n$

(e) $\sum(\frac{2^n}{n!})x^n$ **(f)** $\sum(\frac{1}{(n+1)^2 2^n})x^n$

(g) $\sum(\frac{3^n}{n\cdot 4^n})x^n$ **(h)** $\sum(\frac{(-1)^n}{n^2\cdot 4^n})x^n$

23.2. Repeat Exercise 23.1 for the following:

(a) $\sum\sqrt{n}x^n$ **(b)** $\sum\frac{1}{n\sqrt{n}}x^n$

(c) $\sum x^{n!}$ **(d)** $\sum\frac{3^n}{\sqrt{n}}x^{2n+1}$

23.3. Find the exact interval of convergence for the series in Example 6.

23.4. For $n = 0, 1, 2, 3, \ldots$, let $a_n = [\frac{4+2(-1)^n}{5}]^n$.

(a) Find $\limsup(a_n)^{1/n}$, $\liminf(a_n)^{1/n}$, $\limsup|\frac{a_{n+1}}{a_n}|$ and $\liminf|\frac{a_{n+1}}{a_n}|$.

(b) Do the series $\sum a_n$ and $\sum(-1)^na_n$ converge? Explain briefly.

(c) Now consider the power series $\sum a_nx^n$ with the coefficients a_n as above. Find the radius of convergence and determine the exact interval of convergence for the series.

23.5. Consider a power series $\sum a_nx^n$ with radius of convergence R.

(a) Prove that if all the coefficients a_n are integers and if infinitely many of them are nonzero, then $R \leq 1$.

(b) Prove that if $\limsup|a_n| > 0$, then $R \leq 1$.

23.6. **(a)** Suppose that $\sum a_nx^n$ has finite radius of convergence R and that $a_n \geq 0$ for all n. Show that if the series converges at R, then it also converges at $-R$.

(b) Give an example of a power series whose interval of convergence is exactly $(-1, 1]$.

The next three exercises are designed to show that the notion of convergence of functions discussed prior to Example 8 has many defects.

23.7. For each $n \in \mathbb{N}$, let $f_n(x) = (\cos x)^n$. Each f_n is a continuous function. Nevertheless, show that

(a) $\lim f_n(x) = 0$ unless x is a multiple of π,

(b) $\lim f_n(x) = 1$ if x is an even multiple of π,

(c) $\lim f_n(x)$ does not exist if x is an odd multiple of π.

23.8. For each $n \in \mathbb{N}$, let $f_n(x) = \frac{1}{n}\sin nx$. Each f_n is a differentiable function. Show that

(a) $\lim f_n(x) = 0$ for all $x \in \mathbb{R}$,

(b) but $\lim f_n'(x)$ need not exist [at $x = \pi$ for instance].

23.9. Let $f_n(x) = nx^n$ for $x \in [0, 1]$ and $n \in \mathbb{N}$. Show that

(a) $\lim f_n(x) = 0$ for $x \in [0, 1)$. *Hint*: Use Exercise 9.12.

(b) However, $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = 1$.

§24 Uniform Convergence

We first formalize the notion of convergence discussed prior to Example 8 in the preceding section.

24.1 Definition.
Let (f_n) be a sequence of real-valued functions defined on a set $S \subseteq \mathbb{R}$. The sequence (f_n) *converges pointwise* [i.e., at each point] to a function f defined on S if

$$\lim_{n\to\infty} f_n(x) = f(x) \quad \text{for all} \quad x \in S.$$

We often write $\lim f_n = f$ *pointwise* [*on* S] or $f_n \to f$ *pointwise* [*on* S].

Example 1
All the functions f obtained in the last section as a limit of a sequence of functions were pointwise limits. See Example 8 of §23 and Exercises 23.7–23.9. In Exercise 23.8 we have $f_n \to 0$ pointwise on $\mathbb{R}$, and in Exercise 23.9 we have $f_n \to 0$ pointwise on $[0, 1)$.

Example 2
Let $f_n(x) = x^n$ for $x \in [0, 1]$. Then $f_n \to f$ pointwise on $[0, 1]$ where $f(x) = 0$ for $x \in [0, 1)$ and $f(1) = 1$.

$\frac{1}{\sqrt{n}}$ and its minimum at $-\frac{1}{\sqrt{n}}$. Since $f_n(\pm\frac{1}{\sqrt{n}}) = \pm\frac{1}{2\sqrt{n}}$, we conclude that

$$\lim_{n\to\infty}[\sup\{|f_n(x)| : x \in S\}] = \lim_{n\to\infty} \frac{1}{2\sqrt{n}} = 0.$$

Therefore $f_n \to 0$ uniformly on $\mathbb{R}$ by Remark 24.4.

Example 8

Let $f_n(x) = n^2x^n(1-x)$ for $x \in [0,1]$. Then we have $\lim_{n\to\infty} f_n(1) = 0$. For $x \in [0,1)$ we have $\lim_{n\to\infty} n^2x^n = 0$ by applying Exercise 9.12 [since

$$\frac{(n+1)^2x^{n+1}}{n^2x^n} = \left(\frac{n+1}{n}\right)^2 x \to x],$$

and hence $\lim_{n\to\infty} f_n(x) = 0$. Thus $f_n \to 0$ pointwise on $[0,1]$. Again, to find the maximum and minimum of f_n we set its derivative equal to 0. We obtain $x^n(-1) + (1-x)nx^{n-1} = 0$ or $x^{n-1}[n-(n+1)x] = 0$. Since f_n takes the value 0 at both endpoints of the interval $[0,1]$, it follows that f_n takes it maximum at $\frac{n}{n+1}$. We have

$$f_n\left(\frac{n}{n+1}\right) = n^2\left(\frac{n}{n+1}\right)^n\left(1-\frac{n}{n+1}\right) = \frac{n^2}{n+1}\left(\frac{n}{n+1}\right)^n. \qquad (1)$$

The reciprocal of $(\frac{n}{n+1})^n$ is $(1+\frac{1}{n})^n$, the nth term of a sequence which has limit e. This was mentioned, but not proved, in Example 3 of §7; a proof is given in Theorem 37.11. Therefore we have $\lim(\frac{n}{n+1})^n = \frac{1}{e}$. Since $\lim[\frac{n^2}{n+1}] = +\infty$, we conclude from (1) that $\lim f_n(\frac{n}{n+1}) = +\infty$; see Exercise 12.9(a). In particular, (f_n) does *not* converge uniformly to 0.

Exercises

24.1. Let $f_n(x) = \frac{1+2\cos^2 nx}{\sqrt{n}}$. Prove carefully that (f_n) converges uniformly to 0 on $\mathbb{R}$.

24.2. For $x \in [0,\infty)$, let $f_n(x) = \frac{x}{n}$.

(a) Find $f(x) = \lim f_n(x)$.

(b) Determine whether $f_n \to f$ uniformly on $[0,1]$.

(c) Determine whether $f_n \to f$ uniformly on $[0, \infty)$.

24.3. Repeat Exercise 24.2 for $f_n(x) = \frac{1}{1+x^n}$.

24.4. Repeat Exercise 24.2 for $f_n(x) = \frac{x^n}{1+x^n}$.

24.5. Repeat Exercise 24.2 for $f_n(x) = \frac{x^n}{n+x^n}$.

24.6. Let $f_n(x) = (x - \frac{1}{n})^2$ for $x \in [0, 1]$.

(a) Does the sequence (f_n) converge pointwise on the set $[0, 1]$? If so, give the limit function.

(b) Does (f_n) converge uniformly on $[0, 1]$? Prove your assertion.

24.7. Repeat Exercise 24.6 for $f_n(x) = x - x^n$.

24.8. Repeat Exercise 24.6 for $f_n(x) = \sum_{k=0}^{n} x^k$.

24.9. Consider $f_n(x) = nx^n(1 - x)$ for $x \in [0, 1]$.

(a) Find $f(x) = \lim f_n(x)$.

(b) Does $f_n \to f$ uniformly on $[0, 1]$? Justify.

(c) Does $\int_0^1 f_n(x)\,dx$ converge to $\int_0^1 f(x)\,dx$? Justify.

24.10. (a) Prove that if $f_n \to f$ uniformly on a set S, and if $g_n \to g$ uniformly on S, then $f_n + g_n \to f + g$ uniformly on S.

(b) Do you believe that the analogue of (a) holds for products? If so, see the next exercise.

24.11. Let $f_n(x) = x$ and $g_n(x) = \frac{1}{n}$ for all $x \in \mathbb{R}$. Let $f(x) = x$ and $g(x) = 0$ for $x \in \mathbb{R}$.

(a) Observe that $f_n \to f$ uniformly on $\mathbb{R}$ [obvious!] and that $g_n \to g$ uniformly on $\mathbb{R}$ [almost obvious].

(b) Observe that the sequence $(f_n g_n)$ does not converge uniformly to fg on $\mathbb{R}$. Compare Exercise 24.2.

24.12. Prove the assertion in Remark 24.4.

24.13. Prove that if (f_n) is a sequence of uniformly continuous functions on an interval (a, b), and if $f_n \to f$ uniformly on (a, b), then f is also uniformly continuous on (a, b). *Hint*: Try an $\frac{\epsilon}{3}$ argument as in the proof of Theorem 24.3.

24.14. Let $f_n(x) = \frac{nx}{1+n^2x^2}$.

(a) Show that $f_n \to 0$ pointwise on $\mathbb{R}$.

(b) Does $f_n \to 0$ uniformly on $[0, 1]$? Justify.

(c) Does $f_n \to 0$ uniformly on $[1, \infty)$? Justify.

24.15. Let $f_n(x) = \frac{nx}{1+nx}$ for $x \in [0, \infty)$.

(a) Find $f(x) = \lim f_n(x)$.

(b) Does $f_n \to f$ uniformly on $[0, 1]$? Justify.

(c) Does $f_n \to f$ uniformly on $[1, \infty)$? Justify.

24.16. Repeat Exercise 24.15 for $f_n(x) = \frac{nx}{1+nx^2}$.

24.17. Let (f_n) be a sequence of continuous functions on $[a, b]$ that converges uniformly to f on $[a, b]$. Show that if (x_n) is a sequence in $[a, b]$ and if $x_n \to x$, then $\lim_{n\to\infty} f_n(x_n) = f(x)$.

§25 More on Uniform Convergence

Our next theorem shows that one can interchange integrals and *uniform* limits. The adjective "uniform" here is important; compare Exercise 23.9.

25.1 Discussion.
To prove Theorem 25.2 below we merely use some basic facts about integration which should be familiar [or believable] even if your calculus is rusty. Specifically, we use:

(a) *If g and h are integrable on $[a, b]$ and if $g(x) \le h(x)$ for all $x \in [a, b]$, then $\int_a^b g(x)\,dx \le \int_a^b h(x)\,dx$.* See Theorem 33.4.

We also use the following corollary:

(b) *If g is integrable on $[a, b]$, then*

$$\left| \int_a^b g(x)\,dx \right| \le \int_a^b |g(x)|\,dx.$$

Continuous functions on closed intervals are integrable, as noted in Discussion 19.3 and proved in Theorem 33.2.

Example 5
Show that if the series $\sum g_n$ converges uniformly on a set S, then

$$\lim_{n\to\infty}[\sup\{|g_n(x)| : x \in S\}] = 0. \tag{1}$$

Solution
Let $\epsilon > 0$. Since the series $\sum g_n$ satisfies the Cauchy criterion, there exists N such that

$$n \geq m > N \quad \text{implies} \quad \left|\sum_{k=m}^{n} g_k(x)\right| < \epsilon \quad \text{for all} \quad x \in S.$$

In particular,

$$n > N \quad \text{implies} \quad |g_n(x)| < \epsilon \quad \text{for all} \quad x \in S.$$

Therefore

$$n > N \quad \text{implies} \quad \sup\{|g_n(x)| : x \in S\} \leq \epsilon.$$

This establishes (1). □

Exercises

25.1. Derive 25.1(b) from 25.1(a). *Hint*: Apply (a) twice, once to g and $|g|$ and once to $-|g|$ and g.

25.2. Let $f_n(x) = \frac{x^n}{n}$. Show that (f_n) is uniformly convergent on $[-1, 1]$ and specify the limit function.

25.3. Let $f_n(x) = \frac{n+\cos x}{2n+\sin^2 x}$ for all real numbers x.

(a) Show that (f_n) converges uniformly on $\mathbb{R}$. *Hint*: First decide what the limit function is; then show (f_n) converges uniformly to it.

(b) Calculate $\lim_{n\to\infty} \int_2^7 f_n(x)\,dx$. *Hint*: Don't integrate f_n.

25.4. Let (f_n) be a sequence of functions on a set $S \subseteq \mathbb{R}$, and suppose that $f_n \to f$ uniformly on S. Prove that (f_n) is uniformly Cauchy on S. *Hint*: Use the proof of Lemma 10.9 as a model, but be careful.

25.5. Let (f_n) be a sequence of bounded functions on a set S, and suppose that $f_n \to f$ uniformly on S. Prove that f is a bounded function on S.

25.6. **(a)** Show that if $\sum |a_k| < \infty$, then $\sum a_k x^k$ converges uniformly on $[-1, 1]$ to a continuous function.

(b) Does $\sum_{n=1}^{\infty} \frac{1}{n^2} x^n$ represent a continuous function on $[-1, 1]$?

25.7. Show that $\sum_{n=1}^{\infty} \frac{1}{n^2} \cos nx$ converges uniformly on $\mathbb{R}$ to a continuous function.

25.8. Show that $\sum_{n=1}^{\infty} \frac{x^n}{n^2 2^n}$ has radius of convergence 2 and that the series converges uniformly to a continuous function on $[-2, 2]$.

25.9. **(a)** Let $0 < a < 1$. Show that the series $\sum_{n=0}^{\infty} x^n$ converges uniformly on $[-a, a]$ to $\frac{1}{1-x}$.

(b) Does the series $\sum_{n=0}^{\infty} x^n$ converge uniformly on $(-1, 1)$ to $\frac{1}{1-x}$? Explain.

25.10. **(a)** Show that $\sum \frac{x^n}{1+x^n}$ converges for $x \in [0, 1)$.

(b) Show that the series converges uniformly on $[0, a]$ for each a, $0 < a < 1$.

(c) Does the series converge uniformly on $[0, 1)$? Explain.

25.11. **(a)** Sketch the functions g_0, g_1, g_2 and g_3 in Example 3.

(b) Prove that the function f in Example 3 is continuous.

25.12. Suppose that $\sum_{k=1}^{\infty} g_k$ is a series of continuous functions g_k on $[a, b]$ that converges uniformly to g on $[a, b]$. Prove that

$$\int_a^b g(x)\,dx = \sum_{k=1}^{\infty} \int_a^b g_k(x)\,dx.$$

25.13. Suppose that $\sum_{k=1}^{\infty} g_k$ and $\sum_{k=1}^{\infty} h_k$ converge uniformly on a set S. Show that $\sum_{k=1}^{\infty} (g_k + h_k)$ converges uniformly on S.

25.14. Prove that if $\sum g_k$ converges uniformly on a set S and if h is a bounded function on S, then $\sum h g_k$ converges uniformly on S.

25.15. Let (f_n) be a sequence of continuous functions on $[a, b]$. Suppose that, for each $x \in [a, b]$, $(f_n(x))$ is a nonincreasing sequence of real numbers.

(a) Prove that if $f_n \to 0$ pointwise on $[a, b]$, then $f_n \to 0$ uniformly on $[a, b]$. *Hint:* If not, there exists $\epsilon > 0$ and a sequence (x_n) in $[a, b]$ such that $f_n(x_n) \geq \epsilon$ for all n. Obtain a contradiction.

(b) Prove that if $f_n \to f$ pointwise on $[a, b]$ and if f is continuous on $[a, b]$, then $f_n \to f$ uniformly on $[a, b]$. This is Dini's theorem.

for $k \geq 0$. This tells us that *if* f can be represented by a power series, then that power series must be $\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$. This is the *Taylor series* for f about 0. Frequently, but not always, the Taylor series will agree with f on the interval of convergence. This turns out to be true for many familiar functions. Thus the following relations can be proved:

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k, \qquad \cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}, \qquad \sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

for all x in $\mathbb{R}$. A more detailed study of Taylor series is given in §31.

Exercises

26.1. Prove Theorem 26.4 for $x > 0$.

26.2. **(a)** Observe that $\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}$ for $|x| < 1$; see Example 1.

(b) Evaluate $\sum_{n=1}^{\infty} \frac{n}{2^n}$. Compare with Exercise 14.13(d).

(c) Evaluate $\sum_{n=1}^{\infty} \frac{n}{3^n}$ and $\sum_{n=1}^{\infty} \frac{(-1)^n n}{3^n}$.

26.3. **(a)** Use Exercise 26.2 to derive an explicit formula for $\sum_{n=1}^{\infty} n^2 x^n$.

(b) Evaluate $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ and $\sum_{n=1}^{\infty} \frac{n^2}{3^n}$.

26.4. **(a)** Observe that $e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}$ for $x \in \mathbb{R}$, since we have $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$ for $x \in \mathbb{R}$.

(b) Express $F(x) = \int_0^x e^{-t^2}\, dt$ as a power series.

26.5. Let $f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$ for $x \in \mathbb{R}$. Show that $f' = f$. Do *not* use the fact that $f(x) = e^x$; this is true but has not been established at this point in the text.

26.6. Let $s(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$ and $c(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$ for $x \in \mathbb{R}$.

(a) Prove that $s' = c$ and $c' = -s$.

(b) Prove that $(s^2 + c^2)' = 0$.

(c) Prove that $s^2 + c^2 = 1$.

Actually $s(x) = \sin x$ and $c(x) = \cos x$, but you do *not* need these facts.

26.7. Let $f(x) = |x|$ for $x \in \mathbb{R}$. Is there a power series $\sum a_n x^n$ such that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ for all x? Discuss.

26.8. **(a)** Show that $\sum_{n=0}^{\infty}(-1)^n x^{2n} = \frac{1}{1+x^2}$ for $x \in (-1,1)$. *Hint*: $\sum_{n=0}^{\infty} y^n = \frac{1}{1-y}$. Let $y = -x^2$.

(b) Show that $\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$ for $x \in (-1,1)$.

(c) Show that the equality in (b) also holds for $x = 1$. Use this to find a nice formula for π.

(d) What happens at $x = -1$?

§27 * Weierstrass's Approximation Theorem

Suppose that a power series has radius of convergence greater than 1, and let f denote the function given by the power series. Theorem 26.1 tells us that the partial sums of the power series get uniformly close to f on $[-1,1]$. In other words, f can be approximated uniformly on $[-1,1]$ by polynomials. Weierstrass's approximation theorem is a generalization of this last observation, for it tells us that *any* continuous function on $[-1,1]$ can be uniformly approximated by polynomials on $[-1,1]$. This result is quite different because such a function need not be given by a power series; see Exercise 26.7. The approximation theorem is valid for any closed interval $[a,b]$ and can be deduced from the case $[0,1]$; see Exercise 27.1.

We give the beautiful proof due to S. N. Bernstein. Bernstein was motivated by probabilistic considerations, but we will not use any probability here. One of the attractive features of Bernstein's proof is that the approximating polynomials will be given explicitly. There are more abstract proofs in which this is not the case. On the other hand, the abstract proofs lead to far-reaching and important generalizations. See the treatment in [23] or [36].

We need some preliminary facts about polynomials involving binomial coefficients.

It is worth emphasizing that if f is differentiable on an interval I and if g is differentiable on $\{f(x) : x \in I\}$, then $(g \circ f)'$ is exactly the function $(g' \circ f) \cdot f'$ on I.

Example 5

Let $h(x) = \sin(x^3 + 7x)$ for $x \in \mathbb{R}$. The reader can undoubtedly verify that $h'(x) = (3x^2 + 7)\cos(x^3 + 7x)$ for $x \in \mathbb{R}$ using some automatic technique learned in calculus. Whatever the automatic technique, it is justified by the chain rule. In this case, $h = g \circ f$ where $f(x) = x^3 + 7x$ and $g(y) = \sin y$. Then $f'(x) = 3x^2 + 7$ and $g'(y) = \cos y$ so that

$$h'(x) = g'(f(x)) \cdot f'(x) = [\cos f(x)] \cdot f'(x) = [\cos(x^3 + 7x)] \cdot (3x^2 + 7).$$

We do not want the reader to unlearn the automatic technique, but the reader should be aware that the chain rule stands behind it.

Exercises

28.1. For each of the following functions defined on $\mathbb{R}$, give the set of points at which it is *not* differentiable. Sketches will be helpful.

(a) $e^{|x|}$ **(b)** $\sin|x|$
(c) $|\sin x|$ **(d)** $|x| + |x-1|$
(e) $|x^2 - 1|$ **(f)** $|x^3 - 8|$

28.2. Use the *definition* of derivative to calculate the derivatives of the following functions at the indicated points.

(a) $f(x) = x^3$ at $x = 2$;

(b) $g(x) = x + 2$ at $x = a$;

(c) $f(x) = x^2 \cos x$ at $x = 0$;

(d) $r(x) = \frac{3x+4}{2x-1}$ at $x = 1$.

28.3. **(a)** Let $h(x) = \sqrt{x} = x^{1/2}$ for $x \geq 0$. Use the *definition* of derivative to prove that $h'(x) = \frac{1}{2}x^{-1/2}$ for $x > 0$.

(b) Let $f(x) = x^{1/3}$ for $x \in \mathbb{R}$ and use the definition of derivative to prove that $f'(x) = \frac{1}{3}x^{-2/3}$ for $x \neq 0$.

(c) Is the function f in part (b) differentiable at $x = 0$? Explain.

28.4. Let $f(x) = x^2 \sin\frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$.

(a) Use Theorems 28.3 and 28.4 to show that f is differentiable at each $a \neq 0$ and calculate $f'(a)$. Use, without proof, the fact that $\sin x$ is differentiable and that $\cos x$ is its derivative.

(b) Use the definition to show that f is differentiable at $x = 0$ and that $f'(0) = 0$.

(c) Show that f' is not continuous at $x = 0$.

28.5. Let $f(x) = x^2 \sin \frac{1}{x}$ for $x \neq 0$, $f(0) = 0$, and $g(x) = x$ for $x \in \mathbb{R}$.

(a) Observe that f and g are differentiable on $\mathbb{R}$.

(b) Calculate $f(x)$ for $x = \frac{1}{\pi n}$, $n = \pm 1, \pm 2, \ldots$.

(c) Explain why $\lim_{x \to 0} \frac{g(f(x)) - g(f(0))}{f(x) - f(0)}$ is meaningless.

28.6. Let $f(x) = x \sin \frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$.

(a) Observe that f is continuous at $x = 0$ by Exercise 17.9(c).

(b) Is f differentiable at $x = 0$? Justify your answer.

28.7. Let $f(x) = x^2$ for $x \geq 0$ and $f(x) = 0$ for $x < 0$.

(a) Sketch the graph of f.

(b) Show that f is differentiable at $x = 0$. *Hint*: You will have to use the definition of derivative.

(c) Calculate f' on $\mathbb{R}$ and sketch its graph.

(d) Is f' continuous on $\mathbb{R}$? differentiable on $\mathbb{R}$?

28.8. Let $f(x) = x^2$ for x rational and $f(x) = 0$ for x irrational.

(a) Prove that f is continuous at $x = 0$.

(b) Prove that f is discontinuous at all $x \neq 0$.

(c) Prove that f is differentiable at $x = 0$. *Warning*: You cannot simply claim $f'(x) = 2x$.

28.9. Let $h(x) = (x^4 + 13x)^7$.

(a) Calculate $h'(x)$.

(b) Show how the chain rule justifies your computation in part (a) by writing $h = g \circ f$ for suitable f and g.

28.10. Repeat Exercise 28.9 for the function $h(x) = [\cos x + e^x]^{12}$.

28.11. Suppose that f is differentiable at a, g is differentiable at $f(a)$, and h is differentiable at $g \circ f(a)$. State and prove the chain rule for $(h \circ g \circ f)'(a)$. *Hint*: Apply Theorem 28.4 twice.

28.12. **(a)** Differentiate the function whose value at x is $\cos(e^{x^5-3x})$.

(b) Use Exercise 28.11 or Theorem 28.4 to justify your computation in part (a).

28.13. Show that if f is defined on an open interval containing a, if g is defined on an open interval containing $f(a)$, and if f is continuous at a, then $g \circ f$ is defined on an open interval containing a.

28.14. Suppose that f is differentiable at a. Prove
(a) $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = f'(a)$, **(b)** $\lim_{h\to 0} \frac{f(a+h)-f(a-h)}{2h} = f'(a)$.

28.15. Prove Leibniz' rule

$$(fg)^{(n)}(a) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(a) g^{(n-k)}(a)$$

provided both f and g have n derivatives at a. Here $h^{(j)}$ signifies the jth derivative of h so that $h^{(0)} = h$, $h^{(1)} = h'$, $h^{(2)} = h''$, etc. Also, $\binom{n}{k}$ is the binomial coefficient that appears in the binomial expansion; see Exercise 1.12. *Hint*: Use mathematical induction. For $n = 1$, apply Theorem 28.3(iii).

28.16. Let f be a function defined on an open interval I containing a. Let h be a function defined on an open interval J containing $f(a)$, except at $f(a)$, and suppose that $f(x) \in J$ and $f(x) \neq f(a)$ for all $x \in I \setminus \{a\}$. Then $h \circ f$ is defined on $I \setminus \{a\}$. Use Corollary 20.7 to prove that if $\lim_{x\to a} f(x) = f(a)$ and if $\lim_{y\to f(a)} h(y)$ exists and is finite, then $\lim_{x\to a} h \circ f(x) = \lim_{y\to f(a)} h(y)$.

§29 The Mean Value Theorem

Our first result justifies the following strategy in calculus: To find the maximum and minimum of a continuous function f on an interval $[a, b]$ it suffices to consider (a) the points x where $f'(x) = 0$; (b) the points where f is not differentiable; and (c) the endpoints a and b. These are the candidates for maxima and minima.

shows that

$$g'(y_0) = \frac{1}{nx_0^{n-1}} = \frac{1}{ny_0^{(n-1)/n}} = \frac{1}{n}y_0^{1/n-1}.$$

This shows that the function g is differentiable for $y \neq 0$ and that the rule for differentiating x^n holds for exponents of the form $1/n$; see also Exercise 29.15.

Theorem 29.9 applies to the various inverse functions encountered in calculus. We give one example.

Example 3

The function $f(x) = \sin x$ is one-to-one on $[-\frac{\pi}{2}, \frac{\pi}{2}]$, and it is traditional to use the inverse g of f restricted to this domain; g is usually denoted $\sin^{-1}$ or arcsin. Note that $\mathrm{dom}(g) = [-1, 1]$. For $y_0 = \sin x_0$ in $(-1, 1)$ where $x_0 \in (-\frac{\pi}{2}, \frac{\pi}{2})$, Theorem 29.9 shows that $g'(y_0) = \frac{1}{\cos x_0}$. Since $1 = \sin^2 x_0 + \cos^2 x_0 = y_0^2 + \cos^2 x_0$ and $\cos x_0 > 0$, we may write

$$g'(y_0) = \frac{1}{\sqrt{1 - y_0^2}} \quad \text{for} \quad y_0 \in (-1, 1).$$

Exercises

29.1. Determine whether the conclusion of the Mean Value Theorem holds for the following functions on the specified intervals. If the conclusion holds, give an example of a point x satisfying (1) of Theorem 29.3. If the conclusion fails, state which *hypotheses* of the Mean Value Theorem fail.

(a) x^2 on $[-1, 2]$, **(b)** $\sin x$ on $[0, \pi]$,
(c) $|x|$ on $[-1, 2]$, **(d)** $\frac{1}{x}$ on $[-1, 1]$,
(e) $\frac{1}{x}$ on $[1, 3]$, **(f)** $\mathrm{sgn}(x)$ on $[-2, 2]$.

The function sgn is defined in Exercise 17.10.

29.2. Prove that $|\cos x - \cos y| \leq |x - y|$ for all $x, y \in \mathbb{R}$.

29.3. Suppose that f is differentiable on $\mathbb{R}$ and that $f(0) = 0$, $f(1) = 1$ and $f(2) = 1$.

(a) Show that $f'(x) = \frac{1}{2}$ for some $x \in (0, 2)$.

(b) Show that $f'(x) = \frac{1}{7}$ for some $x \in (0, 2)$.

29.4. Let f and g be differentiable functions on an open interval I. Suppose that a, b in I satisfy $a < b$ and $f(a) = f(b) = 0$. Show that $f'(x) + f(x)g'(x) = 0$ for some $x \in (a, b)$. *Hint*: Consider $h(x) = f(x)e^{g(x)}$.

29.5. Let f be defined on $\mathbb{R}$, and suppose that $|f(x) - f(y)| \leq (x - y)^2$ for all $x, y \in \mathbb{R}$. Prove that f is a constant function.

29.6. Give the equation of the straight line used in the proof of the Mean Value Theorem 29.3.

29.7. **(a)** Suppose that f is twice differentiable on an open interval I and that $f''(x) = 0$ for all $x \in I$. Show that f has the form $f(x) = ax + b$ for suitable constants a and b.

(b) Suppose f is three times differentiable on an open interval I and that $f''' = 0$ on I. What form does f have? Prove your claim.

29.8. Prove (ii)–(iv) of Corollary 29.7.

29.9. Show that $ex \leq e^x$ for all $x \in \mathbb{R}$.

29.10. Let $f(x) = x^2 \sin(\frac{1}{x}) + \frac{x}{2}$ for $x \neq 0$ and $f(0) = 0$.

(a) Show that $f'(0) > 0$; see Exercise 28.4.

(b) Show that f is not increasing on any open interval containing 0.

(c) Compare this example with Corollary 29.7(i).

29.11. Show that $\sin x \leq x$ for all $x \geq 0$. *Hint*: Show that $f(x) = x - \sin x$ is increasing on $[0, \infty)$.

29.12. **(a)** Show that $x < \tan x$ for all $x \in (0, \frac{\pi}{2})$.

(b) Show that $\frac{x}{\sin x}$ is a strictly increasing function on $(0, \frac{\pi}{2})$.

(c) Show that $x \leq \frac{\pi}{2} \sin x$ for $x \in [0, \frac{\pi}{2}]$.

29.13. Prove that if f and g are differentiable on $\mathbb{R}$, if $f(0) = g(0)$ and if $f'(x) \leq g'(x)$ for all $x \in \mathbb{R}$, then $f(x) \leq g(x)$ for $x \geq 0$.

29.14. Suppose that f is differentiable on $\mathbb{R}$, that $1 \leq f'(x) \leq 2$ for $x \in \mathbb{R}$, and that $f(0) = 0$. Prove that $x \leq f(x) \leq 2x$ for all $x \geq 0$.

29.15. Let r be a nonzero rational number $\frac{m}{n}$ where n is a positive integer, m is any nonzero integer, and m and n have no common factors. Let $h(x) = x^r$ where $\mathrm{dom}(h) = [0, \infty)$ if n is even and $m > 0$, dom(h) $= (0, \infty)$ if n is even and $m < 0$, dom(h) $= \mathbb{R}$ if n is odd and $m > 0$, and dom(h) $= \mathbb{R} \setminus \{0\}$ if n is odd and $m < 0$. Show that $h'(x) = rx^{r-1}$ for $x \in$ dom(h), $x \neq 0$. *Hint*: Use Example 2.

29.16. Use Theorem 29.9 to obtain the derivative of the inverse $g =$ arctan of f where $f(x) = \tan x$ for $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

29.17. Let f and g be differentiable on an open interval I and consider $a \in I$. Define h on I by the rules: $h(x) = f(x)$ for $x < a$, and $h(x) = g(x)$ for $x \geq a$. Prove that h is differentiable at a if and only if both $f(a) = g(a)$ and $f'(a) = g'(a)$ hold. *Suggestion*: Draw a picture to see what is going on.

29.18. Let f be differentiable on $\mathbb{R}$ with $a = \sup\{|f'(x)| : x \in \mathbb{R}\} < 1$. Select $s_0 \in \mathbb{R}$ and define $s_n = f(s_{n-1})$ for $n \geq 1$. Thus $s_1 = f(s_0)$, $s_2 = f(s_1)$, etc. Prove that (s_n) is a convergence sequence. *Hint*: To show (s_n) is Cauchy, first show that $|s_{n+1} - s_n| \leq a|s_n - s_{n-1}|$ for $n \geq 1$.

§30 * L'Hospital's Rule

In analysis one frequently encounters limits of the form

$$\lim_{x \to s} \frac{f(x)}{g(x)}$$

where s signifies a, a^+, a^-, ∞ or $-\infty$. See Definition 20.3 concerning such limits. The limit exists and is simply $\frac{\lim_{x\to s} f(x)}{\lim_{x\to s} g(x)}$ provided the limits $\lim_{x\to s} f(x)$ and $\lim_{x\to s} g(x)$ exist and are finite and provided $\lim_{x\to s} g(x) \neq 0$; see Theorem 20.4. If these limits lead to an indeterminate form such as $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then L'Hospital's rule can often be used. Moreover, other indeterminate forms, such as $\infty - \infty$, 1^∞, ∞^0, 0^0 or $0 \cdot \infty$, can usually be reformulated so as to take the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$; see Examples 5–9. Before we state and prove L'Hospital's rule, we will prove a generalized mean value theorem.

30.1 Generalized Mean Value Theorem.
Let f and g be continuous functions on $[a, b]$ that are differentiable on (a, b). Then there exists [at least one] x in (a, b) such that

$$f'(x)[g(b) - g(a)] = g'(x)[f(b) - f(a)]. \tag{1}$$

Neither of the limits $\lim_{x\to 0}(e^x - 1)^{-1}$ or $\lim_{x\to 0} x^{-1}$ exists, so $\lim_{x\to 0} h(x)$ is not an indeterminate form as written. However, $\lim_{x\to 0^+} h(x)$ is indeterminate of the form $\infty - \infty$ and $\lim_{x\to 0^-} h(x)$ is indeterminate of the form $(-\infty) - (-\infty)$. By writing

$$h(x) = \frac{x - e^x + 1}{x(e^x - 1)}$$

the limit $\lim_{x\to 0} h(x)$ becomes an indeterminate of the form $\frac{0}{0}$. By L'Hospital's rule this should be

$$\lim_{x\to 0} \frac{1 - e^x}{xe^x + e^x - 1},$$

which is still indeterminate $\frac{0}{0}$. Note that $xe^x + e^x - 1 \neq 0$ for $x \neq 0$ so that the hypotheses of Theorem 30.2 hold. Applying L'Hospital's rule again, we obtain

$$\lim_{x\to 0} \frac{-e^x}{xe^x + 2e^x} = -\frac{1}{2}.$$

Note that we have $xe^x + 2e^x \neq 0$ for x in $(-2, \infty)$. We conclude that $\lim_{x\to 0} h(x) = -\frac{1}{2}$.

Exercises

30.1. Find the following limits if they exist.

(a) $\lim_{x\to 0} \frac{e^{2x} - \cos x}{x}$ **(b)** $\lim_{x\to 0} \frac{1 - \cos x}{x^2}$

(c) $\lim_{x\to \infty} \frac{x^3}{e^{2x}}$ **(d)** $\lim_{x\to 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x}$

30.2. Find the following limits if they exist.

(a) $\lim_{x\to 0} \frac{x^3}{\sin x - x}$ **(b)** $\lim_{x\to 0} \frac{\tan x - x}{x^3}$

(c) $\lim_{x\to 0}[\frac{1}{\sin x} - \frac{1}{x}]$ **(d)** $\lim_{x\to 0}(\cos x)^{1/x^2}$

30.3. Find the following limits if they exist.

(a) $\lim_{x\to \infty} \frac{x - \sin x}{x}$ **(b)** $\lim_{x\to \infty} x^{\sin(1/x)}$

(c) $\lim_{x\to 0^+} \frac{1 + \cos x}{e^x - 1}$ **(d)** $\lim_{x\to 0} \frac{1 - \cos 2x - 2x^2}{x^4}$

30.4. Let f be a function defined on some interval $(0, a)$, and define $g(y) = f(\frac{1}{y})$ for $y \in (a^{-1}, \infty)$; here we set $a^{-1} = 0$ if $a = \infty$. Show that $\lim_{x\to 0^+} f(x)$ exists if and only if $\lim_{y\to\infty} g(y)$ exists, in which case they are equal.

30.5. Find the limits

(a) $\lim_{x\to 0}(1+2x)^{1/x}$ **(b)** $\lim_{y\to\infty}(1+\frac{2}{y})^y$

(c) $\lim_{x\to\infty}(e^x+x)^{1/x}$

30.6. Let f be differentiable on some interval (c,∞) and suppose that $\lim_{x\to\infty}[f(x)+f'(x)]=L$, where L is finite. Prove that $\lim_{x\to\infty}f(x)=L$ and that $\lim_{x\to\infty}f'(x)=0$. *Hint*: $f(x)=\frac{f(x)e^x}{e^x}$.

30.7. This example is taken from [38] and is due to Otto Stolz, *Math. Annalen* **15** (1879), 556-559. The requirement in Theorem 30.2 that $g'(x)\neq 0$ for x "near" s is important. In a careless application of L'Hospital's rule in which the zeros of g' "cancel" the zeros of f', erroneous results can be obtained. For $x\in\mathbb{R}$, let

$$f(x)=x+\cos x\sin x \quad\text{and}\quad g(x)=e^{\sin x}(x+\cos x\sin x).$$

(a) Show that $\lim_{x\to\infty}f(x)=\lim_{x\to\infty}g(x)=+\infty$.

(b) Show $f'(x)=2(\cos x)^2$ and $g'(x)=e^{\sin x}\cos x[2\cos x+f(x)]$.

(c) Show that $\frac{f'(x)}{g'(x)}=\frac{2e^{-\sin x}\cos x}{2\cos x+f(x)}$ if $\cos x\neq 0$ and $x>3$.

(d) Show that $\lim_{x\to\infty}\frac{2e^{-\sin x}\cos x}{2\cos x+f(x)}=0$ and yet the limit $\lim_{x\to\infty}\frac{f(x)}{g(x)}$ does *not* exist.

§31 Taylor's Theorem

31.1 Discussion.
Consider a power series with radius of convergence $R>0$ [R may be $+\infty$]:

$$f(x)=\sum_{k=0}^{\infty}a_kx^k. \tag{1}$$

By Theorem 26.5 the function f is differentiable in the interval $|x|<R$ and

$$f'(x)=\sum_{k=1}^{\infty}ka_kx^{k-1}.$$

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If $y_n = \frac{1}{x_n}$, then $\lim y_n = +\infty$ [by Theorem 9.10] and we need to show $\lim_{n\to\infty} y_n^k e^{-y_n} = 0$ or

$$\lim_{y\to\infty} y^k e^{-y} = 0. \tag{3}$$

To see (3) note that $e^y \geq \frac{y^{k+1}}{(k+1)!}$ for $y > 0$ by Example 1(a) so that

$$y^k e^{-y} \leq y^k (k+1)!\, y^{-k-1} = \frac{(k+1)!}{y} \quad \text{for} \quad y > 0.$$

The limit (3) also can be verified via k applications of L'Hospital's Rule 30.2.

Just as with power series, one can consider Taylor series that are not centered at 0.

31.8 Definition.
Let f be a function defined on some open interval containing $x_0 \in \mathbb{R}$. If f has derivatives of all order at x_0, then the series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

is called the *Taylor series for f about x_0*.

The theorems in this section are easily transferred to the general Taylor series just defined.

Exercises

31.1. Find the Taylor series for $\cos x$ and indicate why it converges to $\cos x$ for all $x \in \mathbb{R}$.

31.2. Repeat Exercise 31.1 for $\sinh x = \frac{1}{2}(e^x - e^{-x})$ and $\cosh x = \frac{1}{2}(e^x + e^{-x})$.

31.3. In Example 2, why did we apply Theorem 31.3 instead of Corollary 31.4?

31.4. Consider a, b in $\mathbb{R}$ where $a < b$. Show that there exist infinitely differentiable functions f_a, g_b, $h_{a,b}$ and $h^*_{a,b}$ on $\mathbb{R}$ with the following properties. You may assume, without proof, that the sum, product, etc. of infinitely differentiable functions is again infinitely differentiable. The same applies to the quotient provided that the denominator never vanishes.

(a) $f_a(x) = 0$ for $x \le a$ and $f_a(x) > 0$ for $x > a$. *Hint*: Let $f_a(x) = f(x-a)$ where f is the function in Example 3.

(b) $g_b(x) = 0$ for $x \ge b$ and $g_b(x) > 0$ for $x < b$.

(c) $h_{a,b}(x) > 0$ for $x \in (a,b)$ and $h_{a,b}(x) = 0$ for $x \notin (a,b)$.

(d) $h^*_{a,b}(x) = 0$ for $x \le a$ and $h^*_{a,b}(x) = 1$ for $x \ge b$. *Hint*: Use $f_a/(f_a + g_b)$.

31.5. Let $g(x) = e^{-1/x^2}$ for $x \neq 0$ and $g(0) = 0$.

(a) Show that $g^{(n)}(0) = 0$ for all $n = 0, 1, 2, 3, \ldots$. *Hint*: Use Example 3.

(b) Show that the Taylor series for g about 0 agrees with g only at $x = 0$.

31.6. A standard proof of Theorem 31.3 goes as follows. Assume $x > 0$, let M be as in the proof of Theorem 31.3, and let

$$F(t) = f(t) + \sum_{k=1}^{n-1} \frac{(x-t)^k}{k!} f^{(k)}(t) + M \cdot \frac{(x-t)^n}{n!}$$

for $t \in [0, x]$.

(a) Show that F is differentiable on $[0, x]$ and that

$$F'(t) = \frac{(x-t)^{n-1}}{(n-1)!}[f^{(n)}(t) - M].$$

(b) Show that $F(0) = F(x)$.

(c) Apply Rolle's Theorem 29.2 to F to obtain y in $(0, x)$ such that $f^{(n)}(y) = M$.