1. (*pts*) Let $f(x) = \sin\frac{1}{x}$. Use the definition of the limit to prove that $\lim\limits_{x\to 0} f(x)$ does not exist.

Consider sequence $\{x_n\} \to 0$ with $x_n = \frac{1}{2\pi n}$. So $f(x_n) = \sin(2\pi n) = 0 \ \forall n$.

Then $\lim\limits_{n\to\infty} f(x_n) = 0$.

Consider another sequence $\{x_k\} \to 0$ with $x_k = \frac{1}{2\pi k + \frac{\pi}{2}}$

So $f(x_k) = \sin(2\pi k + \frac{\pi}{2}) = 1 \ \forall k$.

Then $\lim\limits_{k\to\infty} f(x_k) = 1$.

Since $\{x_n\}, \{x_k\} \subseteq (-2,0)\cup(0,2)$ where $\{x_n\} \to 0$ and $\{x_k\} \to 0$

$\lim\limits_{k\to\infty} f(x_k) \neq \lim\limits_{n\to\infty} f(x_n)$, $\lim\limits_{x\to 0} f(x)$ D.N.E. by definition.

2. (*pts*) Find the interval of convergence for the following power series

$$\sum \frac{2^n}{n5^{n+1}} x^n$$

$\beta = \limsup\limits_{n\to\infty} |a_n|^{1/n} = \limsup\limits_{n\to\infty} \left|\frac{2^n}{n\,5^{n+1}}\right|^{\frac{1}{n}} = \lim\limits_{n\to\infty} \frac{2}{5}\frac{1}{\sqrt[n]{n}\,5^{1/n}} = \frac{2}{5}$.

Thus, radius of convergence $R = \frac{1}{\beta} = \frac{5}{2}$.

<u>Check endpoints</u>

- Let $x = \frac{5}{2} \Rightarrow \sum \frac{2^n}{n\,5^{n+1}}\left(\frac{5}{2}\right)^n = \sum \frac{1}{5}\frac{1}{n} = \frac{1}{5}\sum \frac{1}{n}$ which is a divergent p-series.

- Let $x = \frac{-5}{2} \Rightarrow \sum \frac{2^n}{n\,5^{n+1}}\left(-\frac{5}{2}\right)^n = \sum \frac{1}{5}\frac{(-1)^n}{n} = \frac{1}{5}\sum (-1)^n \frac{1}{n}$.

Let $a_n = \frac{1}{n}$ which is $+$, $\downarrow$, and $\lim\limits_{n\to\infty} a_n = 0$.

By Alternating Series Test, this is a convergent series.

Therefore, the interval of convergence is

$$\boxed{-\frac{5}{2} \leq x < \frac{5}{2}}$$

3. (*pts*) Let the sequence of functions $\{f_n\}$ be $f_n(x) = x - x^n$ for $x \in [0, 1]$.

(a) Find $f(x)$ such that $\{f_n\} \to f$ on $[0, 1]$.

For $x = 1$, $f_n(1) = 0 \ \forall n \Rightarrow \lim_{n\to\infty} f_n(1) = 0$

For $x \in [0,1)$, $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x - x^n = x$

Define $f(x) = \begin{cases} x & x \in [0,1) \\ 0 & x = 1 \end{cases}$ where $\{f_n\} \to f$ on $[0,1]$

(b) Using the definition, prove $\{f_n\}$ does not converge uniformly to f (found in part a) on $[0, 1]$.

Choose $\varepsilon = \frac{1}{2}$. Let N be given and let $N^* = \lceil N \rceil \in \mathbb{N}$.

Choose $n = N^* + 1$. For $x \neq 1$,

$$|f_n(x) - f(x)| = |x - x^n - x| = x^n \geq \frac{1}{2} \iff x \geq \sqrt[n]{\tfrac{1}{2}}.$$

Choose $x = \sqrt[n]{\frac{1}{2}}$ where we chose $n = \lceil N \rceil + 1 > N$, and we have $|f_n(x) - f(x)| \geq \frac{1}{2} = \varepsilon$.

Thus, $\{f_n\}$ does not converge uniformly to f on $[0,1]$.

4. (*pts*) Let the sequence of functions $\{f_n\}$ be $f_n(x) = \dfrac{1}{1+nx}$ for $x \in [2,\infty)$. Let $f(x) = 0$ for $x \in [2,\infty)$. Using the definition, prove $\{f_n\}$ converges uniformly to f on $x \in [2,\infty)$.

Let $\varepsilon > 0$ be given.

Notice $|f_n(x) - f(x)| = \left|\dfrac{1}{1+nx} - 0\right| = \dfrac{1}{1+nx} < \dfrac{1}{1+2n}$

since $x \geq 2$.

Then,

$$|f_n(x) - f(x)| < \varepsilon \iff \frac{1}{1+2n} < \varepsilon \iff n > \frac{1}{2}\left(\frac{1}{\varepsilon} - 1\right).$$

Choose $N = \frac{1}{2}\left(\frac{1}{\varepsilon} - 1\right)$. Then, we have

$$\forall x \in [2,\infty),\ \forall n > N \implies |f_n(x) - f(x)| < \varepsilon.$$

Therefore, $\{f_n\} \rightrightarrows f$ on $[2,\infty)$.

5. (*pts*) For $x \in [0,1]$, we have the following power series

$$\sqrt{1+x} = \sum \frac{(-1)^n (2n)!}{(1-2n)(n!)^2(4^n)} x^n.$$

Use this fact to build a power series for $\dfrac{1}{\sqrt{1-x^2}}$

Start with $\sqrt{1+x} = \sum_0^\infty \frac{(-1)^n (2n)!}{(1-2n)(n!)^2(4^n)} x^n$

Differentiating both sides, we get

$$\frac{1}{2\sqrt{1+x}} = \sum_1^\infty \frac{(-1)^n (2n)!\, n}{(1-2n)(n!)^2(4^n)} x^{n-1}$$

Substituting $-x^2$ for x & multiplying by 2, we have

$$\frac{1}{\sqrt{1-x^2}} = \sum_1^\infty \frac{(-1)^n (2n)!\, 2n}{(1-2n)(n!)^2(4^n)} (-x^2)^{n-1}$$

6. (*pts*) Prove the following series converges uniformly on $\mathbb{R}$ to a continuous function

$$\sum_{n=1}^\infty \frac{1}{n^2} \cos nx$$

Consider the sequence $\{a_n\}$ where $a_n = \frac{1}{n^2}$. All the terms are nonnegative and $\sum \frac{1}{n^2} < \infty$ since it's a convergent p-series. Notice

$$\left|\frac{1}{n^2}\cos nx\right| \leq \frac{1}{n^2} \quad \forall x \in \mathbb{R}.$$

Thus, the series $\sum \frac{1}{n^2}\cos nx$ converges uniformly on $\mathbb{R}$. Also, the limit is continuous since each partial sum is continuous (addition preserves continuity).

7. (*pts*) Use the definition of the derivative to prove the Quotient Rule.

Let f, g be differentiable at a where $g(a) \neq 0$.
Since $g(a) \neq 0$ and g is continuous at a (from differentiability of g), there exist an open interval I with $a \in I$ such that $g(x) \neq 0$ $\forall x \in I$. For $x \in I$, we have

$$\left(\frac{f}{g}\right)'(a) := \lim_{x\to a} \frac{\frac{f(x)}{g(x)} - \frac{f(a)}{g(a)}}{x-a} = \lim_{x \to a} \frac{f(x)g(a) - f(a)g(x)}{(x-a)\cdot g(x) g(a)}$$

$$= \lim_{x\to a} \frac{f(x)g(a) - f(a)g(a) + f(a)g(a) - f(a)g(x)}{(x-a)\cdot g(x) \cdot g(a)}$$

$$= \lim_{x\to a} \left[g(a) \frac{f(x)-f(a)}{x-a} - f(a) \cdot \frac{g(x)-g(a)}{x-a} \right] \frac{1}{g(x)g(a)} = \frac{g(a)f'(a) - f(a)g'(a)}{[g(a)]^2}$$

Therefore, $\frac{f}{g}$ is differentiable at a.

8. (*pts*) Use the definition of the derivative to show $f(x) = |x| + |x+1|$ is not differentiable at $x = -1$.

Notice by definition of $|x|$, $f(x) = \begin{cases} -2x-1 & x \leq -1 \\ 1 & -1 < x \leq 0 \\ 2x+1 & x > 0 \end{cases}$.

So $$\lim_{x \to -1^-} \frac{f(x) - f(-1)}{x+1} = \lim_{x\to -1^-} \frac{-2x-1-1}{x+1} = \lim_{x \to -1^-} \frac{-2\cancel{(x+1)}}{\cancel{x+1}} = -2$$

and $$\lim_{x\to -1^+} \frac{f(x)-f(-1)}{x+1} = \lim_{x\to -1^+} \frac{1-1}{x+1} = 0$$

Since $$\lim_{x\to -1^+} \frac{f(x)-f(-1)}{x+1} \neq \lim_{x \to -1^-} \frac{f(x)-f(-1)}{x+1},$$

$$f'(-1) := \lim_{x\to -1} \frac{f(x)-f(-1)}{x+1} \text{ does not exist}$$

Thus, f is not differentiable at $x=-1$.

9. (*pts*) Let the sequence of functions $\{f_n\}$ be $f_n(x) = \dfrac{nx}{1+n^2x^2}$ for $x \in [0,1]$. Let $f(x) = 0$ for $x \in [0,1]$. Prove $\{f_n\}$ does not converge uniformly to f on $x \in [0,1]$.

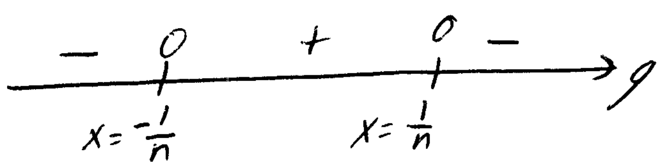
For fixed $n \in \mathbb{N}$. Consider

$$\sup\{|f_n(x) - f(x)| : x \in [0,1]\} = \sup\left\{\frac{nx}{1+n^2x^2} : x \in [0,1]\right\}$$

We need to find max value for $g(x) = \dfrac{nx}{1+n^2x^2}$

$$\overset{\frac{d}{dx}}{\Rightarrow} g'(x) = \frac{n(1+n^2x^2) - nx(2n^2x^2)}{(1+n^2x^2)^2} = \frac{n(1-n^2x^2)}{(1+n^2x^2)^2} = 0$$

$\Rightarrow x = \pm \frac{1}{n}$



Thus, max occurs at $x = \frac{1}{n} \in [0,1]$ and we have

$$\sup\{|f_n(x) - f(x)| : x \in [0,1]\} = \frac{n\frac{1}{n}}{1+n^2(\frac{1}{n})^2} = \frac{1}{2}.$$

So $\lim\limits_{n\to\infty} \sup\{|f_n(x) - f(x)| : x \in [0,1]\} = \lim\limits_{n\to\infty} \frac{1}{2} = \frac{1}{2} \neq 0.$

Therefore, $\{f_n\}$ does not converge uniformly to f on $[0,1]$