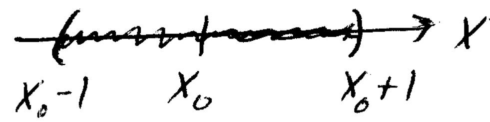
1. (*pts*) Prove that $f(x) = 5x^2 - 7$ is continuous on the interval $(1, \infty)$ by verifying the $\epsilon\delta$-property.

Choose $x_0 \in (1, \infty)$. Let $\varepsilon > 0$ be given.

Note $|f(x) - f(x_0)| = |5x^2 - 7 - (5x_0^2 - 7)| = 5|x^2 - x_0^2| = 5|x - x_0||x + x_0|$

So $|f(x) - f(x_0)| < \varepsilon$ iff $5|x - x_0||x + x_0| < \varepsilon$

Assume $\delta < 1$,



$\Rightarrow x_0 - 1 < x < x_0 + 1 \quad \Rightarrow 2x_0 - 1 < x + x_0 < 2x_0 + 1$

$\Rightarrow |x + x_0| < 2x_0 + 1$ (since $x_0 > 0$)

Then $5|x - x_0||x + x_0| < 5|x - x_0|(2x_0 + 1) < \varepsilon$

iff $|x - x_0| < \dfrac{\varepsilon}{5(2x_0 + 1)}$. Choose $\delta = \min\left\{1, \dfrac{\varepsilon}{5(2x_0+1)}\right\}$.

Thus, if $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$ and f is continuous at x_0. Since x_0 was arbitrary, f is continuous on $(1, \infty)$

2. (*pts*) Prove that $f(x) = 5x^2 - 7$ is not uniformly continuous on the interval $(1, \infty)$ by definition.

Want to show

$\exists \varepsilon > 0, \forall \delta > 0, \exists x, y \in (1, \infty)$ w/ $|x-y| < \delta$ & $|f(x) - f(y)| \geq \varepsilon$

Choose $\varepsilon = 1$ and let $\delta > 0$ be given.

From (1), $|f(x) - f(y)| \geq 1$ iff $5|x-y||x+y| \geq 1$.

Let $y = x + \frac{\delta}{2}$. So $5|x-y||x+y| = \frac{5}{2}\delta|2x + \frac{\delta}{2}|$

& $|f(x) - f(y)| \geq 1$ iff $\frac{5}{2}\delta|2x + \frac{\delta}{2}| \geq 1$

iff $|2x + \frac{\delta}{2}| \geq \frac{2}{5\delta}$ iff $x \geq \frac{1}{2}\left(\frac{2}{5\delta} - \frac{\delta}{2}\right)$ since $x > 0$.

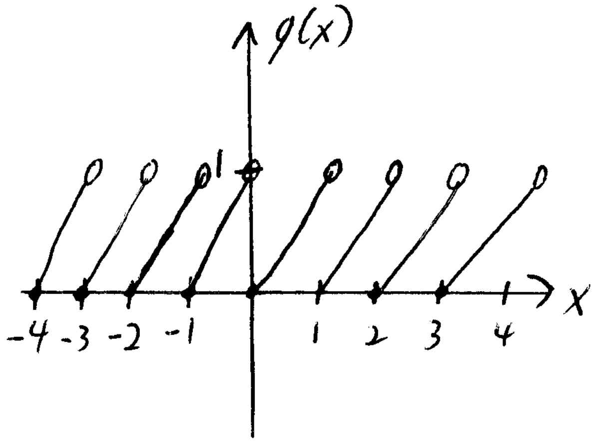
Choose $x = \frac{1}{2}\left(\frac{2}{5\delta} - \frac{\delta}{2}\right)$ & $y = x + \frac{\delta}{2}$ where

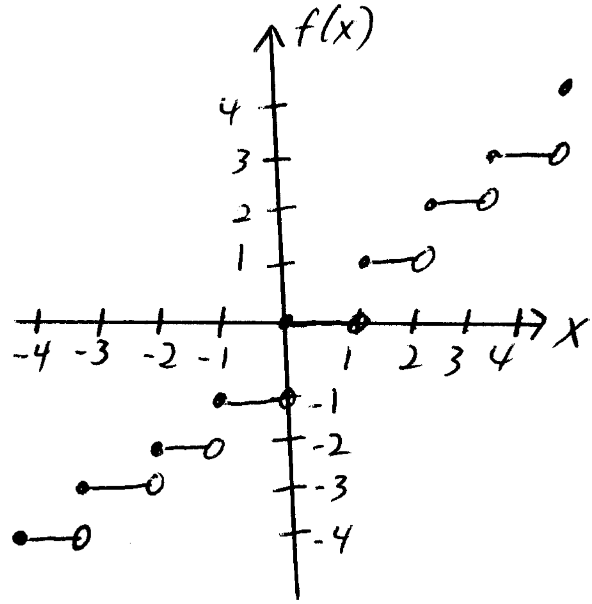
$|x-y| < \delta$ and $|f(x) - f(y)| \geq 1$.

Thus, f is not uniformly continuous on $(0, \infty)$

3. (*pts*) Let $f(x) = \lfloor x \rfloor$ be the floor function (i.e. $f(x)$ is the largest integer less than x, which can be defined as $\lfloor x \rfloor := \max\{p \in \mathbb{Z} : p \leq x\}$). Define the function $g(x) = x - \lfloor x \rfloor$ to be the fractional part of x.

(a) Sketch both functions $f(x)$ and $g(x)$ over the interval $[-4, 4]$, and determine where each function is discontinuous on $\mathbb{R}$.





f and g are both discontinuous at all integers.

(b) Prove $g(x)$ is discontinuous at $x_0 = 0$ using the definition.

Construct the sequence $\{x_n\}$ by $x_n = -\frac{1}{n}$ $\forall n \in \mathbb{N}$. For this sequence, $\{x_n\} \to 0$ and $f(x_n) = 1 - \frac{1}{n}$ $\forall n$.

Then $\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} 1 - \frac{1}{n} = 1 \neq f(0) = 0$.

Hence, f is discontinuous at $x_0 = 0$

4. (*pts*) Prove that $\ln(x+1) = 1 - x$ is solvable.

Note $\ln(x+1) = 1-x \iff \ln(x+1) - 1 + x = 0$

Let $f(x) = \ln(x+1) - 1 + x$ and $m = 0$.

Notice $f(0) = -1 < 0$ and $f(1) = \ln 2 > 0$.

Choose interval $[0,1]$ so that $f(0) < m = 0 < f(1)$.

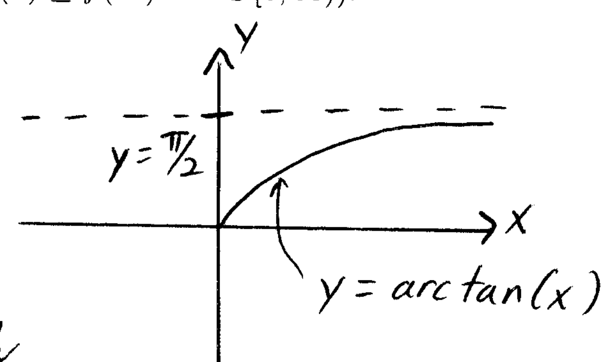
By IMVT, there exists at least one $x \in (0,1)$ with

$f(x) = m = 0 \iff \ln(x+1) - 1 + x = 0 \iff \ln(x+1) = 1-x$

Thus, the equation is solvable.

5. (*pts*) Give an example of a continuous function $f(x)$ bounded on $[0, \infty)$ that does not obtain it's maximum value (i.e. $\nexists x^* \in [0,\infty)$ such that $f(x) \le f(x^*)$ $\forall x \in [0,\infty)$).

Let $f(x) = \arctan(x)$.



It is clearly bounded and continuous (since it's the inverse of a continuous strictly increasing function $\tan x$). But, it never obtains it's maximum of $\frac{\pi}{2}$ on $[0,\infty)$.

6. (*pts*) Which are the following functions on the indicated domain are continuous and/or uniformly continuous or neither? Briefly justify your answer, using any theorem (in the book) you wish.

(a) $f(x) = 2^{x^3}$ on $[-7, 5]$.

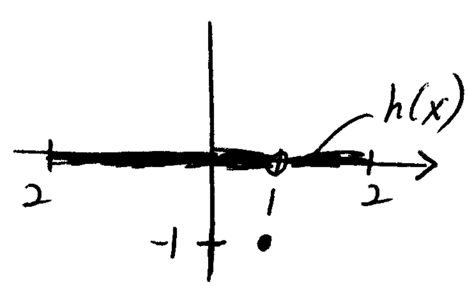
f(x) is continuous because its the composition of two continuous fns 2^x and x^3.
f(x) is uniformly continuous since it's continuous on a bounded domain.

(b) $g(x) = \dfrac{1}{x^3}$ on $(0, 1)$.

g is continuous because it's x^p with $p=-3$ and $x=0$ is not in the domain $(0,1)$.
g is not uniformly continuous since it becomes unbounded at $x=0$ which implies there doesn't exist an continuous extention to $[0,1]$.

(c) $h(x) = \begin{cases} -1 & \text{if } x \neq 1 \\ 0 & \text{if } x = 1 \end{cases}$ on $(-2, 2)$.

The graph of h(x) is



which clearly has a discontinuity at $x=1$.
Hence, f is not continuous on $[-2,2]$. Also, f is not uniformly continuous on $[-2,2]$ since it's not continuous there.

7. (*pts*) Let f be a continuous function with domain (a, b). Prove that if $f(r) = 0$ for each rational number r in (a, b), then $f(x) = 0$ for all $x \in (a, b)$.

Suppose f is a continuous function where $f(r)=0$ $\forall r \in \mathbb{Q} \cap (a,b)$. Let $x \in (a,b)$. If $x_0 \in \mathbb{Q}$, clearly $f(x_0)=0$ by assumption. Consider the case when $x_0 \notin \mathbb{Q}$ (i.e x_0 is irrational)). By the Extra Credit problem, there exists a sequence $\{x_n\} \subseteq \mathbb{Q}$ with $\{x_n\} \to x$. $\{x_n\}$ might not be contained in (a,b), but there must exist a subsequence that is, call it $\{x_{n_k}\} \subseteq (a,b)$. Clearly, $f(x_{n_k})=0$ $\forall k$. Since f is continuous, $f(x_0)=\lim_{k\to\infty} f(x_{n_k})=0$. Since x_0 is arbitrary, $f(x)=0$ $\forall x \in (a,b)$.