Tuesday, 20. October 2020 21:54

Discrete subgroups of small critical exponent

j-with S' wang

$(X, d)$

negatively pinched Hadamard mfd.

$-\kappa^2 \leq K \leq -1$

e.g $\mathbb{H}^n$. $\mathbb{H}^3$

$\Gamma < \mathrm{Isom}(X)$ discrete

$\forall K \subseteq X$ $\#\{\gamma \in \Gamma \mid \gamma(K) \cap K \neq \phi\}$ is finite

$M = X/\Gamma$ $\Gamma$ torsion-free → manifold.

e.g. $M$ is compact

e.g $M = \Sigma_g$

$\Gamma$ is co-compact action.

$x \in X$ $\Gamma(x)$

convex hull of $\Gamma(x) \curvearrowleft \Gamma$ $= \mathrm{Hull}(\Gamma)$

e.g $\mathrm{Hull}(\Gamma)/\Gamma$ is compact

$\Gamma$ is convex co-compact.

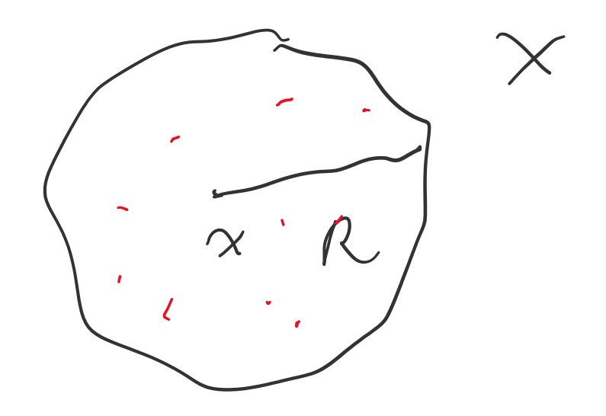
i.e. compactify $M = X/\Gamma$ by adjoining $\partial M$. (ideal boundary)

Critical exponent $\delta(\Gamma)$

Invariant of $\Gamma$

**Def:** $\delta(\Gamma) = \limsup_{R \to \infty} \frac{\log N(R)}{R}$

$N(R) = \#\{\gamma \in \Gamma \mid d(\gamma x, x) \leq R\}$ $x \in X$.



Remark: $0 \leq \delta(\Gamma) \leq n-1$ ← dim X.

**Thm** (L-Shi) Let $\Gamma < \mathrm{Isom}(X)$ be a finitely generated, torsion-free discrete isometry subgroup of X

If $\delta(\Gamma) < D(n, K)$ which a constant depending on n, K, (dimension) $\Rightarrow \Gamma$ is convex cocompact.

**Coro:** $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$ discrete, finitely generated. If $\delta(\Gamma) < D(n)$ $\Rightarrow$ $\Gamma$ is virtually free.

Remark: $\boxed{0 \leq \delta(\Gamma) \leq n-1}$

$\mathbb{H}^3$ $\delta(\Gamma) = 2 \Rightarrow \Gamma$ is not convex compact.

---

$\delta < 1 \Rightarrow$ linear isoperimetric inequality

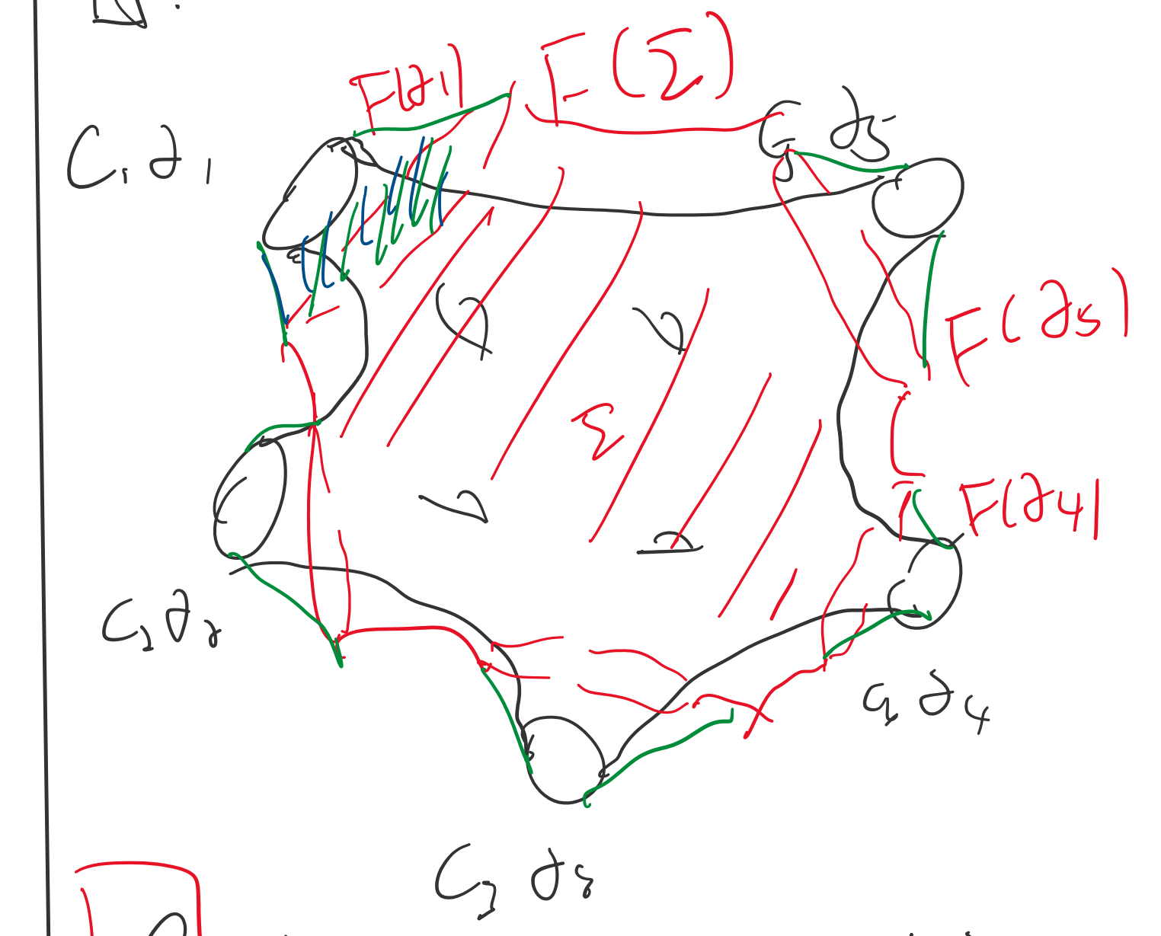
**Thm** (L-Shi) let $\mathcal{J} = \{\partial_1, \cdots \partial_k\}$ a family of smooth loop in M which is linearly dependent in $H_1(M)$. (i.e $\sum_{i=1}^{k} c_i \partial_i = 0$)

If $\boxed{\delta(\Gamma) < 1} \Rightarrow \bigcup_{i=1}^{k} c_i \partial_i$ bounds a surface $\Sigma \subseteq M$ s.t

$$\boxed{\mathrm{Area}(\Sigma)} \leq \frac{4}{1-\delta}\left(\sum_{i=1}^{k} |c_i|\, \ell(\partial_i)\right)$$

$|c_i| \mathrm{NJ}(\partial_i) \ell(\partial_i) \leq$

Pf:



$\boxed{\mathcal{G}} = \{$piecewise-smooth surface bounded by $\cup c_i \partial_i\}$

$f: \Sigma \to M$.

$\underline{A_0} = \inf\{\mathrm{Area}(\Sigma) \mid \Sigma \in \mathcal{G}\}$

$\forall \varepsilon > 0$ $\exists \Sigma_\varepsilon \in \mathcal{G}$

s.t $\mathrm{Area}(\Sigma_\varepsilon) \leq A_0(1+\varepsilon)$

Key tool: Besson-Courtois-Gallot map

$F: X \to X$ $\Gamma$-equivariant

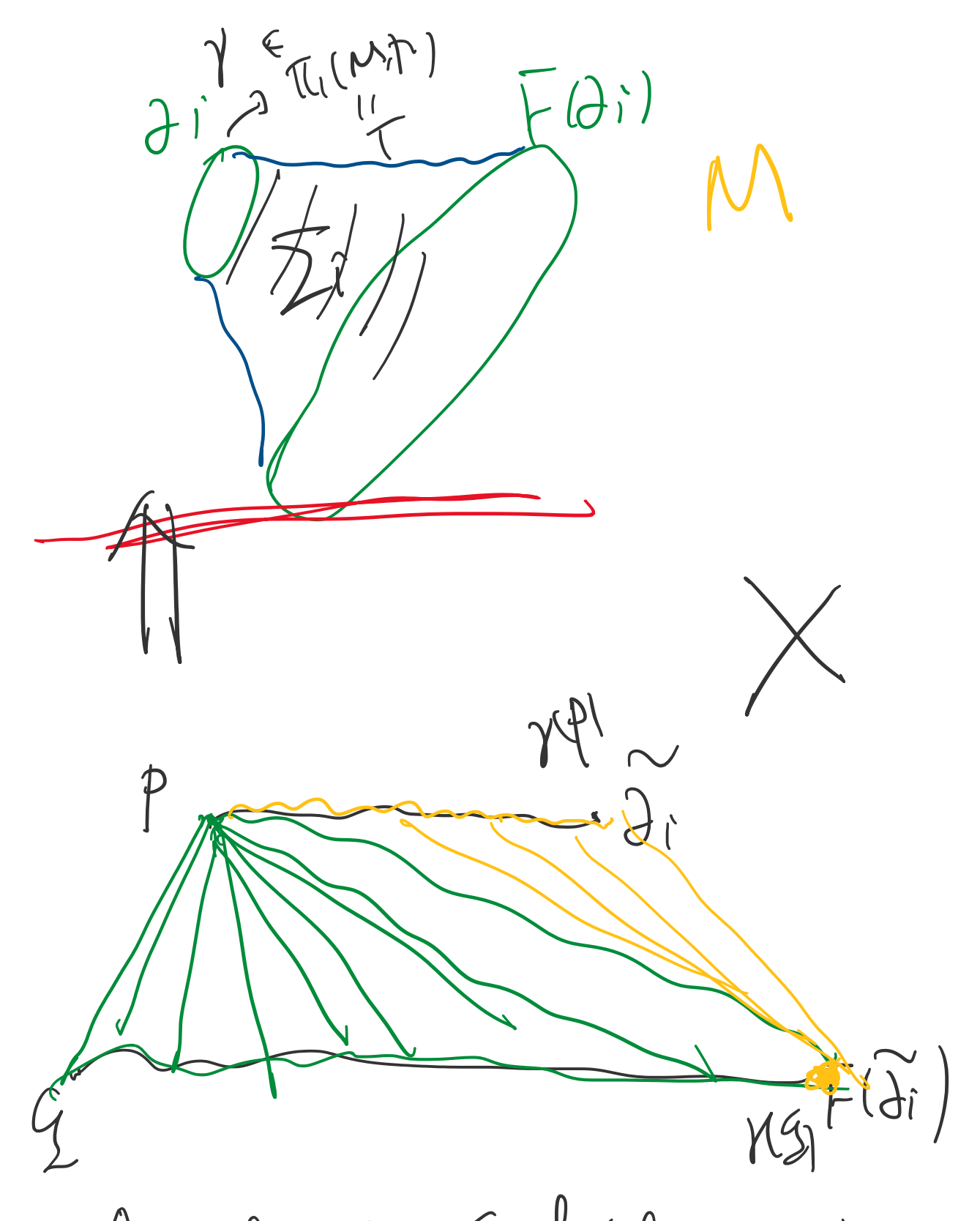
$M \to M$

① $F(\Sigma_\varepsilon)$

$\mathrm{Area}(F(\Sigma_\varepsilon)) \leq \left(\frac{1+\delta}{2}\right)^2 \mathrm{Area}(\Sigma_\varepsilon)$

$\ell(F(\partial_i)) \leq (1+\delta)\ell(\partial_i)$

$F(\partial_i)$ freely homotopic $\partial_i$



$\mathrm{Area}(\mathrm{Cone}) \leq \ell(q, \gamma(q))$

$\underline{A_0 \leq \mathrm{Area}(\Sigma'_\varepsilon) \leq A(F(\Sigma_\varepsilon)) + \boxed{|c_i| \boxed{A(\Sigma_i)}}}$

$\Sigma'_\varepsilon = F(\Sigma_\varepsilon) \cup |c_i| \Sigma_i$

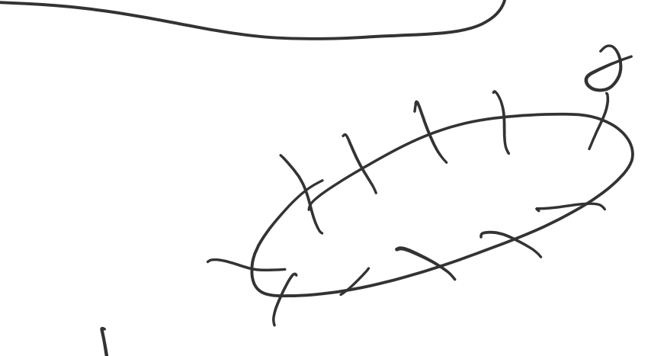
$\leq \left(\frac{1+\delta}{2}\right)^2 A(\Sigma_\varepsilon) + \sum |c_i| (\ell(\partial_i) + \ell(F(\partial_i)))$

$\leq \left(\frac{1+\delta}{2}\right)^2 (1+\varepsilon) A_0 + \sum |c_i| (2+\delta) \ell(\partial_i)$

$A_0 < \frac{4}{1-\delta} \sum |c_i| \ell(\partial_i)$

$\underline{\mathrm{NJ}(\partial, M)}$

$=$ largest r s.t $\exp|_{N_r(\partial, M)}$ is 1-1.



corollary: $\mathcal{J} = \{\partial_1 \cdots \partial_k\}$ ← closed geodesics

mutually $(2(C(\delta)+1))$-apart.

↑ $\frac{4}{1-\delta}$

$k \geq \beta_1(M) \Rightarrow \exists$ a closed geodesic $\partial_i$ s.t $\mathrm{NJ}(\partial_i, M) \leq C(\delta)$

rmk: There is a universal lower bound for the length of closed geodesics in M

Pf: $\{\partial_i\}$ s.t $\ell(\partial_i) \to 0$

$\mathrm{NJ}(\partial_i) \to \infty$

---

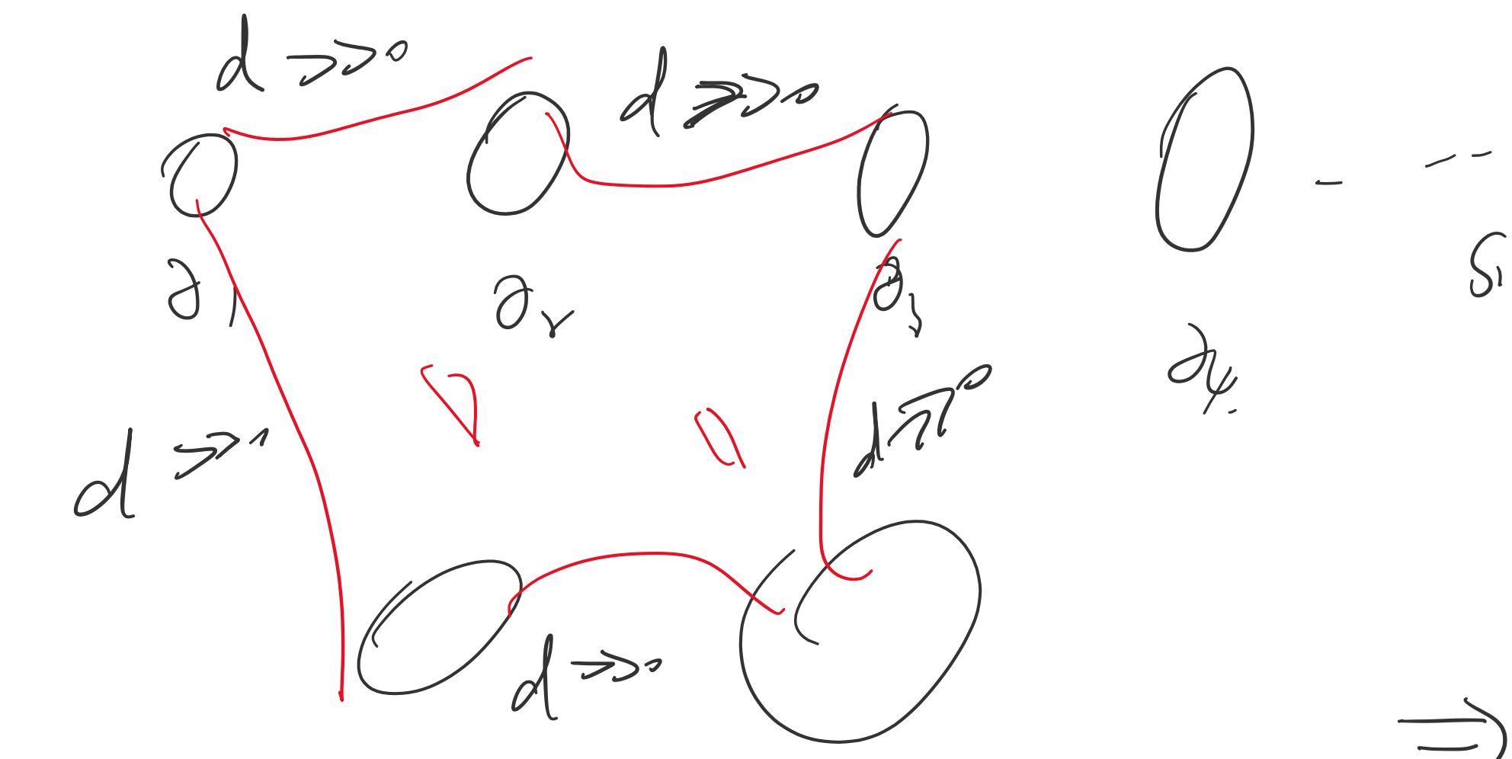
Pf of main thm:

$\delta(\Gamma) < D(n, k) \Rightarrow \Gamma$ is convex cocompact.

Pf: If $\Gamma$ is not convex cocompact.

By (Kapovich-L) $\exists$ a sequence of escaping geodesics $(\partial_i) \subseteq M$

$\forall K \subseteq M.$ $\#\{\partial_i \cap K = \phi\} < \infty$



① each $\partial_i$ is homologically trivial

② how to control $\ell(\partial_i)$ multiplicity ✓

Claim: $\exists$ a sequence of escaping geodesics $(\partial_i) \subseteq M$ of uniformly bounded length $(\delta(\Gamma) < 1)$

s.t $\ell(\partial_i) \leq C$ ← constant

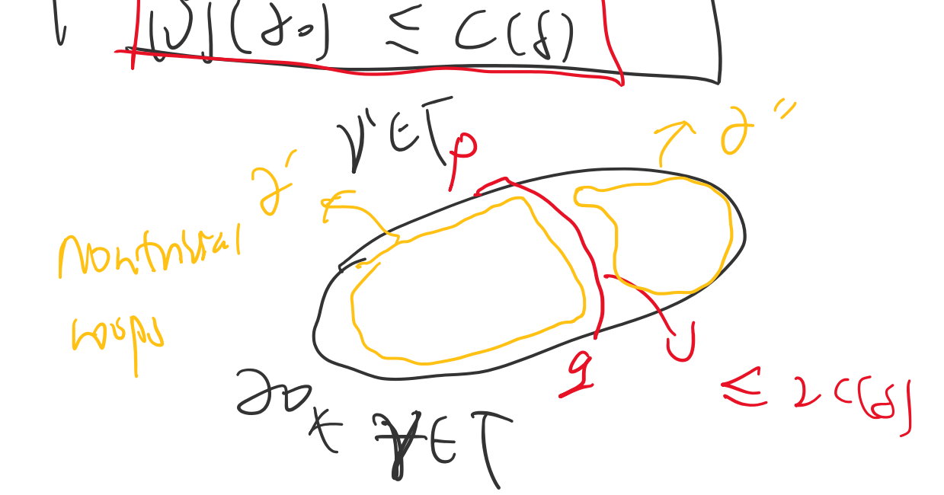
$\mathrm{NJ}(\partial_i) < C(\delta)$

$\lim_{i \to \infty} \ell(\partial_i) \leq 4C(\delta)$

$\Rightarrow \forall \varepsilon > 0.$ $\exists \partial_0$

s.t $\ell(\partial_0) \leq 4C(\delta) + \varepsilon$

$\boxed{\mathrm{NJ}(\partial_0) \leq C(\delta)}$



$\ell(\partial') \leq 4C(\delta)$

$\ell(\partial'')$

$F = \langle \gamma', \gamma \rangle < \Gamma$

$$\delta(\Gamma) \geq \delta(F) \geq \frac{\ln 3}{4C(\delta) + \varepsilon}$$