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Abel Partial Summation Formula

First some notation: For $x \in \mathbb{R}$ let $[x]$ denote the greatest integer less than or equal to x. Thus, for example, $[3.1] = 3$ and $[-1.7] = -2$. We introduce the *fractional part* of x: $\{x\} := x - [x]$. Thus, for example, $\{3.1\} = 0.1$ and $\{-1.7\} = 0.3$. Observe that $0 \le \{x\} < 1$.

Let $\{c_n\}$ be a sequence of complex numbers and $f(x)$ a complex-valued function defined for $x \in \mathbb{R}^+$. We assume f has a continuous derivative in $\mathbb{R}^+$. Define for $x \in \mathbb{R}^+$

$$C(x) = \sum_{1 \le n \le x} c_n$$

Thus, for example, $C(3.1) = c_1 + c_2 + c_3$.

The following is an algebraic identity that can be easily checked. Fix x with $k \le x < k+1$, then

$$\begin{aligned} \sum_{1 \le n \le x} c_n f(n) &= \sum_{1 \le n \le k} c_n f(n) \\ &= C(k)f(k) - \sum_{1 \le n \le k-1} C(n)\left(f(n+1) - f(n)\right) \end{aligned} \tag{1}$$

Now

$$\begin{aligned} \sum_{1 \le n \le k-1} C(n)\left(f(n+1) - f(n)\right) &= \sum_{1 \le n \le k-1} C(n) \int_n^{n+1} f'(t)\,dt \\ &= \sum_{1 \le n \le k-1} \int_n^{n+1} C(t)f'(t)\,dt \quad \text{since } C(t) = C(n), n \le t < n+1 \\ &= \int_1^k C(t)f'(t)\,dt \\ &= \int_1^x C(t)f'(t)\,dt - \int_k^x C(t)f'(t)\,dt \end{aligned} \tag{2}$$

Now

$$\int_k^x C(t)f'(t)\,dt = C(k)\int_k^x f'(t)\,dt = C(k)f(x) - C(k)f(k) = C(x)f(x) - C(k)f(k)$$

Putting these results (1) and (2) together we obtain the *Abel partial summation formula*

$$\boxed{\sum_{1 \le n \le x} c_n f(n) = C(x)f(x) - \int_1^x C(t)f'(t)\,dt}$$

Examples:

1. Let

$$H_x = \sum_{1 \le n \le x} \frac{1}{n}$$

Choose $f(x) = 1/x$ and $c_n = 1$, then

$$C(x) = \sum_{1 \le n \le x} 1 = [x].$$

By the Abel partial summation formula

$$\begin{aligned} H_x &= \frac{[x]}{x} + \int_1^x [t]\frac{1}{t^2}\,dt \\ &= \frac{x-\{x\}}{x} + \int_1^x (t-\{t\})\,\frac{1}{t^2}\,dt \\ &= 1 - \frac{\{x\}}{x} + \log x - \int_1^x \frac{\{t\}}{t^2}\,dt \end{aligned}$$

The integral $\int_1^x \frac{\{t\}}{t^2}\,dt$ converges to a limit as $x \to \infty$. This is so because

$$\left|\int_1^x \frac{\{t\}}{t^2}\,dt\right| \le \int_1^x |\{t\}|\,\frac{1}{t^2}\,dt \le \int_1^x \frac{1}{t^2}\,dt \tag{3}$$

and the last integral is convergent as $x \to \infty$. Thus write

$$\int_1^x \frac{\{t\}}{t^2}\,dt = \int_1^\infty \frac{\{t\}}{t^2}\,dt - \int_x^\infty \frac{\{t\}}{t^2}\,dt$$

By a similar estimate as in (3) we see that for large x

$$\int_x^\infty \frac{\{t\}}{t^2}\,dt = \mathrm{O}(\frac{1}{x})$$

Thus we have shown

$$H_x = \log x + \gamma + \mathrm{O}(\frac{1}{x}), \;\; x \to \infty$$

where

$$\gamma = 1 - \int_1^\infty \frac{\{t\}}{t^2}\,dt.$$

The constant γ is called *Euler's constant*. Its approximate value is $\gamma = 0.5772156649\ldots$. It is an unsolved problem to prove that γ is irrational. From the representation for γ one can derive the series expansion

$$\gamma = 1 - \sum_{n=1}^{\infty}\left[\log(1+\frac{1}{n}) - \frac{1}{n+1}\right]$$

This is a very slowly convergent series: Summing the first 10,000 terms gives $0.577266\ldots$.

2. Consider the sum

$$S_x = \sum_{1\le n\le x} \frac{\log n}{n}$$

Choose

$$c_n = 1, f(x) = \frac{\log x}{x}$$

so that

$$C(x) = [x],\ f'(x) = \frac{1 - \log x}{x^2}$$

The summation formula tells us

$$\begin{aligned} S_x &= [x]\frac{\log x}{x} - \int_1^x [t]\,\frac{1-\log t}{t^2}\,dt \\ &= (x - \{x\})\,\frac{\log x}{x} - \int_1^x (t - \{t\})\,\frac{1-\log t}{t^2}\,dt \\ &= \log x + \mathrm{O}(\frac{\log x}{x}) - \int_1^x \frac{1-\log t}{t}\,dt + \int_1^x \{t\}\frac{1-\log t}{t^2}\,dt \end{aligned}$$

Now use the fact that

$$\int_1^x \frac{1-\log t}{t}\,dt = \log x - \frac{1}{2}(\log x)^2$$

and that

$$\int_1^x \frac{1-\log t}{t^2}\,dt = \frac{\log x}{x}$$

to conclude that

$$\int_1^x \{t\}\,\frac{1-\log t}{t^2}\,dt = \int_1^\infty \{t\}\,\frac{1-\log t}{t^2} + \mathrm{O}(\frac{\log x}{x}), x \to \infty,$$

and hence

$$S_x = \frac{1}{2}(\log x)^2 + c_1 + \mathrm{O}(\frac{\log x}{x}),\ \ x \to \infty$$

where c_1 is a constant given by

$$c_1 = \int_1^\infty \{t\}\,\frac{1-\log t}{t^2}\,dt.$$

3. Let

$$S_x = \sum_{1 \le n \le x} \frac{(\log n)^2}{n}$$

Show that

$$S_x = \frac{1}{3}(\log x)^3 + c_2 + \mathrm{O}(\frac{(\log x)^2}{x}),\ \ x \to \infty.$$

4. Assume $f = f(x)$ is a continuously differentiable function of x with $f(x) \to 0$ as $x \to \infty$. Since $f(x) - f(1) = \int_1^x f(t)\,dt$, and $\lim_{x\to\infty} f(x)$ exists, we know that $\int_1^\infty f'(t)\,dt$ exists (and equals $-f(1)$). We further require that $\int_1^\infty |f'(t)|\,dt < \infty$. We then have

$$\sum_{1 \le n \le x} f(n) = \int_1^x f(t)\,dt + \gamma_f + \mathrm{O}(|f(x)|) + \mathrm{o}(1) \qquad (4)$$

where

$$\gamma_f = f(1) + \int_1^\infty \{t\} f'(t)\, dt.$$

Remark: If f is nonnegative, we can eliminate the o(1) term and simply have an error of $\mathrm{O}(f(x))$.

PROOF: In Abel partial summation choose $c_n = 1$, then

$$\begin{aligned}
\sum_{1\le n\le x} f(n) &= [x]f(x) - \int_1^x [t]f'(t)\, dt \\
&= (x - \{x\})\, f(x) - \int_1^x (t - \{t\})\, f'(t)\, dt \\
&= xf(x) - \int_1^x tf'(t)\, dt + \int_1^x \{t\}f'(t)\, dt - \{x\}f(x)
\end{aligned}$$

Since $|\{t\}f'(t)| \le |f'(t)|$ we have as $x \to \infty$

$$|\int_x^\infty \{t\}f'(t)\, dt| \le \int_x^\infty |f'(t)|\, dt = \mathrm{o}(1) \text{ as } x \to \infty$$

since the integral $\int_1^\infty |f'(t)|dt$ exists. If $f(x)$ is nonnegative then we don't need the absolute values and we get an error of the order $\mathrm{O}(f(x))$.

Observe that the first three examples are special cases of (4).

(a) Choose $f(x) = 1/x$ for example 1.

(b) Choose $f(x) = \log x/x$ for example 2.

(c) Choose $f(x) = (\log x)^2/x$ for example 3.