Lecture #18 The Linear Connection / Gradient

Last time we defined the gradient on vectors and scalars

$$\begin{cases} \nabla f = df \\ \nabla v = \left(dv^i + dx^k \, T^i_{kj} v^j\right) \frac{\partial}{\partial x^i} \end{cases}$$

Arbitrary tensors follow by a Leibnitz property.

<u>For example</u>, for a 1-form ω,

$$\mathrm{trace}(\omega, v) = \omega(v) \quad \text{is a scalar}$$

and $\nabla\omega$ is determined from

$$\nabla(\omega(v)) = d(\omega(v)) = \underbrace{(\nabla\omega)(v, \cdot)}_{\binom{0}{2}\text{-tensor}} + \underbrace{\nabla v(\omega, \cdot)}_{\binom{1}{1}\text{-tensor}}$$

This leads to

$$\begin{array}{ccc} \nabla : \Gamma(T^p_q M) & \longrightarrow & \Gamma(T^p_{q+1} M) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ X & \longmapsto & \nabla X \end{array}$$

subject to

(i) $\nabla(X+Y) = \nabla X + \nabla Y$

(ii) $\nabla(fX) = df \otimes X + f\,\nabla X$

(iii) $\nabla(X \otimes Y) = \nabla X \otimes Y + X \otimes \nabla Y$

(iv) ∇ commutes with trace.

Alternatively we could have taken the above as a definition and then tried to solve for ∇ obeying these axioms.

When $\{e_a\}$, $\{\theta^b\}$ are dual (but not necessarily coordinate bases we call the functions γ^b_{ac}:

$$\nabla e_a = \gamma^b_{ac}\,\theta^c \otimes e_b$$

connexion coefficients.

* When $e_a = \frac{\partial}{\partial x^a}$ "coordinate basis",

$$\gamma^b_{ac} = \Gamma^b_{ac}$$

* The functions γ^b_{ac} are not the components of a $\binom{1}{2}$- tensor

* Even though the Christoffel symbols are symmetric

$$\Gamma^k_{ij} = \Gamma^k_{ji}$$

the same need not apply to the corresponding γ^b_{ac} in a non-coordinate basis.

* From the gradient we can define an affine connexion

$$\nabla : \mathfrak{X}M \times \mathfrak{X}M \longrightarrow \mathfrak{X}M$$
$$(v, u) \longmapsto \nabla_v u = \nabla u(v, \cdot)$$

where $\nabla_v u$ is the directional derivative. This implies

$$\text{(i)} \quad \nabla_{fu+gv} w = f\nabla_u w + g\nabla_v w$$

$$\text{(ii)} \quad \nabla_u (v+w) = \nabla_u v + \nabla_u w$$

$$\text{(iii)} \quad \nabla_u (fv) = u(f)v + f\nabla_u v$$

Notice $\nabla_\cdot$ is the linear connection above
The case $\nabla = d + \Gamma$ gives the Levi-Civita connexion.

Example covariant derivative of a 1-form

$$\nabla_{e_a} \omega = e_a(\omega_b)\theta^b - \omega_b \gamma^b{}_{ac} \theta^c$$

ensures $\nabla\omega(v) = d\omega(v)$

Parallel Transport

Let $\gamma : \mathbb{R} \to M$ be a differentiable curve

and $v : \mathbb{R} \to TM$ the corresponding velocity vector
$t \longmapsto (\gamma(t), \dot{\gamma}(t))$

suppose $u \in \mathfrak{X}M$ some vector field.

Call
$$\boxed{\nabla_v u = \frac{\nabla u}{dt}}$$

If $(\varphi_\alpha \circ \gamma)^i(t) = x^i(t)$ in some chart & $\gamma(0) = p$

$$\nabla_v u\big|_p = v_p^i (\nabla_i u^j)_p \frac{\partial}{\partial x^j} = \dot{x}^i(0) \left(\frac{\partial u^j}{\partial x^i} + \Gamma^j_{ik} u^k \right)_p \frac{\partial}{\partial x^j}\Big|_p$$

$$\Rightarrow \quad \boxed{\frac{\nabla u^j}{dt} = \dot{u}^j + \dot{x}^i \Gamma^j_{ik} u^k}$$

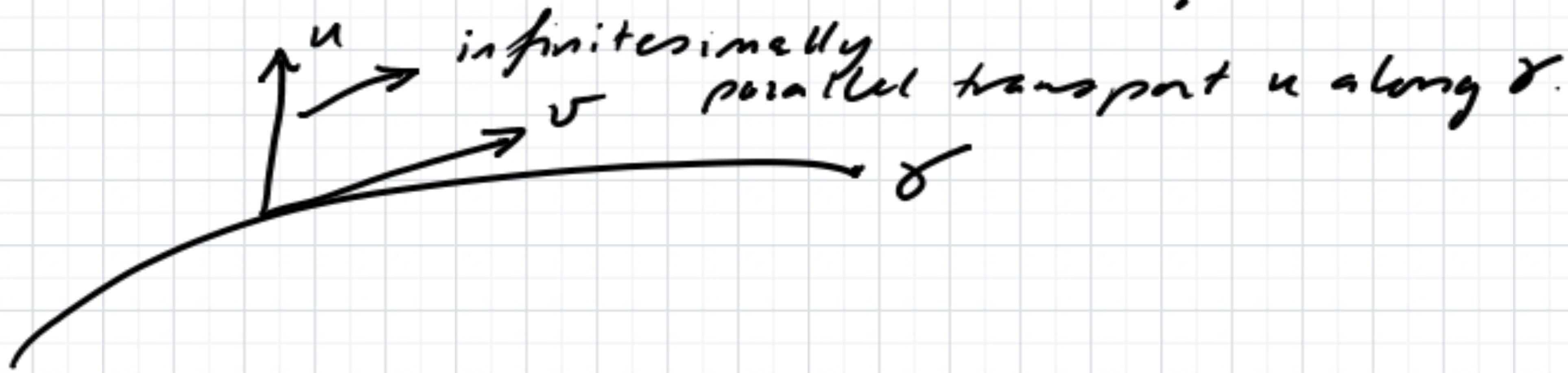
specializing to γ an integral curve of the vector field u, then $u|_\gamma = v$ and $u^j = \dot{x}$. Hence $\frac{\nabla u}{dt} = 0$ becomes

$$\boxed{\ddot{x}^j + \dot{x}^i \dot{x}^k \Gamma^j_{ik} = 0}$$ **Geodesic Equation**

Parallel transport of the tangent vector.

In general

$$\boxed{\frac{\nabla u}{dt} = 0}$$ Is called the <u>parallel transport equation</u>

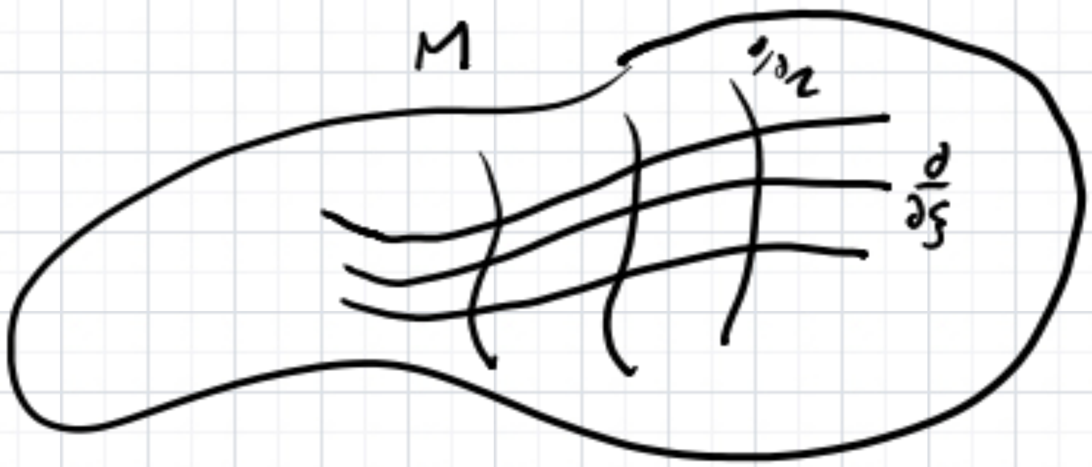


Notice, since γ "follows" from v, we can read this as parallel transport in the direction of v.

Torsion and curvature

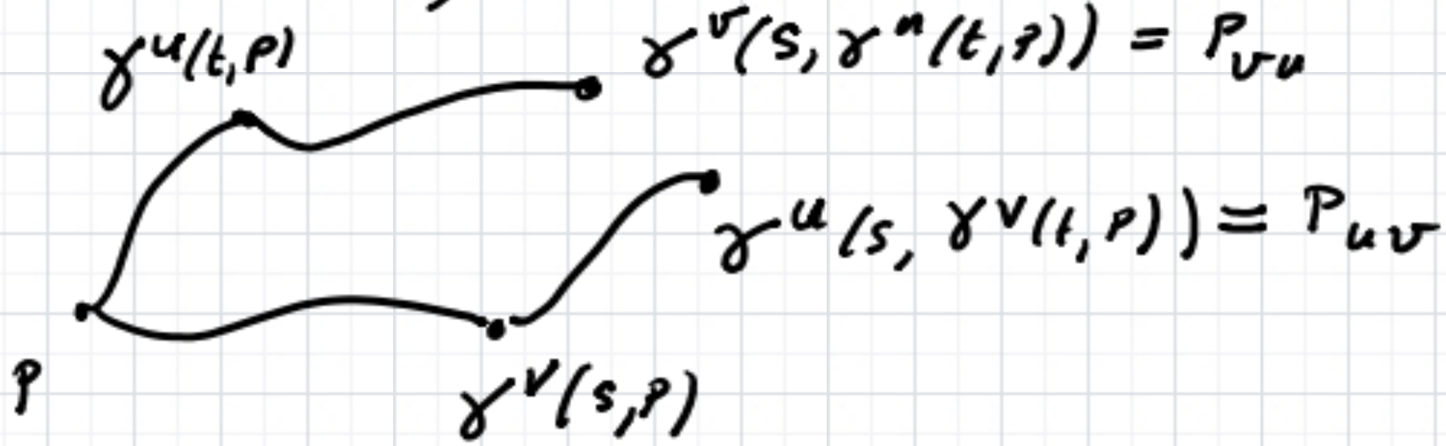
The bracket $[u,v]$ measures the failure of integral curves of u, v to form a coordinate system; since when

$$u = \frac{\partial}{\partial \xi}, \quad v = \frac{\partial}{\partial \eta} \Longleftrightarrow [u,v] = 0$$



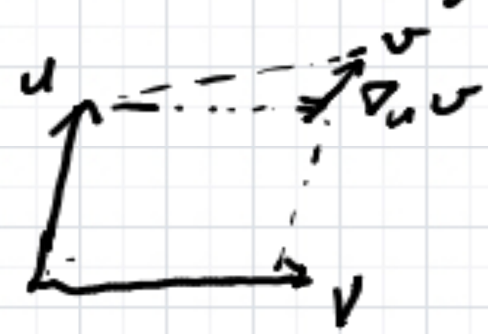
Hence $[u,v]$ is also the closer of quadrilaterals

i.e. Call γ^u, γ^v integral curves of u and v, then

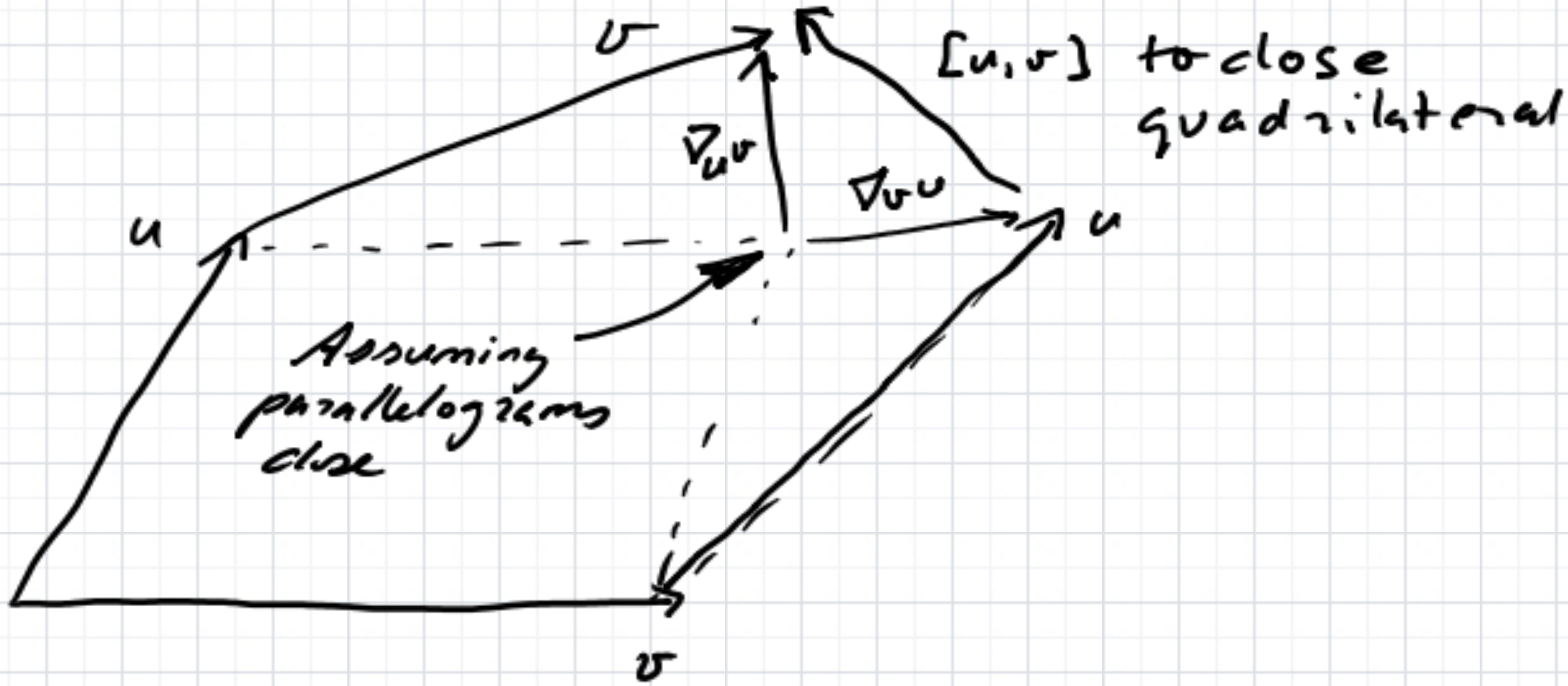


For infinitesimal s,t, $\gamma^{[u,v]}(st, P_{uv}) \doteq P_{vu}$

Now parallel transport $\nabla_u v$ measures whether the vector field v stays parallel along integral curves of u, schematically:



Torsion swap roles of u & v and compare



Torsion tensor, $\binom{1}{2}$-tensor

$$T(\alpha, u, v) = \left(\nabla_u v - \nabla_v u - [u,v]\right)(\alpha)$$

1-form

$$= \langle \alpha, \nabla_u v - \nabla_v u - [u,v] \rangle$$

Notice $T(\alpha, u, v) = T(\alpha, v, u)$

$\Rightarrow \quad T(\alpha, \cdot, \cdot) \in \Lambda^2 M \quad$ a <u>2-form</u>

Review Exercise ① Study the closure of infinitesimal parallelograms built from flows of vector fields.

② Compute the components of the torsion tensor (i) in a coordinate basis (ii) in an arbitrary basis using connexion coefficients.