Gaussian Elimination Worksheet

The aim is to teach yourself how to solve linear systems via Gaussian elimination. First we begin with some theory:

(1) Explain how to convert a linear system of equations to an *augmented matrix* and vice versa. Be sure to state precisely the shape of the augmented matrix needed for r linear equations in s unknowns. Give an example for a simple linear system.

(2) Find out what the *reduced row echelon form* (RREF) of an augmented matrix is. Explain why RREF allows the underlying linear system to be rapidly solved.

(3) There are three *elementary row operations* on augmented matrices. List them, and explain what they do to the underlying linear system.

(4) Gaussian elimination is an algorithm that applies a sequence of elementary row operations to an augmented matrix to achieve RREF. Write a summary of the Gaussian elimination algorithm.

Once you are confident that you understand the Gaussian elimination method, apply it to the following linear systems to find all their solutions. A sample answer might look like this:

converting to augmented matrix

performing row operations

$$\begin{array}{l} x+2y+z=1 \\ x+y-z=3 \end{array} \Leftrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 1 & 1 & -1 & 3 \end{array}\right) \overset{R2-R1}{\sim} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & -1 & -2 & 2 \end{array}\right)$$

$$\overset{-R2}{\sim} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & -2 \end{array}\right) \overset{R1-2R2}{\sim} \left(\begin{array}{ccc|c} 1 & 0 & -3 & 3 \\ 0 & 1 & 2 & -2 \end{array}\right)$$

converting back to a linear system

$$\Leftrightarrow \begin{array}{l} x \quad -3z=3 \\ \quad y+2z=-2 \end{array}$$

$\sim$ means "row equivalent" NOT equality

Let $z=\lambda$ be arbitrary. Then

$$\begin{cases} x=3+3\lambda \\ y=-2-2\lambda \end{cases} \quad //$$

There are many solutions each labeled by the number λ

(1) $x = y + z,\ x + y + z = 3,\ x + 3z = 1.$

(2) $\alpha x + \beta y = 1,\ \gamma x + \delta y = 0$ ($\alpha, \beta, \gamma, \delta$ constant).

(3) $\alpha x + \beta y = 0,\ \gamma x + \delta y = 1$ ($\alpha, \beta, \gamma, \delta$ constant).

(4) $\alpha x + \beta y = u,\ \gamma x + \delta y = v$ ($\alpha, \beta, \gamma, \delta, u, v$ constant).

(5) $y + 2z + 6w = 21,\ x - y - z - 4w = -9,\ 3x - 2y - 6w = -4$