# Math 67A Homework 4 Solutions

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## 1 Chapter 6 CWE

**6.1** Define the map $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x, y) = (x + y, x)$.

(a) Show that $T$ is linear.

(b) Show that $T$ is surjective.

(c) Find $\dim(\mathrm{null}(T))$.

(d) Find the matrix for $T$ with respect to the canonical basis of $\mathbb{R}^2$.

(e) Find the matrix for $T$ with respect to the canonical basis for the domain $\mathbb{R}^2$ and the basis $((1, 1), (1, -1))$ for the target space $\mathbb{R}^2$.

(f) Show that the map $F : \mathbb{R}^2\mathbb{R}^2$ given by $F(x, y) = (x + y, x + 1)$ is not linear.

**Solution** (a) For $T$ to be linear it must satisfy the equality $T(u+av) = T(u)+aT(v)$. I test this with direct computation

$$\begin{aligned} F(x_1 + x_2, y_1 + y_2) =&(x_1 + x_2 + y_1 + y_2, x_1 + x_2) \\ =&(x_1 + y_1, x_1) + (x_2 + y_2, x_2). \end{aligned}$$

(b) I must show that for each $u \in \mathbb{R}^2$ there exists $v \in \mathbb{R}^2$ such that $T(v) = u$. I must solve the equation $(x_1 + y_1, x_1) = (x_2, y_2)$; which has solutions $x_1 = y_2$ and $y_1 = x_2 - y_2$.

(c) The null space of a linear operator is the set of vectors mapped to the zero vector. From part (b) I can find the pre-image of $(0, 0)$, which is $x_1 = 0$, $y_1 = 0 - 0 = 0$, so the kernel of $T$ consists only of the zero vector. The space spanned by the zero vector is zero dimensional so $\dim(\mathrm{null}(T)) = 0$.

(d) The matrix representation of a linear transformation is the matrix whose columns are the images of the bases vectors under that transformation. Hence, I compute $T(1, 0) = (1, 1)$ and $T(0, 1) = (1, 0)$ so

$$M[T] = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

(e) I must find the matrix that transforms the standard basis to the new basis $((1, 1), (1, -1))$. The matrix $P$ that takes the new basis back to the standard basis is

$$P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

so

$$P^{-1} = \frac{1}{2}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix},$$

takes the new basis to the standard one. Thus,

$$P^{-1}T = \frac{1}{2}\begin{pmatrix} -2 & 0 \\ -1 & -1 \end{pmatrix}$$

is the desired matrix.

(f) The map $F$ is not linear because $F(x_1 + x_2, y_1 + y_2) \neq F(x_1, y_1) + F(x_2, y_2)$ as seen by $F(x_1 + x_2, y_1 + y_2) = (x_1 + x_2 + y_1 + y_2, x_1 + x_2 + 1)$ and $F(x_1, y_1) + F(x_2, y_2) = (x_1 + x_2 + y_1 + y_2, x_1 + x_2 + 2)$, which is different.

**6.2** Let $T \in \mathcal{L}(\mathbb{R}^2)$ be defined by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -x \end{pmatrix}, \text{ for all } \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2.$$

(a) Show that $T$ is surjective.

(b) Find $\dim(\mathrm{null}(T))$.

(c) Find the matrix for $T$ with respect to the canonical basis of $\mathbb{R}^2$.

(d) Show that the map $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by $F(x, y) = (x + y, x + 1)$ is not linear.

**Solution** (a) The operator $T$ is surjective if for any $u \in \mathbb{R}^2$ there exists $v \in \mathbb{R}^2$ such that $T(v) = u$. This requires me to solve the equation $(y_1, -x_1) = (x_2, y_2)$, which is solved by $x_1 = -y_2$ , $y_1 = x_2$; thus $T$ is surjective.

(b) By the dimension formula the dimension of the null space is equal to the dimension of the domain minus the dimension of the range. Since $T$ is surjective its range is $\mathbb{R}^2$, which has dimension two. The domain of $T$ is also $\mathbb{R}^2$; thus, the dimension of the null space of $T$ is zero.

(c) The matrix representation of a linear transformation is the matrix whose columns are the images of each basis vector

$$M[T] = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

(d) This is the same as part (f) of problem 1.

**6.3** Consider the complex vector spaces $\mathbb{C}^2$ and $\mathbb{C}^3$ with their canonical bases, and define $S \in \mathcal{L}(\mathbb{C}^2, \mathbb{C}^3)$ be the linear map defined by $S(v) = Av$, where $A$ is the matrix

$$A = M[S] = \begin{pmatrix} i & 1 & 1 \\ 2i & -1 & -1 \end{pmatrix}.$$

Find a basis for $\mathrm{null}(S)$.

**Solution** Reducing a matrix to row echelon form is a way to determine its null space. The matrix form of $S$ is

$$A = M[S] = \begin{pmatrix} i & 1 & 1 \\ 2i & -1 & -1 \end{pmatrix}.$$

Subtracting twice the top row from the bottom row yields

$$A = M[S] = \begin{pmatrix} i & 1 & 1 \\ 0 & -3 & -3 \end{pmatrix}.$$

Adding one third of the bottom row to the top row yields

$$A = M[S] = \begin{pmatrix} i & 0 & 0 \\ 0 & -3 & -3 \end{pmatrix},$$

which corresponds to $ix_1 = 0$ and $-3x_2 - 3x_3 = 0$, so $x_2 = -x_3$ and $x_1 = 0$. Thus the null space of $S$ is spanned by $(0, 1, -1)$ in the standard basis.

**6.4** Give an example of a function $f : \mathbb{R}^2 \to \mathbb{R}$ having the property that for $a \in \mathbb{R}$, $v \in (R)^2$

$$f(av) = af(v).$$

**Solution** The function $f(x, y) = x$ has the desired property.

**6.5** Show that the linear map $T : \mathbb{F}^4 \to \mathbb{F}^2$ is surjective if

$$\text{null}(T) = \{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 | x_1 = 5x_2, x_3 = 7x_4\}.$$

**Solution** The null space of $T$ is spanned by $(5, 1, 0, 0), (0, 0, 7, 1)$, thus it has dimension two. The dimension formula states that the dimension of the range is equal to the dimension of the domain minus the dimension of the null space; in this case this leads to $4 - 2 = 2$. The dimension of the range of $T$ is two and the dimension of $\mathbb{F}^2$ is two; thus, the range of $T$ is all of $\mathbb{F}^2$ so $T$ is surjective.

**6.6** Show that no linear map $T : \mathbb{F}^5 \to \mathbb{F}^2$ can have as its null space the set

$$\{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{F}^5 | x_1 = 3x_2, x_3 = x4 = x_5\}.$$

**Solution** The null space of $T$ spanned by $(3, 1, 0, 0, 0), (0, 0, 1, 1, 1)$, and thus has dimension two. By the dimension counting formula the range of $T$ must have dimension $5 - 2 = 3$; since $\mathbb{F}^2$ has dimension two it has no subspace of dimension three, thus no such $T$ can exist.

**6.7** Describe the set of solutions $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ of the system of equations

$$\begin{aligned} x_1 - x_2 + x_3 &= 0 \\ x_1 + 2x_2 + x_3 &= 0 \\ 2x_1 + x_2 + 2x_3 &= 0. \end{aligned}$$

**Solution** Row reduction is a systematic way to solve a system of linear equations. I begin with the matrix

$$\begin{pmatrix} 1 & -1 & 1 \\ 1 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix}.$$

Subtracting the first row from the second row and twice the first row from the third row yields

$$\begin{pmatrix} 1 & -1 & 1 \\ 0 & 3 & 0 \\ 0 & 3 & 0 \end{pmatrix}.$$

Subtracting the second row from the third yields

$$\begin{pmatrix} 1 & -1 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Adding one third of the second row to top row yields

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This corresponds to $x_1 + x_3 = 0$ and $x_2 = 0$, so the solution space is spanned by $(1, 0, -1)$.

## 2 Chapter 6 PWE

**6.2** Let $V$ and $W$ be vector spaces over $\mathbb{F}$, and suppose that $T \in \mathcal{L}(V, W)$ is injective. Given a linearly independent list $(v_1, ..., v_n)$ of vectors in $V$, prove that the list $(T(v_1), ..., T(v_n))$ is linearly independent in $W$.

**Solution** Let $\alpha_i \in \mathbb{F}$ such that

$$\sum_{i=1}^{n} \alpha_i T(v_i) = 0.$$

Since $T$ is linear this implies

$$T(\sum_{i=1}^{n} \alpha_i v_i) = 0.$$

Since $T$ is injective this implies that

$$\sum_{i=1}^{n} \alpha_i v_i = 0.$$

Since the $v_i$ are linearly independent this implies that $\alpha_i = 0$ for all $i$. Thus

$$\sum_{i=1}^{n} \alpha_i T(v_i) = 0$$

only if $\alpha_i = 0$ for all $i$, so $(T(v_1), ..., T(v_n))$ is linearly independent.

**6.3** Let $U, V$ and $W$ be vector spaces over $\mathbb{F}$, and suppose that the linear maps $S \in \mathcal{L}(U, V)$ and $T\mathcal{L}(V, W)$ are both injective. Prove that the composition map $T \circ S$ is injective.

**Solution** Let $u$ be an element of the kernel of $T \circ S$. Then $S(u)$ is in the kernel of $T$. SInce $T$ is injective I have $S(u) = 0$. Since $S$ is injective this implies $u = 0$, thus $u = 0$ is the only solution to $T \circ Su = 0$, so $T \circ S$ is injective.

**6.4** Let $V$ and $W$ be vector spaces over $\mathbb{F}$, and suppose that $T \in \mathcal{L}(V, W)$ is surjective. Given a spanning list $(v_1, ..., v_n)$ for $V$, prove that $\text{span}(T(v_1), ..., T(v_n)) = W$.

**Solution** I must show that any element of $W$ can be written as a linear combination of $T(v_i)$. Towards that end take $w \in W$. Since $T$ is surjective there exists $v \in V$ such that $w = T(v)$. Since $v_i$ span $V$ there exists $\alpha_i$ such that

$$\sum_{i=1}^{n} \alpha_i v_i = v.$$

Since $T$ is linear

$$T(\sum_{i=1}^{n} \alpha_i v_i) = \sum_{i=1}^{n} \alpha_i T(v_i),$$

hence $w$ is a linear combination of $T(v_i)$. Since $w$ was arbitrary this shows that $T(v_i)$ spans $W$.

**6.5** Let $V$ and $W$ be vector spaces over $\mathbb{F}$ with $V$ finite-dimensional. Given $T \in \mathcal{L}(V, W)$, prove that there is a subspace $U$ of $V$ such that

$$U \cap \text{null}(T) = \{0\} \text{ and } \text{range}(T) = \{T(u) | u \in U\}.$$

**Solution** Let $\{w_i\}$ be a basis of the range of $T$ and for each $w_i$ choose $v_i \in V$ such that $T(v_i) = w_i$. Then $U = \text{span}\{v_i\}$ has the same dimension as the range of $T$, so when $T$ is restricted to acting on $U$ it has trivial kernel, thus $U$ has the desired properties.

**6.6** Let $V$ be a vector space over $\mathbb{F}$, and suppose that there is a linear map $T \in \mathcal{L}(V, V)$ such that both null$(T)$ and range$(T)$ are finite-dimensional subspaces of $V$. Prove that $V$ must also be finite-dimensional.

**Solution** The dimension formula states that the dimension of the domain is equal to the sum of the dimension of the null space and the dimension of the range. If both the null space and range are finite dimensional then the sum of their dimension is finite, so the dimension of the domain, $V$, is also finite.

**6.7** Let $U, V$, and $W$ be finite-dimensional vector spaces over $\mathbb{F}$ with $S \in \mathcal{L}(U, V)$ and $T \in \mathcal{L}(B, W)$. Prove that

$$\dim(\text{null}(T \circ S)) \leq \dim(\text{null}(T)) + \dim(\text{null}(S)).$$

**Solution** If $Su = 0$ then $T \circ Su = 0$, so the null space of $S$ is a subspace of $T \circ S$. On the orthogonal compliment of null$(S)$ the operator $S$ is injective (this is reflected by the dimension formula). What could still happen is that $Su$ could be a non-zero element of the null space of $T$. Since $S$ restricted to null$(S)^\perp$ is injective the maximum dimension of the subspace of $U$ mapped to the kernel of $T$ is the dimension of the kernel of $T$, thus

$$\dim(\text{null}(T \circ S)) \leq \dim(\text{null}(T)) + \dim(\text{null}(S)).$$

**6.8** Let $V$ be a finite dimensional vector space over $\mathbb{F}$ with $S, T \in \mathcal{L}(V, V)$. Prove that $T \circ S$ is invertible if and only if both $S$ and $T$ are invertible.

**Solution** First I show that if $S$ and $T$ are invertible then $T \circ S$ is invertible. The map $S^{-1}T^{-1}$ is an inverse for $T \circ S$ as

$$\begin{aligned} S^{-1}T^{-1}TS &= S^{-1}S \\ &= I \\ &= TT^{-1} \\ &= TSS^{-1}T^{-1}. \end{aligned}$$

Now I prove that if $T \circ S$ is invertible then so are $T, S$ independently. For $T \circ S$ to be surjective $T$ must be surjective, since the domain and target space are the same and are finite dimensional the dimension formula implies that $T$ is injective. For $T \circ S$ to be injective $S$ must be injective, and again the basis dimension formula implies that $S$ is bijective. Thus $T \circ S$ invertible implies that each of $T$, $S$ are injective.

**6.9** Let $V$ be a finite-dimensional vector space over $\mathbb{F}$ with $S, T \in \mathcal{L}(V, V)$ and denote by $I$ the identity map on $V$. Prove that $T \circ S = I$ if and only if $S \circ T = I$

**Solution** If $T \circ S = I$ then $T \circ S$ is invertible, so each of $S$, $T$ are invertible. Thus $T = S^{-1}$ so $S \circ T = S \circ S^{-1} = I$. Likewise, if $S \circ T = I$ then it is invertible so each of $S, T$ are invertible and $T = S^{-1}$.

# 3 Chapter 7 CWE

**7.1** Let $T \in \mathcal{L}(\mathbb{F}^2, \mathbb{F}^2)$ be defined by

$$T(u, v) = (v, u).$$

Compute the eigenvalues and eigenvectors associated with $T$.

**Solution** First I write $T$ in matrix form

$$M[T] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

If $\lambda$ is an eigenvalue of $T$ then the determinant of the matrix $T - I\lambda$ is zero

$$\det \begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 - 1.$$

This is zero when $\lambda = 1$ or $\lambda = -1$, so these are the eigenvalues of $T$. To determine the eigenvectors of $T$ I must solve the linear equation

$$(T - \lambda I)x = 0.$$

In the case $\lambda = 1$ this is $u - v = 0$, or $v = u$ so $(1, 1)$ is an eigenvector with eigenvalue 1. In the case $\lambda = -1$ this is $u + v = 0$ or $u = -v$ so $(1, -1)$ is an eigenvector associated with eigenvalue $-1$.

**7.22** Let $T \in \mathcal{L}(\mathbb{F}^3, \mathbb{F}^3)$ be defined by

$$T(u, v, w) = (2v, 0, 5w).$$

Compute the eigenvalues and associated eigenvectors for $T$.

**Solution** First I write $T$ in matrix form

$$M[T] = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{pmatrix}.$$

Then I compute the determinant of $T - \lambda I$

$$\det \begin{pmatrix} -\lambda & 2 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & 5 - \lambda \end{pmatrix} = -\lambda(-\lambda)(5 - \lambda)$$

which is zero when $\lambda = 0$ ( a repeated root with multiplicity two) or when $\lambda = 5$. To determine the eigenvectors I must solve the linear equation

$$(T - \lambda I)x = 0.$$

When $\lambda = 5$ this is $-5u + 2v = 0$, $-5v = 0$, and $0 = 0$, which has solution $(0, 0, w)$ so $(0, 0, 1)$ is an eigenvector with eigenvalue 5. When $\lambda = 0$ this is $2v = 0$, $0 = 0$, and $5w = 0$, which has solution $(1, 0, 0)$ .