The Deformed Hermitian-Yang-Mills Equation with Calabi Ansatz

By

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To my family, friends, and teachers.

Contents

Abstract

In this dissertation we study the deformed Hermitian-Yang-Mills equation, an equation that can be derived via mirror symmetry as the mirror of the special Lagrangian graph equation. In particular, we are interested in how certain notions of stability associated with the geometric setup relate to existence of a solution. We restrict our study to a certain class of manifolds with large symmetry, with special emphasis on the blowup of complex projective space. Using symmetry, we can rewrite the deformed Hermitian-Yang-Mills equation as an exact ODE with boundary values. This allows us to accomplish two things. First, it allows for a more simple setup in which one can compute the special Lagrangrian angle associated to the equation, and second it allows the stability condition we consider to be expressed in a simple combinatorial matter. With these two observations, we demonstrate that our stability condition forces the boundary values into a configuration where one can then solve the ODE, and thus the deformed Hermitian-Yang-Mills equation.

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CHAPTER 1

Introduction

This dissertation works towards solving the deformed Hermitian-Yang-Mills equation, a problem in complex differential geometry. In this field, both analytic and algebraic methods are used to study geometric problems. Our particular problem arises from string theory.

Physicists observed that Calabi-Yau manifolds often come in pairs $X, \check{X}$ that give equivalent string compactifications of certain string theories. Mirror symmetry relates the symplectic geometry on $\check{X}$ to the complex geometry on X and vice versa, so in some sense, mirror symmetry allows you to exchange the two.

Kontsevich [**16**] proposed that the right way to think of this is that mirror symmetry could be explained as an equivalence of triangulated categories

$$D^b Coh(X) \sim D^{Fuk}(\check{X}),$$

where we have the category of coherent sheaves, a complex object, on the left and the Fukaya category, a symplectic object, on the right. Strominger-Yau-Zaslow [**28**] proposed a mechanism for this equivalence using T-duality and a real Fourier-Mukai transform. A key object in the SYZ setup are special Lagrangians, which we now introduce.

Let $(X^{2n}, J, \Omega, \omega)$ be a compact Calabi-Yau manifold of real dimension $2n$ with complex structure J, holomorphic volume form Ω, and Kahler form ω. A n dimensional submanifold $L \hookrightarrow X$ is Lagrangian if

$$\omega|_L = 0,$$

and special Lagrangian if, in addition, there is a constant $\hat{\theta} \in \mathbb{R}$ such that

$$\mathrm{Im}\left(e^{-i\hat{\theta}}\Omega\right)|_L = 0.$$

Special Lagrangian submanifolds appeared in Harvey-Lawson [**13**] as a special class of calibrated submanifolds that are volume minimizing in their homology class. They have been studied extensively in many settings, including the graphical case [**7**, **21**, **22**, **29**, **30**, **31**, **32**, **33**]. The discovery of mirror symmetry generated further interest in special Lagrangians.

The objects of interest are supersymmetric cycles, physically realistic states, also called D-branes, which are stable under decay. On the symplectic side, a supersymmetric cycle on the "A-model" $(\check{X}, \check{J}, \check{\Omega}, \check{\omega})$ is a special Lagrangian $L \subset \check{X}$ with flat unitary connection on the trivial bundle $L \times \mathbb{C} \to L$. On the complex side, a supersymmetric cycle on the "B-model" (X, J, Ω, ω) is a holomorphic line bundle $(E, \nabla) \to Z$, $Z \subset X$ a complex submanifold, with a unitary connection solving the *deformed Hermitian-Yang-Mills equation*:

$$\operatorname{Im}\left(e^{-i\hat{\theta}}(\omega - F)^{dim_{\mathbb{C}} Z}\right) = 0,\ \hat{\theta} \in \mathbb{R}. \tag{1.0.1}$$

This nonlinear PDE was derived independently by Marino-Minasian-Moore-Stronminger [**19**] and Leung-Yau-Zaslow [**18**]. To reiterate, under mirror symmetry, we have this correspondence between special Lagrangians on one side of mirror symmetry and bundles with connections that solve this nonlinear PDE on the other side.

Because (1.0.1) is defined on complex submanifolds, it makes sense to work on a general compact Kähler manifold, as opposed to a Calabi-Yau manifold. Also, it is not necessary to restrict ourselves to the line bundle setting. Therefore, in this dissertation, we will set up the deformed Hermitian-Yang-Mills (dHYM) equation as follows. Let (X, ω) be a compact Kähler manifold, and $[\alpha] \in H^{1,1}(X, \mathbb{R})$ a real cohomology class. The class $[\alpha]$ solves the *deformed Hermitian-Yang-Mills* (dHYM) equation if it admits a representative $\alpha \in [\alpha]$ satisfying

$$\operatorname{Im}\left(e^{-i\hat{\theta}}(\omega + i\alpha)^n\right) = 0, \tag{1.0.2}$$

where $e^{i\hat{\theta}} \in S^1$ is a fixed constant. Fixing $\alpha_0 \in [\alpha]$, by the $\partial\bar{\partial}$-Lemma, any other representative of this class can be written as $\alpha = \alpha_0 + i\partial\bar{\partial}\phi$ for some real function ϕ, and so (1.0.2) is an elliptic, fully nonlinear equation for ϕ. We can approach the equation with the following question:

Question 1.0.1. *When does there exist a smooth representative α of a fixed class $[\alpha]$ solving* (1.0.2)*?*

One necessary condition on $[\alpha]$ we see right away is we need $\int_X (\omega + i\alpha)^n \neq 0$. We are interested in finding further algebro-geometric obstructions to the existence of solutions to the dHYM equation. Furthermore, we can look at necessary conditions for a solution to exist and see if these are sufficient conditions for existence.

Proposition 1.0.2. *[**10**] Suppose $c_1(L)$ admits a solution of the deformed Hermitian-Yang-Mills equation with $\hat{\theta} \in ((n-2)\frac{\pi}{2}, n\frac{\pi}{2})$. Then for every irreducible analytic subvariety $V \subset X$ of dimension $1 \leq p < n$ we have*

$$Im\left(\int_V e^{-\sqrt{-1}(\hat{\theta}-(n-p)\frac{\pi}{2})}(\omega + \sqrt{-1}\alpha)^p\right) > 0. \tag{1.0.3}$$

*To make aesthetic contact with other stability in geometry and physics [**5**], we use central charge notation. Define*

$$Z_{[\alpha],[\omega]}(L) = -\int_V e^{-\sqrt{-1}(\omega+\sqrt{-1}\alpha)} \tag{1.0.4}$$

then we must have

$$Im\left(\frac{Z_{[\alpha],[\omega]}(L)}{Z_{[\alpha],[\omega]}(X)}\right) > 0. \tag{1.0.5}$$

Notice that (1.0.2) remains unchanged if one adds 2π to the constant $\hat{\theta}$, and so a priori this fixed constant is only S^1 valued. We will see later that if a solution to the equation exists, its formulation allows us to uniquely lift $\hat{\theta}$ to $\mathbb{R}$. However, if one does not yet know existence, determining a lift of the angle $\hat{\theta}$ can be challenging, and this contributes to many analytic difficulties in solving (1.0.2). One way around this is to specify a lift in advance. The simplest case is when we have a "large angle" assumption, and $\hat{\theta} \in \left((n-2)\frac{\pi}{2}, n\frac{\pi}{2}\right)$. This condition is known as "supercritical phase". In this setting, Collins-Jacob-Yau [**9**] conjecture the above necessary condition is sufficient.

Conjecture 1.0.3 (Collins-Jacob-Yau [**9**]). *The cohomology class $[\alpha] \in H^{1,1}(X, \mathbb{R})$ on a compact Kähler manifold (X, ω) admits a solution to the deformed Hermitian-Yang-Mills equation*

(1.0.2) *(with supercritical phase) if and only if* $Z(X) \neq 0$*, and for all analytic subvarieties* $V \subset X$*,*

$$\operatorname{Im}\left(\frac{Z_{[\alpha][\omega]}(V)}{Z_{[\alpha][\omega]}(X)}\right) > 0. \tag{1.0.6}$$

We prove this conjecture on the blowup of complex projective space. We find a stability condition, which is a generalization of (1.0.6) in the non supercritical phase case, and demonstrate that stability is sufficient for existence of a solution.

THEOREM 1.0.1. *Let* X *be the blowup of* $\mathbb{P}^n$ *at a point. Let* $[\omega]$ *be any Kähler class on* X*, and* $[\alpha]$ *any real cohomology class. Then if* $Z(X) \neq 0$*, and if for each* $k \in \{1, ..., n-1\}$ *all analytic subvarieties* $V^k \subset X$ *of dimension* k *satisfy either*

$$\operatorname{Im}\left(\frac{Z_{[\alpha][\omega]}(V^k)}{Z_{[\alpha][\omega]}(X)}\right) > 0 \qquad \text{or} \qquad \operatorname{Im}\left(\frac{Z_{[\alpha][\omega]}(V^k)}{Z_{[\alpha][\omega]}(X)}\right) < 0, \tag{1.0.7}$$

then $[\alpha]$ *admits a solution to the deformed Hermitian-Yang-Mills equation.*

We reiterate that for different dimensions k, we allow for the inequality in (1.0.7) to be either positive or negative. However, for a fixed k, all subvarieties of that dimension must give the same sign. We note that in the supercritical phase case, only the strictly positive inequality is possible, and so our condition (1.0.7) reduces to (1.0.6), proving Conjecture 1.0.3 in this case.

To prove our theorem, we make use of the fact that on X, both $[\omega]$ and $[\alpha]$ admit representatives that satisfy a particular symmetry called *Calabi Symmetry*. Originally studied by Calabi to construct examples of extremal Kähler metrics [**8**], this symmetry has since been employed to study many other geometric equations, including the Kähler Ricci flow [**23**, **24**, **25**, **26**], metric flips [**27**], and the inverse σ_k equations [**12**]. The advantage of working with Calabi Symmetry is that allows us to write equation (1.0.2) as an ODE over a closed interval in $\mathbb{R}$, with a two sided boundary conditions determined by the classes $[\omega]$ and $[\alpha]$. Thus the question of existence is reduced to solving the boundary valued ODE. Of course, by existence and uniqueness of solutions to ODEs we can always find a solution matching one boundary value, so the difficulty is determining when the other boundary value matches up. This is where stability comes into play, and we use (1.0.7) to force the boundary values into certain configurations where a solution will always exist.

While this theorem demonstrates that (1.0.7) is a sufficient condition for existence, it is not clear it is necessary. One can check that, outside of the supercritical phase case, (1.0.7) does not match the necessary condition for existence presented in [**9**]. To elaborate, let the average angle of a subvariety V^k be defined by the argument $\int_{V^k}(\omega + i\alpha)^k$, and denote this argument by $\hat{\Theta}_{V^k}$. In [**9**] it is demonstrated that any class that solves (1.0.2) must satisfy

$$\hat{\Theta}_{V^k} > \hat{\theta} - (n-k)\frac{\pi}{2}.$$

In fact, assuming supercritical phase the above inequality is equivalent to (1.0.6). However, outside of supercritical phase, one needs to specify a unique lift of $\hat{\theta}$ to $\mathbb{R}$, before a necessary condition similar to the above can be generalized. If such a lift exists, then again a solution to equation (1.0.2) will imply

$$\hat{\theta} + (n-k)\frac{\pi}{2} > \hat{\Theta}_{V^k} > \hat{\theta} - (n-k)\frac{\pi}{2}. \tag{1.0.8}$$

When $k = n-1$, we find the above inequality is a stronger condition than (1.0.7), whereas for $k < n-1$ the conditions fail to match. Nevertheless, we are able to demonstrate:

THEOREM 1.0.2. *Let X be the blowup of $\mathbb{P}^n$ at a point. Let $[\omega]$ be any Kähler class on X, and $[\alpha]$ any real cohomology class. Then $[\alpha]$ admits a solution to the deformed Hermitian-Yang-Mills equation if and only if*

(1) *The average angle $\hat{\theta}$ has a unique lift to $\mathbb{R}$.*
(2) *For every divisor $V^{n-1} \subset X$, the average angle $\hat{\Theta}_{V^{n-1}}$ satisfies* (1.0.8).

Here we see the importance of finding a lift of $\hat{\theta}$. In general, finding a purely algebraic method for lifting $\hat{\theta}$, which only depends on the classes $[\omega]$ and $[\alpha]$, would greatly aid our understanding of the relationship between solvability of (1.0.2) and stability. In this light, one could view condition (1.0.7) as algebraic condition which specifies a lift of $\hat{\theta}$, which then leads to a solution of the equation. Therefore, it would be interesting to develop more such methods of lifting $\hat{\theta}$ in general.

The dissertation is organized as follows. Chapter 2 covers some preliminary knowledge. In Chapter 3 we reformulate the deformed Hermitian-Yang-Mills equation and introduce the Calabi Symmetry ansatz, and show how solutions to refomulation correspond to solutions of an exact ODE.

In Chapter 4 we explicitly compute the inequalities arising from an algebraic stability condition for all subvarieties of X. We then show how these inequalities define regions in $\mathbb{R}^2$ where the graph of our ODE is given, and prove a key proposition relating the slopes of the boundaries of these regions. This proposition is used in Chapter 5 to limit the initial configurations of boundary values for our ODE, which we use to prove that stability is sufficient for existence of a solution. Chapter 6 discusses how $\hat{\theta}$ can be lifted from S^1 to $\mathbb{R}$ without appealing to existence of a solution, assuming (1.0.7) is satisfied for all subvarieties. We then prove Theorem 1.0.2. We conclude the dissertation in Chapter 7 on the current progress on solving the equation on a more general manifold that is an extension of the blowup of complex projective space.

CHAPTER 2

Preliminary Material

Let M be a smooth manifold. This introduction will briefly outline definitions and theorems regarding Kähler manifolds, mainly following [**20**].

Complex manifolds are differentiable manifolds with a holomorphic atlas $\{(U_\alpha, \phi_\alpha)\}$. Using these charts, near any point p in a complex manifold M of dimension n there exists a holomorphic coordinate system $z_1, ..., z_n$ consisting of complex valued functions with $z_\alpha(p) = 0$ for each α. Complex manifolds are necessarily of even dimension.

Holomorphic transition functions are the key difference between complex manifolds and real manifolds of even dimension. But before discussing complex manifolds, we look at almost complex manifolds. They are even dimensional manifolds that possess some properties of complex manifolds but are not complex.

Recall that a function $F = f + ig : U \subset \mathbf{C} \to \mathbf{C}$ is called holomorphic if it satisfies the Cauchy Riemann equations $\frac{\partial f}{\partial x} = \frac{\partial g}{\partial y}$ and $\frac{\partial f}{\partial y} = -\frac{\partial g}{\partial x}$. If j is the endomorphism of $\mathbf{R}^2$ corresponding to multiplication by i on $\mathbf{C}$ with $\mathbf{R}^2$ identified to $\mathbf{C}$ via $z = x + iy \mapsto (x, y)$, j can be expressed in the canonical basis as

$$j = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

If we view F as a real function from R^2 to R^2, the differential of F is

$$F_* = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}. \end{pmatrix}$$

$\mathbf{C}^n$ can be similarly identified to $\mathbf{R}^{2n}$ via $(z_1, \ldots, z_n) = (x_1 + iy_1, \ldots, x_n + iy_n) \mapsto (x_1, \ldots, x_n, y_1, \ldots, y_n)$ If j_n is the endomorphism of $\mathbf{R}^{2n}$ corresponding to multiplication by i on $\mathbf{C}^n$,

$$j = \begin{pmatrix} 0 & -Id_{n\times n} \\ Id_{n\times n} & 0 \end{pmatrix}.$$

A function $F : M \to \mathbf{C}$ is holomorphic if $F \cdot \phi_\alpha^{-1}$ is holomorphic for every U_α in our atlas.

Definition 2.0.1. *A (1,1)-tensor $j : TM \to TM$ on a smooth (real) manifold M which satisfies $J^2 = -Id$ is called an almost complex structure. The pair (M, J) is an almost complex manifold.*

In other words, at any given point $p \in (M, J)$, there is an endomorphism $J_p : T_pM \to T_pM$ which acts like multiplication by $\sqrt{-1}$. An endomorphism of an odd dimensional vector space has a real eigenvalue, which cannot square to $\sqrt{-1}$, so the dimension of M must be even.

To diagonalize the endomorphism J, we complexify the tangent space. Define

$$TM^{\mathbf{C}} := TM \otimes_{\mathbf{R}} \mathbf{C}.$$

All real endomorphisms and differential operators can be extended from TM to $TM^{\mathbf{C}}$ by $\mathbf{C}$-linearity. Let $T^{1,0}M$ and $T^{0,1}$ denote the eigenbundle of $TM^{\mathbf{C}}$ corresponding to the eigenvalue i and $-i$, respectively, of J. $TM^{\mathbf{C}}$ then decomposes into $\pm i$ eigenspaces:

$$T^{1,0}M = \{X - iJX | X \in TM\}, \;\; T^{0,1}M = \{X + iJX | X \in TM\}, \;\; TM^{\mathbf{C}} := T^{1,0}M \oplus T^{0,1}M.$$

There are an equal number of i and $-i$ eigenvalues so at any given point p we can choose a basis of $2n$ real vector fields in the tangent space T_pM such that the almost complex structure takes the form

$$J_p = \begin{pmatrix} 0 & Id_{n\times n} \\ -Id_{n\times n} & 0 \end{pmatrix},$$

or in a basis of complex vector fields on $T_pM^{\mathbf{C}}$

$$J_p = \begin{pmatrix} iId_{n\times n} & 0 \\ 0 & -iId_{n\times n} \end{pmatrix}.$$

For $T_pM^{\mathbf{C}}$ we can choose a basis of complex vector fields $\frac{\partial}{\partial z_\alpha}, \alpha = 1, \ldots, n$, and their complex conjugates $\frac{\partial}{\partial \bar{z}_\alpha}, \alpha = 1, \ldots, n$. The duals can be denoted by dz_α and $d\bar{z}_\alpha$. In local coordinates, we can write

$$J_p = i\frac{\partial}{\partial z_\alpha} \otimes dz_\alpha - i\frac{\partial}{\partial d\bar{z}_\alpha}.$$

A subbundle $E \subset TM$ of the tangent bundle TM is integrable if, for any two vector fields X and Y taking values in E, the Lie bracket $[X, Y]$ takes values in E as well.

Theorem 2.0.1. *(Newlander Nirenberg) Let (M, J) be an almost complex manifold. If J is integrable, the manifold M is complex.*

Complex coordinates can be decomposed into their real and imaginary parts, $z_\alpha = x_\alpha + iy_\alpha$. With

$$dz_\alpha = dx_\alpha + idy_\alpha, \quad d\bar{z}_\alpha = d\bar{x}_\alpha - d\bar{y}_\alpha,$$

any element of the cotangent bundle $\Lambda^1_{\mathbb{C}}M$ can be written uniquely as a sum

$$\sum_\alpha a_\alpha dz_\alpha + b_\alpha d\bar{z}_\alpha.$$

Let $\Lambda^{1,0}M$ and $\Lambda^{0,1}M$ be the space of complex differential forms spanned by dz_α and $d\bar{z}_\alpha$, respectively, and $\Lambda^1_{\mathbb{C}}M = \Lambda^{1,0}M \oplus \Lambda^{0,1}M$.

The decomposition extends to higher degree forms

$$\Lambda^k_{\mathbb{C}}M = \bigoplus_{p+q=k} \Lambda^{p,q}M,$$

where $\Lambda^{p,q}M$ is locally spanned by

$$dz_{\alpha_1} \wedge \ldots \wedge dz_{\alpha_p} \wedge d\bar{z}_{\beta_1} \wedge \ldots \wedge d\bar{z}_{\beta_q}.$$

The decomposition of forms gives a decomposition of the exterior derivative $d = \partial + \bar{\partial}$, where

$$\partial : \Lambda^{p,q}M \to \Lambda^{p+1,q}M$$

$$\bar{\partial} : \Lambda^{p,q}M \to \Lambda^{p,q+1}M$$

satisfying

$$\partial^2 = 0, \quad \bar{\partial}^2 = 0, \quad \text{and} \quad \partial\bar{\partial} + \bar{\partial}\partial = 0.$$

THEOREM 2.0.2. *(The local $i\partial\bar{\partial}$- Lemma) Let $\omega \in \Lambda^{1,1}M \cup \Lambda^2 M$ be a real 2-form of type (1,1) on a complex manifold M. The ω is closed if and only if every point $x \in M$ has an open neighborhood U such that $\omega|_U = i\partial\bar{\partial}u$ for some real function u on U.*

We can then define the *(p,q)*-Dolbeault cohomology group of M

$$H^{p,q}(M) = \frac{\ker\, \bar{\partial} : \Lambda^{p,q}M \to \Lambda^{p,q+1}M}{\mathrm{Im}\, \bar{\partial} : \Lambda^{p,q-1}M \to \Lambda^{p,q}M}$$

Let M be a complex manifold and let $\pi : E \to M$ be a complex vector bundle over M. E is a holomorphic vector bundle if there exists a trivialization with holomorphic transition functions. An operator $\bar{\partial} : \mathcal{C}^\infty(\Lambda^{p,q}E) \to \mathcal{C}^\infty(\Lambda^{p,q+1}E)$ on a complex vector bundle E is called a holomorphic structure if $\bar{\partial}^2 = 0$ and satisfies the Leibniz rule:

$$\bar{\partial}(\omega \wedge \sigma) = (\bar{\partial}\omega) \wedge \sigma + (-1)^{p+q}\omega \wedge (\bar{\partial}\sigma), \forall \omega \in \mathcal{C}^\infty(\Lambda^{p,q}M), \sigma \in \mathcal{C}^\infty(\Lambda^{p,q}E)$$

A complex vector bundle E is holomorphic if and only if it has a holomorphic structure $\bar{\partial}$.

A Hermitian structure H on E is a smooth field of Hermitian products on the fibers of E, that is, for every $X \in M$, $H : E_x \times E_x \to \mathbf{C}$ satisfies

- $H(u, v)$ is $\mathbf{C}$-linear in u for every $v \in E_x$
- $H(u, v) = \overline{H(v, u)}$ for all $u, v \in E_x$
- $H(u, u) > 0$ for all $u \neq 0$
- $H(u, v)$ is a smooth function on M for every smooth section u, v of E.

If we take a trivialization (U_i, ϕ_i) of E and a partition of unity f_i subordinate to the open cover $\{U_i\}$ of M and for every $x \in U_i$, let $(H_i)_x$ denote the pull-back of the Hermitian metric on $\mathbf{C}^k$ by the $\mathbf{C}$-linear map $\phi|_{E_x}$, $H := \sum f_i H_i$ is a well defined Hermitian structure on E.

THEOREM 2.0.3. *For every Hermitian structure H in a holomorphic vector bundle E with holomorphic structure $\bar{\partial}$ there exits a unique H-connection ∇, called the Chern connection, such that $\nabla^{0,1} = \bar{\partial}$.*

A Hermitian metric on an almost complex manifold (M, J) is a Riemannian metric h such that $h(X, Y) = h(JY, JX), \forall X, Y \in TM$. The fundamental form of a Hermitian metric is defined by $\Omega(X, Y) := h(JX, Y)$. Every almost complex manifold admits Hermitian metrics. Choose an arbitrary Riemannian metric g and let $h(X, Y) := g(X, Y) + g(JX, JY)$.

If z_α are the holomorphic coordinates on a complex Hermitian manifold (M^{2n}, J, h) we can denote the coefficients of the metric tensor in these local coordinates:

$$h_{\alpha\bar{\beta}} := h\left(\frac{\partial}{\partial z_\alpha}, \frac{\bar{\partial}}{\partial \bar{z}_\beta}\right).$$

The fundamental form is then given by

$$\Omega = i \sum_{\alpha,\beta=1}^{n} h_{\alpha\bar{\beta}} dz_\alpha \wedge d\bar{z}_{\bar{\beta}}.$$

If the fundamental 2-form Ω of a complex Hermitian manifold is closed, then by the $i\partial\bar{\partial}$-lemma, we get locally a real function u such that $\Omega = i\partial\bar{\partial}u$. In local coordinates, this is

$$h_{\alpha\bar{\beta}} = \frac{\partial^2 u}{\partial z_\alpha \partial \bar{z}_\beta}.$$

Definition 2.0.2. Let M be a complex manifold with Hermitian metric h on the holomorphic tangent bundle. The fundamental 2-form is defined by $\omega = i \sum_{\alpha,\beta=1}^{n} h_{\alpha,\bar{\beta}} dz_\alpha \wedge d\bar{z}_\beta$, where $h_{\alpha,\bar{\beta}} := h\left(\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial \bar{z}^\beta}\right)$. If ω is closed, then M is called a *Kähler manifold*, h the *Kähler metric*, and ω the *Kähler form.*

A Hermitian manifold (M, h, J) has two natural linear connections, the Levi Civita connection ∇ and the Chern connection $\bar{\nabla}$ on TM as a Hermitian vector bundle.

Theorem 2.0.4. *The Chern connection and the Livi Civita connection coincide if and only if h is Kähler.*

Definition 2.0.3. A subset V of an open set $U \subset \mathbb{C}^n$ is an *analytic variety* in U if, for any $p \in U$, there exists a neighborhood U' of p in U such that $V \cup U'$ is the common zero locus of a single nonzero holomorphic function f.

CHAPTER 3

Background Material

Let (X, ω) be a compact Kähler manifold, and $[\alpha] \in H^{1,1}(X, \mathbb{R})$ a real cohomology class. We study the *deformed Hermitian-Yang-Mills* equation, which as stated in the introduction seeks a representative $\alpha \in [\alpha]$ satisfying

$$\mathrm{Im}(e^{-i\hat{\theta}}(\omega + i\alpha)^n) = 0$$

for a fixed constant $e^{i\hat{\theta}} \in S^1$. Integrating the above equation we see the angle $\hat{\theta}$ must be the argument of the complex number

$$\zeta_X := \int_X (\omega + i\alpha)^n.$$

By the $\partial\bar{\partial}$-Lemma, ζ_X is independent of a choice of representatives of the classes $[\omega]$ and $[\alpha]$. Thus we see a simple necessary class condition for existence is that $\zeta_X \neq 0$.

We reformulate the deformed Hermitian-Yang-Mills equation as follows. Given a representative $\alpha \in [\alpha]$, let $\lambda_1, ..., \lambda_n$ denote the real eigenvalues of the Hermitian endomorphism $\omega^{-1}\alpha$. Then, at a fixed point where $\omega^{-1}\alpha$ is diagonal, we see

$$\mathrm{Im}\left(e^{-i\hat{\theta}}\frac{(\omega + i\alpha)^n}{\omega^n}\right) = \mathrm{Im}\left(e^{-i\hat{\theta}}\prod_{k=1}^n(1 + i\lambda_k)\right).$$

We denote the angle of the complex number $\prod_{k=1}^n(1 + i\lambda_k)$ by $\Theta_\omega(\alpha)$, which can be computed as follows:

$$\begin{aligned}
\Theta_\omega(\alpha) &= -i\log\frac{\prod_{k=1}^n(1+i\lambda_k)}{|\prod_{k=1}^n(1+i\lambda_k)|}\\
&= -i\log\frac{\prod_{k=1}^n(1+i\lambda_k)}{(\prod_{k=1}^n(1+i\lambda_k)\prod_{k=1}^n(1-i\lambda_k))^{\frac{1}{2}}}\\
&= -\frac{i}{2}\log\frac{\prod_{k=1}^n(1+i\lambda_k)}{\prod_{k=1}^n(1-i\lambda_k)}.
\end{aligned}$$

By the complex formulation of arctangent, we arrive at

$$\Theta_\omega(\alpha) = \sum_{k=1}^{n} \arctan(\lambda_k).$$

Thus equation (1.0.2) is equivalent to

$$\Theta_\omega(\alpha) = \hat{\theta} \qquad \mod 2\pi. \tag{3.0.1}$$

The advantage of this formulation is that the pointwise angle $\Theta_\omega(\alpha)$ is real valued and lies in $(-n\frac{\pi}{2}, n\frac{\pi}{2})$, while $e^{i\hat{\theta}}$ is only valued in S^1. Thus a solution of the deformed Hermitian-Yang-Mills equation specifies a unique lift of $\hat{\theta}$ to $\mathbb{R}$. We refer to such a lift as a *branch* of the equation.

We now turn to some general results. Interested readers can refer to [**9**, **10**, **15**] for complete proofs.

Solutions of the deformed Hermitian-Yang-Mills equation minimize the volume functional given by the map

$$\mathfrak{a} \ni \alpha \longmapsto V_\omega(\alpha) := \int_X r_\omega(\alpha)\omega^n.$$

Proposition 3.0.1. *[**15**] Define $\hat{r} \geq 0$ by*

$$\hat{r} = \left| \int_X (\omega + \sqrt{-1}\alpha)^n \right|.$$

Then we have $V_\omega(\alpha) \geq \hat{r}$. Furthermore, a smooth form α minimizes $V_\omega(\cdot)$ if and only if α solves the deformed Hermitian-Yang-Mills equation. In this case, the minimum value of V_ω is precisely equal to $\hat{r} > 0$.

If we let A and A_ϵ be the matrix associated to $\alpha = \alpha_0 + i\partial\bar{\partial}u$ and $\alpha_\epsilon = \alpha_0 + i\partial\bar{\partial}u + \epsilon i\partial\bar{\partial}\phi$, respectively, then the linearization of $\Theta_\omega(\cdot)$ is an elliptic second order operator since $arctan(\cdot) : \mathbb{R} \to \mathbb{R}$ is strictly increasing:

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} \Theta_\omega(A_\epsilon) = \sum \frac{1}{1+\lambda_i^2} \partial_i \bar{\partial}_i \phi$$

Lemma 3.0.2. *[**15**] Solutions of the deformed Hermitian-Yang-Mills equation are unique, up to addition of a constant.*

A more general result is

Lemma 3.0.3. *[**10**] Suppose ω is a Kähler form, and $\alpha \in \mathfrak{a}$ has the property that $osc_X \Theta_\omega(\alpha) < \pi$. Then*

(1) $\int_X (\omega + \sqrt{-1}\alpha)^n \in \mathbb{C}^*$
(2) *Let $\theta_\alpha \in (-n\frac{\pi}{2}, n\frac{\pi}{2})$ be defined by*

$$\int_X (\omega + \sqrt{-1}\alpha)^n \in \mathbb{R}_{>0} e^{\sqrt{-1}\theta_\alpha},\ \theta_\alpha \in [\inf_X \theta_\omega(\alpha), \sup_X \theta_\omega(\alpha)]$$

If α' is another representative of the class $\mathfrak{a}$ with $osc_X \Theta_\omega(\alpha') < \pi$, then we have $\theta_\alpha = \theta'_\alpha$.

Definition 3.0.4. *If we assume that there exists some $\alpha \in \mathfrak{a}$ with $osc_X \Theta_\omega(\alpha) < \pi$, we will define $\theta = \hat{\theta}$ as above to be the lifted angle. Since this is independent of the choice of α, we will drop the subscript α.*

Theorem 3.0.5. *Suppose that (X, ω) has non-negative orthogonal bisectional curvature. Let L $\to$ X be an ample line bundle. Let h_0 be a positively curved metric on L. Then for k sufficiently large a solution of the deformed Hermitian-Yang-Mills equation exists.*

In this dissertation we construct solutions to the deformed Hermitian-Yang-Mills equation in a specific geometric setup, where we can take advantage of large symmetry. Specifically, let X be the Kähler manifold defined by blowing up $\mathbb{P}^n$ at one point x_0. Let E denote the exceptional divisor, and H the pullback of the hyperplane divisor from $\mathbb{P}^n$. These two divisors span $H^{1,1}(X, \mathbb{R})$, and any Kähler class will lie in $a_1[H] - a_2[E]$ with $a_1 > a_2 > 0$. Normalizing, assume X admits a Kähler form ω in the class

$$[\omega] = a[H] - [E],$$

with $a > 1$. Furthermore, assume our class $[\alpha]$ satisfies

$$[\alpha] = p[H] - q[E],$$

for a choice of $p, q \in \mathbb{R}$.

Calabi introduced the following ansatz in [**8**]. On $X \backslash (H \cup E) \cong \mathbb{C}^n \backslash \{0\}$ define the radial coordinate

$$\rho = \log(|z|^2).$$

Any function $u(\rho) \in C^\infty(\mathbb{R})$ that satisfies $u'(\rho) > 0$, $u''(\rho) > 0$, has the property that its complex Hessian $\omega = i\partial\bar{\partial}u$ defines a Kähler form on $\mathbb{C}^n \backslash \{0\}$. In order for ω to extend to a Kähler form on X in the class $a[H] - [E]$, we need u to satisfy the following boundary asymptotics. Define the functions $U_0, U_\infty : [0, \infty) \to \mathbb{R}$ via

$$U_0(r) := u(\log r) - \log r \qquad \text{and} \qquad U_\infty(r) := u(-\log r) + a\log r.$$

Then we need both U_0 and U_∞ to extend by continuity to a smooth function at $r = 0$, with both $U_0'(0) > 0$ and $U_\infty'(0) > 0$. In particular this fixes the following asymptotic behavior of u:

$$\lim_{\rho \to -\infty} u'(\rho) = 1, \qquad \lim_{\rho \to \infty} u'(\rho) = a,$$

This ensures that $\omega = i\partial\bar{\partial}u$ extends to a Kähler form on X and lies in the correct class.

Similarly, for any function $v(\rho) \in C^\infty(\mathbb{R})$, the Hessian $i\partial\bar{\partial}v(\rho)$ defines a $(1,1)$ form α on $\mathbb{C}^n \backslash \{0\}$. In order for α to extend to X in the class $[\alpha]$, we require asymptotics of the same form, without any positivity assumptions since $[\alpha]$ need not be a Kähler class. As above, we define the functions $V_0, V_\infty : [0, \infty) \to \mathbb{R}$ via

$$V_0(r) := v(\log r) - q\log r \qquad \text{and} \qquad V_\infty(r) := v(-\log r) + p\log r,$$

and specify that V_0 and V_∞ extend by continuity to a smooth function at $r = 0$. As a result $v(\rho)$ satisfies:

$$\lim_{\rho \to -\infty} v'(\rho) = q, \qquad \lim_{\rho \to \infty} v'(\rho) = p. \tag{3.0.2}$$

Then $i\partial\bar{\partial}v$ extends to a smooth (1,1) form on X in the class $[\alpha]$.

Given this setup, the deformed Hermitian-Yang-Mills equation reduces to an ODE. In particular, for a given function $u(\rho)$ satisfying the Calabi ansatz above (which defines our background

Kähler form), we need to find a function $v(\rho)$ of a single real variable ρ. Working on the coordinate patch $X\backslash(H\cup E)\cong\mathbb{C}^n\backslash\{0\}$, we have

$$\omega = i\partial\bar{\partial}u = \left(\frac{u'}{e^{\rho}}\delta_{jk} + (u''-u')\frac{\bar{z}^j z^k}{e^{2\rho}}\right) dz^j\wedge d\bar{z}^k,$$

and

$$\alpha = i\partial\bar{\partial}v = \left(\frac{v'}{e^{\rho}}\delta_{jk} + (v''-v')\frac{\bar{z}^j z^k}{e^{2\rho}}\right) dz^j\wedge d\bar{z}^k.$$

With the above formulas, one can easily check that the eigenvalues of $\omega^{-1}\alpha$ are $\frac{v'}{u'}$ with multiplicity $(n-1)$, and $\frac{v''}{u''}$ with multiplicity one (for instance, see [**12**]).

In fact, before we write down the deformed Hermitian-Yang-Mills equation in this setting, we can simplify our picture further. Because $u''>0$, the first derivative u' is monotone increasing, allowing us to view u' as a real variable, denoted by x, which ranges from 1 to a. We then write v' as a graph f over $x\in(1,a)$:

$$f(x) = f(u'(\rho)) = v'(\rho).$$

Taking the derivative of both sides, we see by the chain rule

$$f'(x)u''(\rho) = v''(\rho).$$

Working in the coordinate x, the eigenvalues of $\omega^{-1}\alpha$ are

$$\frac{v'}{u'} = \frac{f}{x}\,(\text{with multiplicity}\, n-1) \qquad \text{and} \qquad \frac{v''}{u''} = f'.$$

Note that as $x\to 1$, then $\rho\to-\infty$, while $x\to a$ implies $\rho\to\infty$. Thus the asymptotics of $v(\rho)$ given by (7.0.2) are equivalent to

$$\lim_{x\to 1^+} f(x) = q, \qquad \lim_{x\to a^-} f(a) = p,$$

and we extend $f(x)$ to the boundary $[1,a]$ by continuity.

We now reformulate our problem into this setup. Using the explicit formulas for the eigenvalues of $\omega^{-1}\alpha$, need to find a real function $f:[1,a]\to\mathbb{R}$ with boundary values $f(1)=q$, and $f(a)=p$, satisfying the ODE

$$\operatorname{Im}\left(e^{-i\hat{\theta}}(1+i\frac{f}{x})^{n-1}(1+if')\right)=0. \tag{3.0.3}$$

Since x is always positive, multiplying by x^{n-1} will not change the equation, so we rewrite the ODE as

$$\operatorname{Im}\left(e^{-i\hat{\theta}}(x+if)^{n-1}(1+if')\right)=0.$$

Observe that this ODE is exact

$$\begin{aligned}\operatorname{Im}\left(e^{-i\hat{\theta}}(x+if)^{n-1}(1+if')\right) &= \operatorname{Im}\left(e^{-i\hat{\theta}}\frac{d}{dx}\frac{(x+if)^n}{n}\right)\\ &= \frac{d}{dx}\operatorname{Im}\left(e^{-i\hat{\theta}}\frac{(x+if)^n}{n}\right)=0.\end{aligned}$$

Thus we are looking for a function $f(x)$ so that the graph $(x, f(x))$ lies on a level curve of

$$\Phi(x,y) := \operatorname{Im}\left(e^{-i\hat{\theta}}(x+iy)^n\right). \tag{3.0.4}$$

Figure 3.1 below shows a level set $\Phi(x,y)=c$ for some $c\neq 0$, in the case that $n=11$. The n dotted lines represent the level set $\Phi(x,y)=0$. Thus we see $\Phi(x,y)=c$ consists of n disjoint curves lying in alternating sectors, asymptotic to the lines given by $\Phi(x,y)=0$. Solutions to the deformed Hermitian-Yang-Mills equations are graphical portions of the level set that lie over $[1,a]$. Solutions of the equation for different branches can be found by rotating by $2\pi/n$.

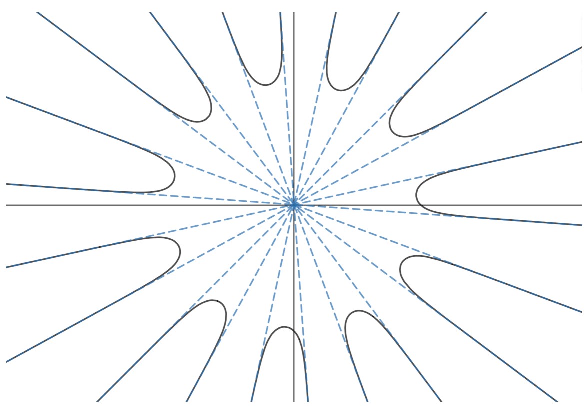


FIGURE 3.1. Graph of a level set $\Phi(x,y)=c$, in the case $n=11$.

CHAPTER 4

Stability

We now turn to the stability condition that guarantees existence of a solution of (1.0.2). This provides a coherent algebraic framework that is simple to interpret from initial conditions, without any assumptions on explicit representatives of $[\omega]$ or $[\alpha]$. Our condition was first introduced in [**9**], where it was demonstrated to be necessary for existence, as well as sufficient in complex dimension 2. In this paper, we use "central charge" notation to highlight possible connections with Bridgeland stability conditions. We refer the reader to [**10**, **11**] for a more detailed discussion of stability and algebraic obstructions to solutions of the deformed Hermitian-Yang-Mills equations in general, and only focus in this paper on our specific geometric setup.

As stated in the introduction, for an analytic subvariety $V \subset X$, we define the following complex number:

$$Z_{[\alpha][\omega]}(V) := -\int_V e^{-i\omega+\alpha},$$

where by convention we only integrate the term in the expansion of order $\dim(V)$.

Definition 4.0.1. *The pair $[\omega], [\alpha]$ is stable if, for each $k \in \{1, ..., n-1\}$ all analytic subvarieties $V^k \subset X$ of dimension k satisfy either for all analytic subvarieties $V \subset X$,*

$$\mathrm{Im}\left(\frac{Z_{[\alpha][\omega]}(V^k)}{Z_{[\alpha][\omega]}(X)}\right) > 0 \qquad \text{or} \qquad \mathrm{Im}\left(\frac{Z_{[\alpha][\omega]}(V^k)}{Z_{[\alpha][\omega]}(X)}\right) < 0. \tag{4.0.1}$$

This definition only makes sense if $Z_{[\alpha][\omega]}(X) \neq 0$, which is equivalent to our assumption that $\zeta_X \neq 0$. Now, because of our specific geometric setup, the inequality (4.0.1) can be explicitly computed in terms of a, p, and q, for each analytic subvariety of X.

Recall that H is the pullback of the hyperplane divisor, and E is the exceptional divisor, and that these divisors do no intersect. We begin by computing ζ_X explicitly:

$$\begin{aligned}
\zeta_X := \int_X (\omega + i\alpha)^n &= (a[H] - [E] + i(p[H] - q[E]))^n \\
&= (a+ip)^n [H]^n + (1+iq)^n (-1)^n [E]^n \\
&= (a+ip)^n - (1+iq)^n,
\end{aligned}$$

where the last line follows since $[E]^n = (-1)^{n-1}$. Again by assumption $\zeta_X \neq 0$, which is the same as requiring a, p, and q do not simultaneously satisfy

$$|a+ip| = |1+iq| \qquad \text{and} \qquad |\arg(a+ip) - \arg(1+iq)| = \frac{2\pi m}{n} \tag{4.0.2}$$

for some $m \in \mathbb{Z}$. We remark that this does not provide a major constraint on which classes we consider. Given a choice of q, there are only a finite number of points $a + ip$ that satisfy $(a+ip)^n = (1+iq)^n$.

We now check stability for H^{n-k} and $(-1)^{n-k-1}E^{n-k}$ for $k \in \{1, ..., n-1\}$, where k represents the dimension of each subvariety. Here we multiply E^{n-k} by $(-1)^{n-k-1}$ so that when this variety is viewed as a divisor of $(-1)^{n-k}E^{n-(k+1)}$ it is effective. We compute

$$\begin{aligned}
Z_{[\alpha][\omega]}(H^{n-k}) &= -\int_{H^{n-k}} (-i)^k (\omega + i\alpha)^k \\
&= -\int_{H^{n-k}} i^{-k} (a[H] - [E] + i(p[H] - q[E]))^k \\
&= -i^{-k} (a+ip)^k [H]^k [H]^{n-k} \\
&= -i^{-k} (a+ip)^k.
\end{aligned}$$

Next we see

$$
\begin{aligned}
Z_{[\alpha][\omega]}((-1)^{n-k-1}E^{n-k}) &= -\int_{(-1)^{n-k-1}E^{n-k}} (-i)^k(\omega+i\alpha)^k \\
&= -\int_{(-1)^{n-k-1}E^{n-k}} i^{-k}(a[H]-[E]+i(p[H]-q[E]))^k \\
&= -i^{-k}(-1)^k(1+iq)^k[E]^k(-1)^{n-k-1}[E]^{n-k} \\
&= -i^{-k}(-1)^{n-1}(1+iq)^k[E]^n \\
&= -i^{-k}(1+iq)^k,
\end{aligned}
$$

since as above $[E]^n = (-1)^{n-1}$. We also can compute the charge of our manifold X, and note

$$
Z_{[\alpha][\omega]}(X) = -\int_X (-i)^n(\omega+i\alpha)^n = -(i)^{-n}\zeta_X = -(i)^{-n}r_Xe^{i\hat{\theta}},
$$

for some fixed real number r_X. Since $r_X > 0$, we can multiply (4.0.1) by r_X without changing the sign of the inequality, and so we note

$$
r_X\mathrm{Im}\left(\frac{Z_{[\alpha][\omega]}(V^k)}{Z_{[\alpha][\omega]}(X)}\right) = \mathrm{Im}\left(\frac{r_XZ_{[\alpha][\omega]}(V^k)}{-i^{-n}r_Xe^{i\hat{\theta}}}\right) = \mathrm{Im}\left(-i^ne^{-i\hat{\theta}}Z_{[\alpha][\omega]}(V^k)\right).
$$

Thus, plugging in our formulas for H^{n-k} and $(-1)^{n-k-1}E^{n-k}$ gives either

$$
\mathrm{Im}\left(i^{n-k}e^{-i\hat{\theta}}(a+ip)^k\right) > 0,
$$

and

$$
\mathrm{Im}\left(i^{n-k}e^{-i\hat{\theta}}(1+iq)^k\right) > 0,
$$

or the above with the inequality flipped. Summing up we have:

LEMMA 4.0.2. *Given a choice of classes $[\omega] = a[H]-[E]$ and $[\alpha] = p[H]-q[E]$ on X, denote complex numbers $z_1 = (1+iq)$ and $z_2 = (a+ip)$. Then the pair $[\omega],[\alpha]$ is stable if and only if, for all $k \in \{1,...,n-1\}$,*

$$
\mathrm{Im}\left(i^{n-k}e^{-i\hat{\theta}}(z_\ell)^k\right) > 0 \qquad \text{or} \qquad \mathrm{Im}\left(i^{n-k}e^{-i\hat{\theta}}(z_\ell)^k\right) < 0 \tag{4.0.3}
$$

for $\ell \in \{1,2\}$.

We now turn to some preliminary results about the structure of the inequalities defined in (4.0.3). Let z be the standard coordinate on $\mathbb{C}$, and choose a branch cut along the negative x-axis, so that $-\pi \leq \arg(z) < \pi$. For each $k \in \{1, ..., n\}$, consider the set defined by

$$\mathcal{R}_k := \{z \in \mathbb{C} |\, \mathrm{Im}\left(i^{n-k} e^{-i\hat{\theta}} z^k\right) = 0 \text{ and } -\frac{\pi}{2} \leq \arg(z) < \frac{\pi}{2}\},$$

which consists of k-rays emanating from the origin. Even though the stability conditions above are only defined for $k \leq n-1$, it is useful for our proof to also consider the rays determined by the $k = n$ case. Now, denote these rays via $\{r_k^1, r_k^2, ..., r_k^k\}$, numbered so that

$$\frac{\pi}{2} > \arg(r_k^1) > \arg(r_k^2) > \cdots > \arg(r_k^k) \geq -\frac{\pi}{2}.$$

By definition of the map $z \mapsto z^k$, we see that these rays are all $\frac{\pi}{k}$ rotations of each other, i.e. $\arg(r_k^{j+1}) - \arg(r_k^j) = \frac{\pi}{k}$. Next, we define a *sector* to be the space between (but not including) two adjacent rays. Again, by the behavior of $z \mapsto z^k$, we see that the space

$$\mathcal{S}_k := \{z \in \mathbb{C} \,|\, \mathrm{Im}\left(i^{n-k} e^{-i\hat{\theta}} z^k\right) > 0 \text{ and } -\frac{\pi}{2} \leq \arg(z) < \frac{\pi}{2}\}$$

consists of alternating sectors, i.e. each ray bounds one and only one sector in $\mathcal{S}_k$. See Figure 4.1 below.

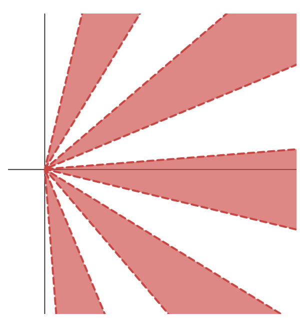


FIGURE 4.1. The set $\mathcal{S}_k$, in the case $k = 10$.

Furthermore, consider the set

$$\mathcal{S}_k^- := \{z \in \mathbb{C} \,|\, \mathrm{Im}\left(i^{n-k} e^{-i\hat{\theta}} z^k\right) < 0 \text{ and } -\frac{\pi}{2} \leq \arg(z) < \frac{\pi}{2}\}.$$

Now, if we write a ray r_k^j as $\mathbb{R}_+ e^{i\phi_k^j}$, we see the sets of rays can be identified with sets of angles, i.e. $\mathcal{R}_k \cong \{\phi_k^1, ..., \phi_k^k\}$. We conclude this section with a combinatorial argument that plays a key role in the proof of Theorem 1.0.1.

PROPOSITION 4.0.1. *For any $k \in \{2, ..., n\}$, the rays in the sets $\mathcal{R}_k$ and $\mathcal{R}_{k-1}$ alternate, and $\mathcal{R}_k$ contains the rays with the largest and smallest argument. In particular:*

$$\frac{\pi}{2} > \phi_k^1 > \phi_{k-1}^1 > \phi_k^2 > \phi_{k-1}^2 > \cdots > \phi_{k-1}^{k-2} > \phi_k^{k-1} > \phi_{k-1}^{k-1} \geq \phi_k^k \geq -\frac{\pi}{2}.$$

Furthermore, if the last inequality is strict, i.e. $\phi_k^k > -\frac{\pi}{2}$, then $\phi_{k-1}^{k-1} > \phi_k^k$ as well.

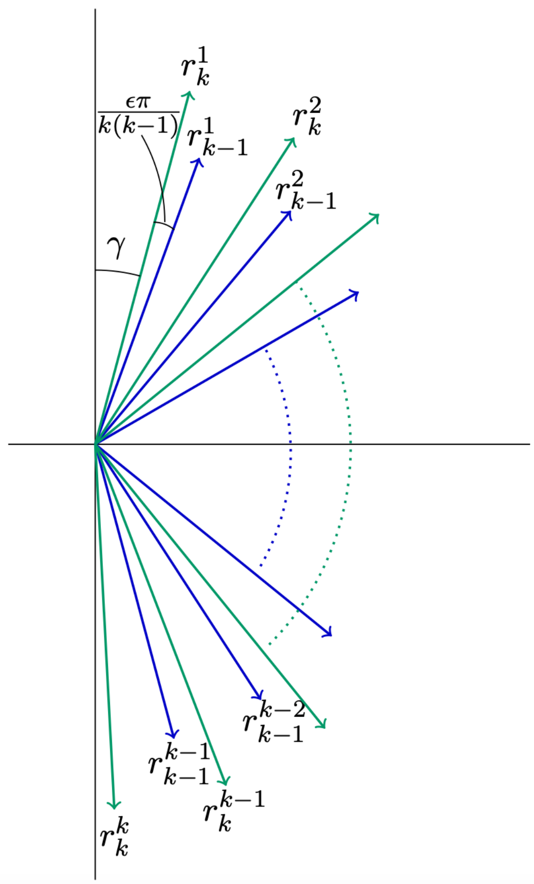


FIGURE 4.2. The alternating condition for rays in sets $\mathcal{R}_k$ and $\mathcal{R}_{k-1}$.

PROOF. Pick two angles ϕ_k^ℓ and ϕ_{k-1}^j from $\mathcal{R}_k$ and $\mathcal{R}_{k-1}$, respectively. It will be convenient to express these angles by their distance to $\frac{\pi}{2}$, so we set $\phi_k^\ell = \frac{\pi}{2} - \gamma^\ell$ and $\phi_{k-1}^j = \frac{\pi}{2} - \sigma^j$.

Now, since ϕ_k^ℓ specifies a ray in the set $\mathcal{R}_k$, by definition we have

$$\operatorname{Im}\left(e^{i\frac{\pi}{2}(n-k)}e^{-i\hat{\theta}}e^{ik\phi_k^\ell}\right) = \operatorname{Im}\left(e^{i\frac{\pi}{2}(n-k)}e^{-i\hat{\theta}}e^{ik(\frac{\pi}{2}-\gamma^\ell)}\right) = 0$$

This equation holds if and only if

$$\frac{n\pi}{2} - \hat{\theta} = k\gamma^\ell + q\pi \tag{4.0.4}$$

for some $q \in \mathbb{Z}$. Next, since ϕ_{k-1}^j lies in $\mathcal{R}_{k-1}$ we have

$$\operatorname{Im}\left(e^{i\frac{\pi}{2}(n-k+1)}e^{-i\hat{\theta}}e^{i(k-1)(\frac{\pi}{2}-\sigma^j)}\right) = 0,$$

which is equivalent to

$$\frac{n\pi}{2} - \hat{\theta} - (k-1)\sigma^j = p\pi$$

for some $p \in \mathbb{Z}$. Plugging in (4.0.4) gives that for all ℓ, j, there exists an $m \in \mathbb{Z}$ so that

$$k\gamma^\ell - (k-1)\sigma^j = m\pi. \tag{4.0.5}$$

This is the key equation relating our angles ϕ_k^ℓ and ϕ_{k-1}^j.

First we prove the result in the special case that $\phi_k^k = -\frac{\pi}{2}$. In this case $\gamma^k = \pi$, and plugging this into (4.0.5) we see that $\sigma^{k-1} = \pi$ solves the equation for $m = 1$. This implies $\phi_{k-1}^{k-1} = -\frac{\pi}{2}$ as well. To see the rays satisfy the alternation condition, note that all rays in $\mathcal{R}_k$ are $\frac{\pi}{k}$ rotations of each other, and furthermore both $\mathcal{R}_k$ and $\mathcal{R}_{k-1}$ contain the negative $y-$axis. As a result

$$\phi_k^\ell = \frac{\pi}{2} - \frac{\ell\pi}{k} \qquad \text{and} \qquad \phi_{k-1}^j = \frac{\pi}{2} - \frac{j\pi}{k-1},$$

for $\ell \in \{1, ..., k\}$ and $j \in \{1, ..., k-1\}$, from which the alternating condition is clear.

We now turn to the general case, and assume that $\phi_k^k > -\frac{\pi}{2}$. As above write $\phi_k^1 = \frac{\pi}{2} - \gamma^1$ and $\phi_{k-1}^1 = \frac{\pi}{2} - \sigma^1$. Since the rays in $\mathcal{R}_k$ are $\frac{\pi}{k}$ rotations of each other, and ϕ_k^1 is the first ray to the right of the positive $y-$axis, we know $0 < \gamma^1 < \frac{\pi}{k}$ (since $\gamma^1 = \frac{\pi}{k}$ corresponds to the special case $\phi_k^k = -\frac{\pi}{2}$). Similarly we know $0 < \sigma^1 < \frac{\pi}{k-1}$. Returning to (4.0.5), and using that $k\gamma^1 < \pi$, we know that for some $m \in \mathbb{Z}$

$$\sigma^1 = \frac{k\gamma^1 - m\pi}{k-1} < \frac{\pi(1-m)}{k-1}.$$

Since $\sigma^1 > 0$ we must have $m \leq 0$. Furthermore, using that $k\gamma^1 > 0$ gives

$$\sigma^1 = \frac{k\gamma^1 - m\pi}{k-1} > \frac{-m\pi}{k-1}.$$

Yet because we know $\sigma^1 < \frac{\pi}{k-1}$, m can not be strictly negative. Thus $m = 0$, giving

$$\sigma^1 = \frac{k\gamma^1}{k-1}. \tag{4.0.6}$$

Now that we have an equation specifying σ^1, we can write down the following general forms for our angles ϕ_k^ℓ and ϕ_{k-1}^j. Specifically,

$$\phi_k^\ell = \frac{\pi}{2} - \gamma^1 - (\ell - 1)\frac{\pi}{k} \qquad \text{and} \qquad \phi_{k-1}^j = \frac{\pi}{2} - \frac{k\gamma^1}{k-1} - (j-1)\frac{\pi}{k-1}.$$

This is equivalent to

$$\gamma^\ell = \gamma^1 + (\ell - 1)\frac{\pi}{k} \qquad \text{and} \qquad \sigma^j = \frac{k\gamma^1}{k-1} + (j-1)\frac{\pi}{k-1}.$$

For all ℓ, j this gives an explicit solution to (4.0.5), with $m = \ell - j$.

To complete the proof, we demonstrate the alternating condition, which states for $j \in \{1, ..., k-1\}$,

$$\phi_k^j > \phi_{k-1}^j > \phi_k^{j+1}.$$

Using our explicit angle formulas this can be written as

$$-\gamma^1 - (j-1)\frac{\pi}{k} > -\frac{k\gamma^1}{k-1} - (j-1)\frac{\pi}{k-1} > -\gamma^1 - j\frac{\pi}{k},$$

which is equivalent to

$$(j-1)\frac{\pi}{k-1} - (j-1)\frac{\pi}{k} > \gamma^1 - \frac{k\gamma^1}{k-1} > (j-1)\frac{\pi}{k-1} - j\frac{\pi}{k}.$$

Multiplying through by $k-1$ gives

$$(j-1)\pi - (j-1)\pi\frac{k-1}{k} > -\gamma^1 > (j-1)\pi - j\pi\frac{k-1}{k}.$$

Simplifying, and multiplying by -1, we arrive at

$$-\frac{(j-1)\pi}{k} < \gamma^1 < \pi(\frac{k-j}{k}),$$

which certainly holds for all $j \in \{1, ..., k-1\}$, assuming that $0 < \gamma^1 < \frac{\pi}{k}$. This completes the proof of the proposition. $\square$

CHAPTER 5

Proof of Theorem (1.0.1)

In this section we prove our main result, and construct a solution to the deformed Hermitian-Yang-Mills equation assuming stability of the pair $[\omega], [\alpha]$.

Recall that on X equation (1.0.2) on be reformulated using Calabi symmetry. Specifically we are looking for a real function $f : [1, a] \to \mathbb{R}$ with boundary values $f(1) = q$, and $f(a) = p$, satisfying

$$\mathrm{Im}\left(e^{-i\hat{\theta}}(1 + i\frac{f}{x})^{n-1}(1 + if')\right) = 0.$$

We saw above that this ODE is exact, and can be integrated to give level curves defined by (3.0.4). Thus we need a function f that satisfies the boundary condition and lies on one of these level curves. For this to be possible, we need the specified boundary points $(1, q)$ and (a, p) to lie on the same level set.

Lemma 5.0.1. *For any choice of $[\omega]$ and $[\alpha]$, the fixed boundary points $(1, q)$ and (a, p) lie on the same level set of*

$$\Phi(x, y) := \mathrm{Im}\left(e^{-i\hat{\theta}}(x + iy)^n\right)$$

Proof. Recall the complex number $\zeta_X = \int_X (\omega + i\alpha)^n$, which in our case is computed to be $(a + ip)^n - (1 + iq)^n$. Set $\zeta_X = r_X e^{i\hat{\theta}}$. Taking the complex conjugate gives $r_X e^{-i\hat{\theta}} = (a - ip)^n - (1 - iq)^n$. Rearranging terms we see

$$e^{-i\hat{\theta}} = \frac{(a - ip)^n - (1 - iq)^n}{r_X}.$$

We then have

$$\begin{aligned}\Phi(a, p) &= \mathrm{Im}\left(\frac{(a - ip)^n - (1 - iq)^n}{r_X}(a + ip)^n\right)\\ &= \mathrm{Im}\left(\frac{(a^2 + p^2)^n}{r_X} - \frac{(a + ip)^n(1 - iq)^n}{r_X}\right).\end{aligned}$$

The first term inside of the imaginary part above is real, so

$$\Phi(a,p) = -\mathrm{Im}\left(\frac{(a+ip)^n(1-iq)^n}{r_X}\right).$$

In exactly the same fashion we see

$$\Phi(1,q) = \mathrm{Im}\left(\frac{(a-ip)^n(1+iq)^n}{r_X}\right).$$

Since $\mathrm{Im}(z) = -\mathrm{Im}(\bar{z})$ it follows that $\Phi(a,p) = \Phi(1,q)$, which completes the proof of the lemma. $\square$

Thus $(1,q)$ and (a,p) always lie on the same level set, which we denote by $\Phi(x,y) = \Phi(a,p) = \Phi(1,q) = c$. We now need to analyze when these points can be connected by a portion of the level set which stays graphical. Note that each level set is made up of several components. If $c = 0$, then the level set consists of n lines through the origin, each line $\frac{\pi}{n}$ rotation of the next. Since $a > 1 > 0$, in this case the points $a + ip$ and $1 + iq$ each lie on a ray in $\mathcal{R}_n$ (although we do not know yet if they lie on the same ray).

If $c \neq 0$, then the level set looks like n distinct curves lying in alternating sectors (see Figure 3.1). In order for there to exists a function lying on a level curve connecting $(1,q)$ to (a,p), the boundary points need to be on the same component of the level set, which we now prove.

Proposition 5.0.1. *If the classes $[\omega], [\alpha]$ are stable in the sense of Lemma 4.0.2, then the points $(1,q)$ and (a,p) both lie on the same component of the level set $\Phi(x,y) = c$.*

Proof. Set $z_1 = (1+iq)$ and $z_2 = (a+ip)$. We argue by contradiction, and assume that z_1 and z_2 do not lie on the same component of the level set. As a first step we show that there exists a ray $r^j_{n-1} \in \mathcal{R}_{n-1}$ lying between z_1 and z_2. To see this, note that if $c = 0$, then by assumption z_1 and z_2 lie on distinct rays in $\mathcal{R}_n$. Applying Proposition 4.0.1 for $k = n$ we see exists a ray $r^j_{n-1} \in \mathcal{R}_{n-1}$ between z_1 and z_2.

In the case that $c \neq 0$, the level set looks like n distinct curves lying in alternating sectors with angle $\frac{\pi}{n}$. If z_1 and z_2 do not lie on the same component, since the components are in alternating sectors, there exists at least one empty sector between the sector containing z_1 and the sector containing z_2. The boundary of this empty sector consists of two rays r^{j+1}_n and r^j_n, and thus these two rays lie between z_1 and z_2. Applying Proposition 4.0.1 for $k = n$ proves existence of a ray r_{n-1}^j between r_n^{j+1} and r_n^j, and thus r_{n-1}^j lies between z_1 and z_2.

We now apply an induction argument and show that if there exists a ray $r_k^j \in \mathcal{R}_k$ lying between z_1 and z_2, then there exists a ray $r_{k-1}^\ell \in \mathcal{R}_{k-1}$ lying between z_1 and z_2 as well. Note that by the stability assumption, either z_1 and z_2 both lie in $\mathcal{S}_k$, or they both lie in $\mathcal{S}_k^-$ (depending on whether the inequality is positive or negative). The key to this proposition is that in either case, the sets containing both z_1 and z_2 consists of alternating sectors. Specifically, given that there exists a ray r_k^j lying between z_1 and z_2, then z_1 and z_2 must lie in different sectors of $\mathcal{S}_k$ (or $\mathcal{S}_k^-$). Because these sectors alternate, there must be an empty sector between z_1 and z_2. The boundary of this empty sector consists of two rays in $\mathcal{R}_k$, which we denote by $r_k^{\ell+1}$ and r_k^ℓ. These two rays lie between z_1 and z_2, and Proposition 4.0.1 gives that the ray r_{k-1}^ℓ lies between z_1 and z_2 as well.

Thus, given that there exists a ray r_{n-1}^j between z_1 and z_2, applying the induction argument $n-2$ times gives that the ray r_1^1 lies between z_1 and z_2. However, the ray r_1^1 divides the space $\{z \in \mathbb{C} | -\frac{\pi}{2} \leq \arg(z) < \frac{\pi}{2}\}$ into two regions, $\mathcal{S}_1$ and ${\mathcal{S}_1}^c$. Thus it is impossible that z_1 and z_2 are both in $\mathcal{S}_1$ (or $\mathcal{S}_1^-$), while also lying on opposite sides of r_1^1. This gives a contradiction, proving the proposition.

We remark that the proof may end sooner in the special case that r_1^1 is the negative $y-$axis. In this case, the ray r_2^2 is also the negative $y-$axis (see the proof of Proposition 4.0.1), so in fact the ray r_2^1 must divide the space $\{z \in \mathbb{C} | -\frac{\pi}{2} \leq \arg(z) < \frac{\pi}{2}\}$ into two regions. Thus the contradiction occurs at this step, with $k = 2$, rather than $k = 1$. $\square$

To finish the proof of the Theorem 1.0.1, we need to show that there exists a function $f(x)$ with $f(1) = q$ and $f(a) = p$, so that the graph of the function lies on the level curve $\Phi(x, y) = c$. We have just demonstrated that the points $(1, q)$ to (a, p) lie on the same component of the level set $\Phi(x, y) = c$, so all that remains to be shown is that the level curve connecting $(1, q)$ to (a, p) does not have vertical slope.

First, if $c = 0$, then the level curves of $\Phi(x, y) = 0$ consist of n rays in $\mathcal{R}_n$. The above proposition shows that $(1, q)$ to (a, p) lie on the same ray r_n^j. Since the ray never has vertical slope, in this case we see right away that there exists a linear function $f(x)$ with $f(1) = q$ and $f(a) = p$, proving the theorem.

In general, the points where the tangent line to $\Phi(x, y) = c$ has vertical slope are given by

$$\frac{\partial}{\partial y}\Phi(x,y) = \frac{\partial}{\partial y}\text{Im}\left(e^{-i\hat{\theta}}(x+iy)^n\right) = \text{Im}\left(ine^{-i\hat{\theta}}(x+iy)^{n-1}\right) = 0.$$

Dividing by n and writing $z = x + iy$, these points satisfy

$$\text{Im}\left(ie^{-i\hat{\theta}}z^{n-1}\right) = 0,$$

and so by definition of $\mathcal{R}_{n-1}$ we see they lie on a ray r^j_{n-1} (see Figure 5.1). Thus in order to show that the level curve connecting $(1, q)$ to (a, p) does not have vertical slope, the curve can not pass over a ray r^j_{n-1}. By our stability assumption, both z_1 and z_2 can not be on opposite sides of the ray r^j_{n-1}. As a result the level curve connecting $(1, q)$ to (a, p) does not have vertical slope, and thus there exists a $f(x)$ with $f(1) = q$ and $f(a) = p$ that solves the ODE (3.0.3). Thus we have demonstrated that if the classes $[\omega]$, $[\alpha]$ are stable, a solution to the deformed Hermitian-Yang-Mills equation exists. This concludes the proof of Theorem 1.0.1.

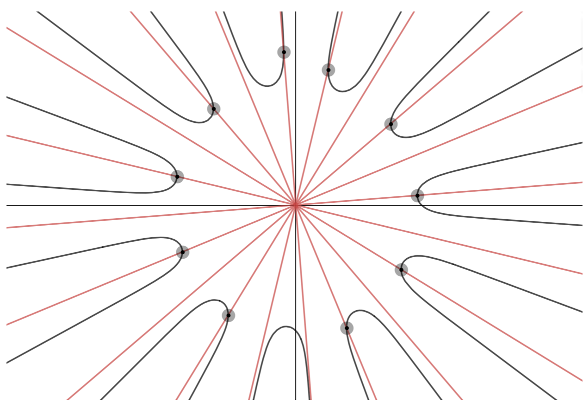


FIGURE 5.1. The intersection of a level set $\Phi(x, y) = c$ with the lines defined by $\text{Im}\left(ie^{-i\hat{\theta}}z^{n-1}\right) = 0$ occurs where the level set has vertical slope.

CHAPTER 6

Lifting the Average Angle

Recall that the average angle $\hat{\theta}$ is defined to be the argument of $\zeta_X = (a+ip)^n - (1+iq)^n$, which is a priori only S^1 valued (note that changing $\hat{\theta}$ by 2π does not effect equation (1.0.2)). This is in contrast to the pointwise angle $\Theta_\omega(\alpha)$, which as a sum of arctangents lifts to $\mathbb{R}$. Since (1.0.2) can be reformulated as (3.0.1), a solution to (3.0.1) specifies a unique lift of $\hat{\theta}$ to $\mathbb{R}$. A slightly weaker (but nevertheless analytic) assumption to specify a lift would be the existence of a representative α_0 that the point-wise angle $\Theta_\omega(\alpha_0)$ has oscillation less that π. This leads to the following question: is it possible to identify how $\hat{\theta}$ lifts to $\mathbb{R}$ from the initial data a, p and q alone, without needing to know existence of a specific representative of $[\alpha]$?

In general the answer is no, but there are special cases in which a lift exists. Collins-Xie-Yau consider the following situation in [**10**]. Define a path $\gamma(t) : [0,1] \to \mathbb{C}$ via

$$\gamma(t) = \int_X (\omega + it\alpha)^n.$$

At the starting time $\gamma(0) = \mathrm{Vol}(X) = a^n - 1$ is a positive real number, which we define to have zero argument. Also $\gamma(1) = \zeta_X$. Then, as long as $\gamma(t) \in \mathbb{C}^*$ for all $t \in [0,1]$, letting t run from 0 to 1, we can count the number of times $\gamma(t)$ winds around the origin to define a lift of $\hat{\theta}$ to $\mathbb{R}$.

Unfortunately there are examples where the angle $\hat{\theta}$ is well defined, but $\gamma(t)$ passes through the origin, so $\hat{\theta}$ can not be lifted using this method. We construct such an example in dimension 3. First, fix a real number $q > \sqrt{3}$. Define an angle $\theta = \frac{2\pi}{3} - \arctan(q)$, and set $a = (\sqrt{q^2+1})\cos(\theta)$ and $p = -(\sqrt{q^2+1})\sin(\theta)$. Note that the choice $q > \sqrt{3}$ ensures $a > 1$. By construction $1 + iq$ and $a + ip$ now satisfy (4.0.2) for $k = 1$, and therefore $(a+ip)^3 = (1+iq)^3$. To complete our example, consider the initial data

$$[\omega] = a[H] - [E] \qquad \text{and} \qquad [\alpha] = 2p[H] - 2q[E],$$

with a and p defined as above. Now, initially $\gamma(1) \neq 0$, since the arguments of $1 + i2q$ and $a + i2p$ are greater than $\frac{2\pi}{3}$ apart, while $\gamma(\frac{1}{2}) = 0$. Of course, one could always choose another path that avoids the origin, however then the lift will depend on the choice of the path.

We remark that similar examples where the lift can not be defined exist in dimension 3 or higher. In dimension 2, the angle $\hat{\theta}$ always lifts, since the arguments of $1+itq$ and $a+itp$ can never be distance π apart, so the path $\gamma(t)$ never passes through the origin. This is a special case of the fact that on a general Kähler surface, the angle $\hat{\theta}$ always lifts by the Hodge Index Theorem [**10**].

One difficulty with the above method is that even if a lift of $\hat{\theta}$ exists, in practice it can be hard to verify. Due to the specific geometry of our setup, we introduce a another notion of a lifted angle.

Assume that $\hat{\theta}$ lies in the branch cut $-\pi \leq \hat{\theta} < \pi$. Suppose that for a given choice of $[\omega]$ and $[\alpha]$, we have

$$|\arg(a + ip) - \arg(1 + iq)| < \frac{\pi}{n}. \tag{6.0.1}$$

We now lift $\hat{\theta}$ to $\mathbb{R}$ as follows. It is easy to see there exists two smooth functions $\rho_1(t), \rho_2(t) : [0, 1] \to [0, 1]$, so that

$$|\arg(a + i\rho_1(t)p) - \arg(1 + i\rho_2(t)q)| < \frac{\pi}{n}.$$

In this case, the complex numbers $(a + i\rho_1(t)p)^n$ and $(1 + i\rho_2(t)q)^n$ lie in the same half-plane, and so the path $\tilde{\gamma}(t) = (a+i\rho_1(t)p)^n - (1+i\rho_2(t)q)^n$ never passes through the origin and has a winding number $k \in \mathbb{Z}$. We then define the lift of $\hat{\theta}$ (denoted $\hat{\Theta}_X$), by

$$\hat{\Theta}_X := \hat{\theta} + 2\pi k \in (-n\frac{\pi}{2}, n\frac{\pi}{2}). \tag{6.0.2}$$

Again we emphasize that this lifted angle depends only on a, p and q, and not on any representatives of the classes $[\omega]$ and $[\alpha]$. One advantage of using the above lifted angle is that our stability implies such a lift exists.

PROPOSITION 6.0.1. *Suppose the pair $[\omega], [\alpha]$ is stable in the sense of Lemma 4.0.2. Then the angle $\hat{\theta}$ has a well defined lift $\hat{\Theta}_X$ given by* (6.0.2).

PROOF. By the induction argument given in Proposition 5.0.1, we know from our stability assumption that the two points $(a + ip)$ and $(1 + iq)$ can not have two rays from $\mathcal{R}_n$ between them.

Since the rays in $\mathcal{R}_n$ are all $\frac{\pi}{n}$ rotations of each other, this verifies (6.0.1), which allows us to define $\hat{\Theta}_X$. $\square$

We expect that in general, being able to determine the lifted angle and specifying the branch will be a key step to solving the deformed Hermitian Yang Mills equation. This expectation is motivated by Theorem 1.0.2, which shows the importance of the lifted angle in our specific case.

First, we note that for any subvariety H^{n-k} or $(-1)^{n-k-1}E^{n-k}$, the lifted restricted angle is always well defined. Specifically, if we assume $z_1 = 1 + iq$ and $z_2 = a + ip$ always have arguments in $(-\frac{\pi}{2}, \frac{\pi}{2})$, then the lifted angle associated to each subvariety is given by

$$\hat{\Theta}_{(-1)^{n-k-1}E^{n-k}} = k\arg(z_1) \qquad \text{and} \qquad \hat{\Theta}_{H^{n-k}} = k\arg(z_2). \tag{6.0.3}$$

We now present the proof of Theorem 1.0.2.

To begin, assume that for a given choice of a, p, q there exists a lifted angle $\hat{\Theta} \in \mathbb{R}$. Furthermore assume that V^{n-1} (which can be either H or E) satisfies

$$\hat{\Theta}_X + \frac{\pi}{2} > \hat{\Theta}_{V^{n-1}} > \hat{\Theta}_X - \frac{\pi}{2}. \tag{6.0.4}$$

Using (6.0.3) this implies

$$\frac{1}{n-1}\left(\hat{\Theta}_X + \frac{\pi}{2}\right) > \arg(z_\ell) > \frac{1}{n-1}\left(\hat{\Theta}_X - \frac{\pi}{2}\right).$$

for $\ell \in \{1, 2\}$. Thus the difference between $\arg(z_1)$ and $\arg(z_2)$ is at most $\frac{\pi}{n-1}$. By Lemma 5.0.1 both z_1 and z_2 lie on the same level set of Φ, and since this level set consists of curves in alternating sectors with angle $\frac{\pi}{n}$, the angle bound of $\frac{\pi}{n-1}$ tells us that either z_1 and z_2 lie in the same component of the level set, or they lie on two adjacent components with an empty sector in between. We must rule out the latter possibility.

Note that (6.0.4) implies

$$\pi > \frac{\pi}{2} - \hat{\Theta}_X + (n-1)\arg(z_\ell) > 0. \tag{6.0.5}$$

This is equivalent to

$$\mathrm{Im}\left(ie^{-i\hat{\theta}}(z_\ell)^{n-1}\right) > 0$$

for $\ell \in \{1,2\}$, which is just stability in the sense of Lemma 4.0.2 for $k = n-1$. So z_1 and z_2 lie in $\mathcal{S}_{n-1}$. Right away this rules out the possibility that they lie on distinct adjacent rays in $\mathcal{R}_n$, since any two such rays will never both be contained in $\mathcal{S}_{n-1}$. We can also rule out the case where z_1 and z_2 lie in two adjacent components which are not rays. In this case, there will be exactly two rays in $\mathcal{R}_n$ between z_1 and z_2, and thus by Proposition 4.0.1 at least one ray in $\mathcal{R}_{n-1}$. Yet because the sectors in $\mathcal{S}_{n-1}$ alternate, there must in fact be two rays in $\mathcal{R}_{n-1}$ between z_1 and z_2. But this is impossible if the difference between $\arg(z_1)$ and $\arg(z_2)$ is at most $\frac{\pi}{n-1}$.

Thus z_1 and z_2 lie in the same component of the level set of Φ. Furthermore, just as in the proof of Theorem 1.0.1, stability in the sense of Lemma 4.0.2 for $k = n-1$ rules out the possibility of a vertical slope on the level curve connecting z_1 and z_2, and so a solution to the deformed Hermitian-Yang-Mills equation exists.

Conversely, suppose for a given a, p, q there exists a solution to equation (3.0.1). As explained above, because the pointwise angle is a sum of arctangents, a solution to (3.0.1) specifies a uniques lift $\hat{\Theta} \in \mathbb{R}$. Additionally, restricting a solution to either H or E, we lose one arctangent from the sum that makes up the pointwise angle. Since the image of arctangent lies in $(-\frac{\pi}{2}, \frac{\pi}{2})$, the average angle on each of these divisors must satisfy (6.0.4). For details see Lemma 8.2 in [**9**]. This completes the proof of Theorem 1.0.2.

We conclude the paper by noting the distinction between the stability from Conjecture 1.0.3 and our stability in the sense of Lemma 4.0.2. Although the original conjecture is only stated for the supercritical phase case, It is not too difficult to see, looking at the proof of Proposition 8.3 in [**9**], that it can be generalized to any phase as

$$\hat{\Theta}_X + (n-k)\frac{\pi}{2} > \hat{\Theta}_{V^k} > \hat{\Theta}_X - (n-k)\frac{\pi}{2},$$

provided that all associated phase angles lift. Thus one difference we see right away is that Conjecture 1.0.3 requires all lifted angles to exist, while this is not true of our stability. Furthermore, we see the above inequality forces z_1 and z_2 between two rays, whereas Lemma 4.0.2 places them in alternating sectors. When $k = n-1$, we see this as a stronger condition. However, when $k < n-1$, the rays fail to match up. It would be interesting to explore this phenomenon more in the future.

CHAPTER 7

Further Exploration

We can look at a more general manifold that is an extension the blowup of $\mathbb{P}^n$. The following setup comes from [**12**].

Let $E = \mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus(m+1)}$ be a vector bundle over a projective space $\mathbb{P}^n$, where $\mathcal{O}_{\mathbb{P}^n}$ is the trivial line bundle and $\mathcal{O}_{\mathbb{P}^n}(-1)$ is the tautological line bundle. Let

$$X_{m,n} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus(m+1)})$$

be the projectiviaztion of E. $X_{m,n}$ is a $\mathbb{P}^{m+1}$ bundle over $\mathbb{P}^n$ with $\pi : X_{m,n} \to \mathbb{P}^n$ the bundle map. Note that $X_{0,n}$ is $\mathbb{P}^{n+1}$ blown up at one point. Let D_∞ be the divisor in $X_{m,n}$ given by $\mathbb{P}(\mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus(m+1)})$ and D_0 be the divisor in $X_{m,n}$ given by $\mathbb{P}(\mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus m})$. The additive divisor group $N^1(X_{m,n})$ is spanned by $[D_0]$ and $[D_\infty]$. We also define the divisor D_H by the pullback of the divisor on $\mathbb{P}^n$ associated to $\mathcal{O}_{\mathbb{P}^n}(1)$. Then

$$[D_\infty] = [D_0] + [D_H].$$

To consider the Calabi Symmetry, let ω_{FS} be the Fubini-Study metric on $\mathbb{P}^n$. We define the radial coordinate

$$\rho = \log(|z|^2).$$

Let h be the Hermitian metric on $\mathcal{O}_{\mathbb{P}^n}(-1)$ such that $Ric(h) = -\omega_{FS}$. Under the local trivialization of E, we have

$$e^\rho = h(z)|\zeta|^2, \qquad \zeta = (\zeta_1, \zeta_2, \cdots, \zeta_{m+1}),$$

where $h(z)$ is a local representation of h. If an inhomogeneous coordinate $z = (z_1, z_2, \cdots, z_n)$ on $\mathbb{P}$ is chosen, we have

$$h(z) = 1 + |z|^2.$$

Let $u(\rho) \in C^\infty(\mathbb{R})$ and consider Kähler metrics of the following type on $X_{1,1}$:

$$\omega = a\pi^*\omega_{FS} + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u(\rho). \tag{7.0.1}$$

ω is Kähler if and only if $a > 0$, $u' > 0$, $u'' > 0$, and the asymptotic behavior of u satisfies:

$$u_0(r) := u(\ln r) \qquad \text{and} \qquad u_\infty(r) := u(-\ln r) + b\ln r$$

with u_0 and u_∞ extendable by continuity to a smooth function at $r = 0$ with both $u_0'(0) > 0$ and $u_\infty'(0) > 0$. This fixes the following asymptotic behavior of u:

$$\lim_{\rho\to-\infty} u'(\rho) = 0, \qquad \lim_{\rho\to\infty} u'(\rho) = a,$$

and ensures that $\omega = a\pi^*\omega_{FS} + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u(\rho)$ extends to a Kähler form on $X_{1,1}$ and lies in the correct class.

Similarly, for any $v(\rho) \in C^\infty(\mathbb{R})$, if α is of the form 7.0.1 we require asymptotics of the same form but without any positivity assumptions since $[\alpha]$ need not be a Kähler class. As above, we define the functions $v_0, v_\infty : [0,\infty) \to \mathbb{R}$ via

$$v_0(r) := v(\log r) - q\log r \qquad \text{and} \qquad v_\infty(r) := v(-\log r) + p\log r,$$

and specify that v_0 and v_∞ extend by continuity to a smooth function at $r = 0$. As a result $v(\rho)$ satisfies:

$$\lim_{\rho\to-\infty} v'(\rho) = 0, \qquad \lim_{\rho\to\infty} v'(\rho) = p. \tag{7.0.2}$$

Furthermore,

$$[\omega] \in [D_H] + a[D_\infty]$$

$$[\alpha] \in q[D_H] + p[D_\infty].$$

Note that

$$\omega = (1 + u'(\rho))\omega_{FS} + \frac{\sqrt{-1}}{2\pi} he^{-\rho}(u'\delta_{\alpha\beta} + he^{-\rho}(u'' - u')\xi^{\bar{\alpha}}\xi^{\beta})\nabla\xi^{\alpha} \wedge \nabla\xi^{\bar{\beta}}$$
$$\alpha = (q + v'(\rho))\omega_{FS} + \frac{\sqrt{-1}}{2\pi} he^{-\rho}(v'\delta_{\alpha\beta} + he^{-\rho}(v'' - v')\xi^{\bar{\alpha}}\xi^{\beta})\nabla\xi^{\alpha} \wedge \nabla\xi^{\bar{\beta}},$$

where $\nabla\xi_i = d\xi_i + h^{-1}\partial h\xi_i$.

Because $u'' > 0$, the first derivative u' is monotone increasing, allowing us to view u' as a real variable, denoted by x, which ranges from 0 to a. We then write v' as a graph f over $x \in (0, a)$:

$$f(x) = f(u'(\rho)) = v'(\rho).$$

Taking the derivative of both sides, we see by the chain rule

$$f'(x)u''(\rho) = v''(\rho).$$

We now restrict the case where $m = n = 1$. Working in the coordinate x, the eigenvalues of $\omega^{-1}\alpha$ are

$$\frac{v'}{u'} = \frac{f}{x}, \frac{q+v'}{1+u'} = \frac{q+f}{1+x}, \qquad \text{and} \qquad \frac{v''}{u''} = f'.$$

Note that as $x \to 0$, then $\rho \to -\infty$, while $x \to a$ implies $\rho \to \infty$. Thus the asymptotics of $v(\rho)$ given by (7.0.2) are equivalent to

$$\lim_{x \to 0^+} f(x) = 0, \qquad \lim_{x \to a^-} f(a) = p,$$

and we extend $f(x)$ to the boundary $[0, a]$ by continuity.

Given this setup, the deformed Hermitian-Yang-Mills equation again reduces to an ODE. In particular, for a given function $u(\rho)$ satisfying the Calabi ansatz above (which defines our background Kähler form), we need to find a function $v(\rho)$ of a single real variable ρ.

We can now reformulate our problem into this setup. Using the explicit formulas for the eigenvalues of $\omega^{-1}\alpha$, we need to find a real function $f : [0, a] \to \mathbb{R}$ with boundary values $f(0) = 0$ and $f(a) = p$ satisfying the ODE

$$\text{(7.0.3)} \qquad \text{Im}\left(e^{-i\hat{\theta}}\left(1 + i\frac{q+f}{1+x}\right)\left(1 + i\frac{f}{x}\right)(1 + if')\right) = 0.$$

Since x is always positive, multiplying by $x(1+x)$ will not change the equation so we rewrite the ODE as

$$\mathrm{Im}\left(e^{-i\hat{\theta}}(1+x+i(q+f))(x+if)(1+if')\right)=0.$$

Observe that this ODE is exact

$$\begin{aligned}\mathrm{Im}\left(e^{-i\hat{\theta}}(1+x+i(q+f))(x+if)(1+if')\right) &= \mathrm{Im}\left(e^{-i\hat{\theta}}\frac{d}{dx}\left((1+iq)\frac{(x+if)^2}{2}+\frac{(x+if)^3}{3}\right)\right)\\ &= \frac{d}{dx}\mathrm{Im}\left(e^{-i\hat{\theta}}\left((1+iq)\frac{(x+if)^2}{2}+\frac{(x+if)^3}{3}\right)\right).\end{aligned}$$

We are looking for a function $f(x)$ so that the graph $(x, f(x))$ lies on a level curve of

$$\Phi(x,y) := \mathrm{Im}\left(e^{-i\hat{\theta}}\left((1+iq)\frac{(x+if)^2}{2}+\frac{(x+if)^3}{3}\right)\right)=0.$$

Figure 7.1 shows 4 different level sets for various choices of a, p, q.

We explicitly compute the charge of $X_{1,1}$ and its subvarieties as defined in (4.0.1). For $X_{1,1}$, we compute

$$\begin{aligned}\int_{X_{1,1}}\omega^3 &= \int_{X_{1,1}}(1+u'(\rho))h^2e^{-2\rho}u'u''f\left(\omega_{FS}\wedge\prod_{j=1}^{2}\frac{i}{2\pi}d\xi^j\wedge d\xi^{\bar{j}}\right)\\ &= \left(\frac{1}{\pi}\right)^2(2\pi)(\pi)(\pi)(2\pi)\frac{1}{2}\left[\frac{(u')^2}{2}+\frac{(u')^3}{3}\right]\Bigg|_{-\infty}^{\infty}\\ &= \left(\frac{1}{\pi}\right)^2(2\pi)(\pi)(\pi)(2\pi)\frac{1}{2}\left[\frac{a^2}{2}+\frac{a^3}{3}\right],\end{aligned}$$

$$\begin{aligned}\int_{X_{1,1}}\omega^2\wedge\alpha = \int_{X_{1,1}}&\left(\frac{\sqrt{-1}}{2\pi}\right)^2(he^{-\rho})^2[2(1+u'(\rho))(2u'v'+u'he^{-\rho}(v''-v')(|\xi^1|^2+|\xi^2|^2)\\ &+v'he^{-\rho}(u''-u')(|\xi^1|^2+|\xi^2|^2))+(q+v'(\rho))2((u')^2+u'he^{-\rho}(u''-u')(|\xi^1|^2+|\xi^2|^2))]\\ &\omega_{FS}\wedge\xi^1\wedge\xi^{\bar{1}}\wedge\xi^2\wedge\xi^{\bar{2}}\\ &= 2\left(\frac{1}{\pi}\right)^2(2\pi)(\pi)(\pi)(2\pi)\frac{1}{2}\left[(u'v')+q\left(\frac{(u')^2}{2}\right)+((u')^2v')\right]\Bigg|_{-\infty}^{\infty}\\ &= 2\left(\frac{1}{\pi}\right)^2(2\pi)(\pi)(\pi)(2\pi)\frac{1}{2}\left[(ap)+\left(\frac{qa^2}{2}\right)+(a^2p)\right].\end{aligned}$$

We can compute $\int_{X_{1,1}} \alpha^3$ and $\int_{X_{1,1}} \omega \wedge \alpha^2$ from these two and see that

$$\zeta_{X_{1,1}} = \int_X (\omega + i\alpha)^3 = \frac{(a+ip)^3}{3} + \frac{(a+ip)^2}{2}(1+iq).$$

We have similar computations for the subvarieties of $X_{1,1}$:

$$\int_{D_0} (\omega + i\alpha)^2 = \frac{(a+ip)^2}{2} + (1+iq)(a+ip) = a + \frac{a^2}{2} - \left(qp + \frac{p^2}{2}\right) + i(p + qa + ap),$$
$$\int_{D_H} (\omega + i\alpha)^2 = (a+ip)^2 = a^2 - p^2 + 2iap,$$
$$\int_{D_\infty} (\omega + i\alpha)^2 = \int_{D_0} (\omega + i\alpha)^2 + \int_{D_H} (\omega + i\alpha)^2 = \frac{3a^2}{2} + a - \left(qp + \frac{3p^2}{2}\right) + i(p + qa + 3ap),$$
$$\int_{D_0^2} (\omega + i\alpha) = 1 + iq,$$
$$\int_{D_0 \cap D_H} (\omega + i\alpha) = a + ip.$$

We do not need to compute $\int_{D_H^2} (\omega + i\alpha)$ since D_H does not self intersect. We then compute the following stability inequalities defined in (4.0.1):

$$\mathrm{Im}\left(\frac{Z(V)}{Z(X_{1,1})}\right) = \mathrm{Im}\left(\frac{-\int_V (-i)^k (\omega + i\alpha)^k}{-(-i)^3 \zeta_{X_{1,1}}}\right) > 0$$

$$\mathrm{Im}\left((i+m)\int_V (-i)^k (\omega + i\alpha)^k\right) > 0, \text{where } \mathrm{m} := \frac{\mathrm{ap} + \frac{\mathrm{qa}^2}{2} + \mathrm{a}^2\mathrm{p} - \frac{\mathrm{qp}^2}{2} - \frac{\mathrm{p}^3}{3}}{\frac{\mathrm{a}^3}{3} + \frac{\mathrm{a}^2}{2} - \mathrm{qap} - \frac{\mathrm{p}^2}{2} - \mathrm{ap}^2}$$

D_H gives the following stability inequality:

$$\mathrm{Im}\left((i+m)\int_{D_H} (-i)^2 (\omega + i\alpha)^2\right) = \mathrm{Im}\left((i+m)(-1)(a+ip)^2\right) > 0$$
$$a^2 - p^2 + 2map < 0$$

D_0 gives the following stability inequality:

$$\mathrm{Im}\left((i+m)\int_{D_0} (-i)^2 (\omega + i\alpha)^2\right) = \mathrm{Im}\left((i+m)(-1)\left(\frac{(a+ip)^2}{2} + (1+iq)(a+ip)\right)\right) > 0$$
$$a + \frac{a^2}{2} - \left(qp + \frac{p^2}{2}\right) + m(p + qa + ap) < 0$$

D_∞ gives the following stability inequality:

$$\mathrm{Im}\left((i+m)\int_{D_\infty}(-i)^2(\omega+i\alpha)^2\right) = \mathrm{Im}\left((i+m)(-1)\left(\frac{3(a+ip)^2}{2}+(1+iq)(a+ip)\right)\right) > 0$$

$$a+\frac{3a^2}{2}-\left(qp+\frac{3p^2}{2}\right)+m(p+qa+3ap) < 0$$

D_0^2 gives the following stability inequality:

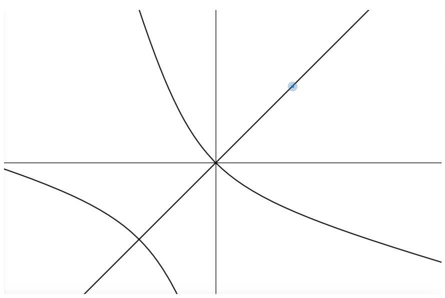
$$\mathrm{Im}\left((i+m)\int_{D_\infty}(-i)(\omega+i\alpha)\right) = \mathrm{Im}((i+m)(-i)(1+iq)) > 0$$

$$m-q < 0$$

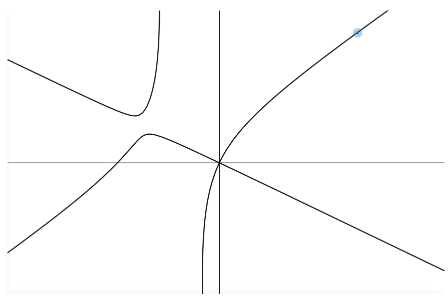
$D_0 \cap D_H$ gives the following stability inequality:

$$\mathrm{Im}\left((i+m)\int_{D_0\cap D_H}(-i)(\omega+i\alpha)\right) = \mathrm{Im}((i+m)(-i)(a+ip)) > 0$$

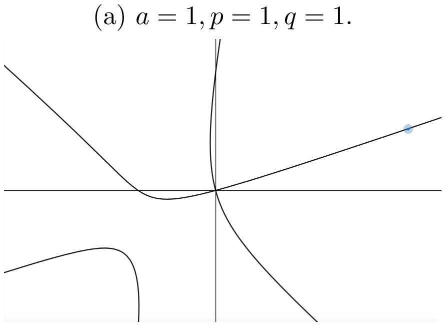
$$ma-p < 0$$



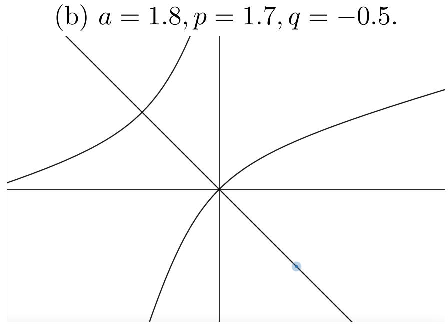
(a) $a=1, p=1, q=1$.



(b) $a=1.8, p=1.7, q=-0.5$.



(c) $a=2.5, p=0.8, q=0.5$.



(d) $a=1, p=-1, q=-1$.

FIGURE 7.1. Graphs of the level set $\Phi(x,y)=0$ for various a,p,q. The dot is the coordinate (a,p).

We make some observations about the level sets and stability regions. First, we note two properties of the D_H stability region.

LEMMA 7.0.1. *The D_H stability region contains the y-axis.*

PROOF. We define d_m and n_m to be the denominator and numerator of m, respectively.

$$d_m := \frac{a^3}{3} + \frac{a^2}{2} - qap - \frac{p^2}{2} - ap^2, \qquad n_m := ap + \frac{qa^2}{2} + a^2p - \frac{qp^2}{2} - \frac{p^3}{3}.$$

We rewrite the D_H stability inequality:

$$x^2 + 2mxy - y^2 = x^2 + 2\frac{n_m}{d_m}xy - y^2 < 0$$
$$d_m x^2 + 2n_m xy - d_m y^2 > 0.$$

Note that d_m is less than zero since $\theta \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$. For $x = 0$,

$$-d_m y^2 > 0$$

for all y. $\square$

LEMMA 7.0.2. *D_H stability region is bounded by two perpendicular lines with slopes $\frac{-1}{\pm\sqrt{m^2+1}+m}$.*

PROOF. We factor the D_H stability region equation:

$$\begin{aligned} x^2 + 2mxy - y^2 &= (x + my)^2 - (m^2 + 1)y^2 \\ &= ((x + my) + \sqrt{m^2 + 1}y)((x + my) - \sqrt{m^2 + 1}y) \\ &= (x + (\sqrt{m^2 + 1} + m)y)(x + (-\sqrt{m^2 + 1} + m)y). \end{aligned}$$

The boundary of the D_H stability region is given by $x^2+2mxy-y^2 = 0$, so we have $y = \frac{-1}{\pm\sqrt{m^2+1}+m}x$. This is the same stability region from the case on the blowup of $\mathbb{P}^n$. $\square$

We next note some properties of the D_0 stability region.

LEMMA 7.0.3. *The boundary of the D_0 stability region has slant asymptotes with slopes $\frac{-1}{\pm\sqrt{m^2+1}+m}$.*

PROOF. The boundary of the D_0 stability region is:

$$\frac{1}{2}(x^2 - y^2) + x - qy + m(xy + qx + y) = 0$$

If we define the slopes as $k := \lim_{x\to\infty} \frac{y}{x}$,

$$\lim_{x\to\infty} \frac{1}{x^2}(x^2 - y^2 + 2x - 2qy + 2mxy + 2mqx + 2my) = 1 - k^2 + 2mk = 0$$

Solutions for $k^2 - 2mk - 1 = 0$ are $\frac{-1}{\pm\sqrt{m^2+1}+m}$. $\square$

It is then easy to check that the slopes of the slant asymptotes of the boundary of the D_0 stability region are perpendicular to each other.

LEMMA 7.0.4. *The D_0 stability region contains the positive y-axis.*

PROOF. First, the y-intercepts of the boundary of the D_0 stability region are ≤ 0. We set the D_0 stability equation to zero:

$$\frac{1}{2}(x^2 - y^2) + x - qy + m(xy + qx + y) = 0$$

Setting $x = 0$ gives

$$-y^2 - 2qy + 2my = 0.$$

Solutions for $-y^2 - 2qy + 2my = -y(y + 2(q - m)) = 0$ are $y = 0$, $2(m - q)$. $m - q < 0$ is the D_0^2 stability equation, so the y-intercepts of the boundary are ≤ 0.

Next, look at the D_0 stability equation and let $x = 0$:

$$\frac{1}{2}(x^2 - y^2) + x - qy + m(xy + qx + y) = -\frac{1}{2}y^2 - qy + my < 0.$$

A sufficiently large choice of y satisfies this inequality. Since the boundary of the D_0 stability region only crosses the y-axis for non-positive values means that the D_0 stability region contains the positive y-axis. $\square$

We now note some properties of the level set $\Phi(x, y) = 0$.

LEMMA 7.0.5. *The slant asymptotes of the level set $\Phi = 0$ have slopes that are $\frac{\pi}{3}$ rotations of each other.*

PROOF.

$$\Phi(x,y) = \frac{1}{3}(3x^2y - y^3) - \frac{m}{3}(x^3 - 3xy^2) + \frac{1}{2}(2xy + q(x^2 - y^2)) - \frac{m}{2}(x^2 - y^2 - 2qxy) = 0$$

If we define the slopes as $k := \lim_{x\to\infty} \frac{y}{x}$,

$$\frac{1}{3}(3k - k^3) - \frac{m}{3}(1 - 3k^2) = 0$$
$$-k^3 + 3mk^2 + 3k - m = 0.$$

This is the same equation as the slope asymptotes for the blowup of $\mathbb{P}^3$. □

In the $X_{1,1}$ case, the slant asymptotes do not pass through the origin in general. An easy example where this happens is when the level set is a straight line passing though the foci, vertices, and center of a hyperbola, shown in (7.0.8).

LEMMA 7.0.6. *The y-intercepts of the level set $\Phi = 0$ are ≤ 0.*

PROOF.

$$\Phi(x,y) = \frac{1}{3}(3x^2y - y^3) - \frac{m}{3}(x^3 - 3xy^2) + \frac{1}{2}(2xy + q(x^2 - y^2)) - \frac{m}{2}(x^2 - y^2 - 2qxy) = 0$$

For $x = 0$,

$$\frac{1}{2}(-y^3) + \frac{1}{2}(-qy^2) - \frac{m}{2}(-y^2) = -y^2(\frac{1}{3}y + \frac{1}{2}q - \frac{1}{2}m) = 0.$$

Then

$$y = 0, \frac{3}{2}(m - q).$$

$m - q < 0$ is the D_0^2 stability equation, so the y-intercepts are ≤ 0. □

If we parameterize $\Phi(x,y)$ by converting to polar coordinates, we can derive a few more properties of the level set at 0 .

$$\begin{aligned}\Phi(x,y) &= \frac{1}{3}(3x^2y - y^3) - \frac{m}{3}(x^3 - 3xy^2) + \frac{1}{2}(2xy + q(x^2 - y^2)) - \frac{m}{2}(x^2 - y^2 - 2qxy) \\ &= \frac{1}{3}r^3(\sin 3\theta - m\cos 3\theta) + \frac{1}{2}r^2((1 + mq)\sin 2\theta + (q - m)\cos 2\theta).\end{aligned}$$

For the level set at 0, we get

$$r(\theta) = \frac{3}{2}\frac{((m-q)\cos 2\theta - (1+mq)\sin 2\theta)}{(\sin 3\theta - m\cos 3\theta)}.$$

$r(\theta)$ has period 2π and $r(\theta) = 0$ when $\theta = \frac{1}{2}\left(n\pi + \arctan\frac{m-q}{1+mq}\right)$. Define

$$P_n := \frac{1}{2}\left(n\pi + \arctan\frac{m-q}{1+mq}\right).$$

LEMMA 7.0.7. *The level set $\Phi = 0$ intersects itself at the origin and the slopes at the intersection are perpendicular.*

PROOF. Using the formula

$$\frac{dy}{dx} = \frac{r'(\theta)\sin\theta + r(\theta)\cos\theta}{r'(\theta)\cos\theta - r(\theta)\sin\theta},$$

we compute the slope when $\Phi = 0$ crosses the origin.

$$\begin{aligned}
\frac{dy}{dx}(0,0) &= \frac{r'(P_0)\sin(P_0) + r(P_0)\cos(P_0)}{r'(P_0)\cos(P_0) - r(P_0)\sin(P_0)}\\
&= \frac{\sin\left(\frac{1}{2}\arctan\frac{m-q}{1+mq}\right)}{\cos\left(\frac{1}{2}\arctan\frac{m-q}{1+mq}\right)}\\
\frac{dy}{dx}(0,0) &= \frac{r'(P_1)\sin(P_1) + r(P_1)\cos(P_1)}{r'(P_1)\cos(P_1) - r(P_1)\sin(P_1)}\\
&= \frac{\sin(P_1)}{\cos(P_1)}\\
&= \frac{\sin\left(\frac{1}{2}\left(\pi + \arctan\frac{m-q}{1+mq}\right)\right)}{\cos\left(\frac{1}{2}\left(\pi + \arctan\frac{m-q}{1+mq}\right)\right)}\\
&= \frac{\cos\left(\frac{1}{2}\arctan\frac{m-q}{1+mq}\right)}{-\sin\left(\frac{1}{2}\arctan\frac{m-q}{1+mq}\right)},
\end{aligned}$$

so we have that the two slopes are perpendicular at the origin. □

There is also the special case when $\Phi = 0$ is a hyperbola and a line passing through the hyperbola's foci, vertices, and center. We have found two choices of a, p, q of when this occurs, but there may be more.

LEMMA 7.0.8. *The level set $\Phi = 0$ is a hyperbola and a line passing though the hyperbola's foci, vertices, and center when $q = \frac{p}{a}$ or $m = \frac{3q-q^3}{1-3q^2}$.*

PROOF. We begin with the case when $q = \frac{p}{a}$.

$$\begin{aligned}
\Phi(x,y) &= \frac{1}{3}\left(3x^2y - y^3\right) - \frac{m}{3}\left(x^3 - 3xy^2\right) + \frac{1}{2}\left(2xy + q(x^2 - y^2)\right) - \frac{m}{2}\left(x^2 - y^2 - 2qxy\right) = 0 \\
&= \frac{1}{6}\frac{2a+3}{6a}\left[a\left(a^2 - 3p^2\right)\left(6x^2y - 2y^3 + 6xy + 3\frac{p}{a}\left(x^2 - y^2\right)\right)\right. \\
&\left. -p\left(3a^2 - p^2\right)\left(2x^3 - 6xy^2 + 3\left(x^2 - y^2\right) - 6\frac{p}{a}xy\right)\right] \\
&= \frac{1}{6}\frac{2a+3}{6a}\left[2\left((3a^3 - ap^2)x^2 + 8a^2pxy + (3a^3 + 3ap^2)x + (3ap^2 - a^3)y^2\right.\right. \\
&\left.\left. +(3a^2p + 3p^3)y\right)\right]\left(y - \frac{p}{a}x\right).
\end{aligned}$$

The line equation is

$$y - \frac{p}{a}x = 0,$$

and the hyperbola equation is

$$(3a^3 - ap^2)x^2 + 8a^2pxy + (3a^3 + 3ap^2)x + (3ap^2 - a^3)y^2 + (3a^2p + 3p^3)y = 0,$$

or

$$\left(\sqrt{\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{p}{2a}\right)^2} - \sqrt{\left(x + \frac{3}{2}\right)^2 + \left(y + \frac{3p}{2a}\right)^2}\right)^2 - \frac{\left(a^2 + p^2\right)}{a^2} = 0.$$

The line intersects the hyperbola at the origin and $(-1, -\frac{p}{a})$.

We look at the other case when $m = \frac{3q-q^3}{1-3q^2}$. Then

$$\begin{aligned}
\Phi(x,y) &= \frac{1}{6}\left[2d\left(3x^2y - y^3\right) + 3d\left(2xy + q\left(x^2 - y^2\right)\right) - 2n\left(x^3 - 3xy^2\right) - 3n\left(x^2 - y^2 - 2qxy\right)\right] = 0 \\
&= \frac{1}{6}\left[(1 - 3q^2)\left(6x^2y - 2y^3 + 6xy + 3q(x^2 - y^2)\right)\right. \\
&\left. - \left(3q - q^3\right)\left(2x^3 - 6xy^2 + 3(x^2 - y^2) - 6qxy\right)\right] \\
&= \frac{1}{6}\left[-2(q^2 - 3)x^2 + 16qxy + (6q^2 + 6)x + (6q^2 - 2)y^2 + 6q(q^2 + 1)y\right](y - qx).
\end{aligned}$$

The line equation is

$$y - qx = 0,$$

and the hyperbola equation is

$$-2(q^2-3)x^2+16qxy+(6q^2+6)x+(6q^2-2)y^2+6q(q^2+1)y=0,$$

or

$$\left(\sqrt{\left(x-\frac{1}{2}\right)^2+\left(y-\frac{q}{2}\right)^2}-\sqrt{\left(x+\frac{3}{2}\right)^2+\left(y+\frac{3q}{2}\right)^2}\right)^2-(q^2+1)=0.$$

The line intersects the hyperbola at the origin and $(-1,-q)$. □

From experimenting with different values of a, p, and q, there are a few more observations to make which have yet to be proved. Except for the special case where $\Phi=0$ is a hyperbola and a line, $\Phi=0$ has three components: one component that does not pass through the origin and approaches two slant asymptotes that are $\frac{\pi}{3}$ rotations of each other, and two components that each pass through the origin and approach two slant asymptotes that are $\frac{2\pi}{3}$ rotations of each other. Using polar coordinates, we can find the two components that pass though the origin. If the component passes though the origin, we will call the parameterization that starts at the origin and approaches the asymptote with the greatest and smallest slope in the positive x direction the highest and the lowest branch, respectively, and the parameterization that passes though the origin and approaches the middle asymptote in the positive x direction the middle branch. For now we will ignore the case when one of the three asymptotes is vertical.

The following conjectures have been observed but not yet proved.

CONJECTURE 7.0.9. *Branches do not cross the D_H stability region boundaries.*

CONJECTURE 7.0.10. *The middle branch and the D_H stability region intersect only at the origin.*

These two conjectures are hinted at by (4.0.1), since one of the asymptotes of $\Phi=0$ is sandwiched between the boundary and outside the D_0 stability region for large positive x values.

CONJECTURE 7.0.11. *The critical points of $\Phi=0$ have negative x coordinate values. More specifically, critical points occur in the strip $-\frac{3}{2}<x\leq 0$.*

If these three conjectures are shown, the middle branch is obviously ruled out as a solution. If it exists, the lowest branch is also not a solution by (7.0.6) and (7.0.7) since it has a critical point. It then follows that

Conjecture 7.0.12. *The highest branch connecting the origin to* (a, p) *is a solution to deformed Hermitian-Yang-Mills equation assuming stability of the pair* $[\omega]$, $[\alpha]$.

We hope to prove these conjectures in future work.

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