Singularity Formation along the Line Bundle Mean Curvature Flow

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Abstract

The line bundle mean curvature flow is a complex analogue of the mean curvature flow for Lagrangian graphs, with fixed points solving the deformed Hermitian-Yang-Mills equation. In this paper we construct two distinct examples of singularities along the flow. First, we find a finite time singularity, ruling out long time existence of the flow in general. Next we show long time existence of the flow with a Calabi symmetry assumption on the blowup of P^n, if one assumes supercritical phase. Using this, we find an example where a singularity occurs at infinite time along the destabilizing subvariety in the semi-stable case.

Chapter 1

Introduction

This dissertation works towards constructing singularities of the line bundle mean curvature flow, which is a geometric flow analogue to a complex mean curvature flow of Lagrangian graphs. It arises as a tool to construct solutions of the deformed Hermitian-Yang-Mills (dHYM) equation, which comes from string theory.

String theory predicts that a Calab-Yau manifold (M, ω, Ω) comes with a mirror $(\hat{M}, \hat{\omega}, \hat{\Omega})$. Kontsevich [17] proposed that the mirror symmetry can be explained homologically, as an equivalence of the triangular categories:

$$D^b Coh(M) \sim D^{Fuk}(\tilde{M}),$$

where on the left is the derived category of coherent sheaves, and on the right is the derived Fukaya category. Later, Strominger-Yau-Zaslow [24] proposed that the mirror symmetry can be explained geometrically. Loosely speaking, M should be obtained by $\tilde{M}$ via T-duality, and the symplectic geometry of $\tilde{M}$ should be interchangeable with the complex geometry of M by fiberwise Fourier-Mukai-type transformations.

In the case when the pair M and $\hat{M}$ have semi-flat dual torus fibration, Leung-Yau-Zaslow gives an explicit correspondence [20]: the supersymmetric cycles in $\hat{M}$ are special Lagrangian submanifolds, which map via the fiberwise Fourier–Mukai transform to a holomorphic submanifold Z of M, with a holomorphic line bundle (L, ∇_A) that satisfies the dHYM equation:

$$\operatorname{Im}\left(e^{-i\hat{\theta}}(\omega - F_A)^{\dim Z}\right) = 0, \quad \hat{\theta} \in \mathbb{R}, \tag{1.1}$$

where $F_A \in H^{1,1}(Z)$ is the curvature form with respect to the connection ∇_A on L.

We can generalise (1.1) to any compact Kähler manifold X, as opposed to Calabi-Yau manifold. Furthermore, we do not have to restrict to the case of a line bundle. Instead, let $[\alpha_0]$ be a real $(1,1)$ Dolbeault cohomology class. The dHYM equation seeks a representative $\alpha \in [\alpha_0]$ satisfying

$$\operatorname{Im}\left(e^{-i\hat{\theta}}(\omega + \sqrt{-1}\alpha)^n\right) = 0 \tag{1.2}$$

for a fixed constant $e^{i\hat{\theta}} \in S^1$, where n is the complex dimension of X.

The Line Bundle Mean Curvature Flow

Extensive work has been done to develop the relationship between the existence of solutions to (1.2) and notions of geometric stability [2, 3, 4, 5, 8, 15, 23]. One direction is to utilize a parabolic method motivated by the mean curvature flow. A submanifold will reduce its volume along the mean curvature flow and become minimal if it converges. The analogue here is to allow a metric to flow along a line bundle version of mean curvature. This flow will reduce a volume functional, and the hope is that it will reach a solution upon convergence. Under certain geometric assumption, this has been verified [16], but much work remains to be done in general.

To write down the flow precisely, let ϕ_t be a smooth function on X and $\alpha_t = \alpha_0 + \sqrt{-1}\partial\bar{\partial}\phi_t$ be a closed 2-form representing $[\alpha_0]$. The line bundle mean curvature flow is defined by

$$\dot{\phi}_t = \sum_k \arctan(\lambda_k) - \hat{\theta} \tag{1.3}$$

where λ_k's are the eigenvalues of $\omega^{-1}\alpha_t$. When the flow converges to a stationary solution, by choosing a coordinates at a point so that $\omega^{-1}\alpha$ is diagonal, we have

$$\hat{\theta} = \sum_k \arctan(\lambda_k) = \arg \prod_k \left(1 + \sqrt{-1}\lambda_k\right) = \arg\det(1 + \sqrt{-1}\omega^{-1}\alpha)^n \tag{1.4}$$

which is a reformulation of (1.2). Hence, convergence of the line bundle mean curvature flow results in a solution of the dHYM equation.

Here, we can already see a particularly challenging aspect of the equation. In (1.2), the constant

$e^{i\hat{\theta}}$ is a priori only S^1-valued, but (1.3) requires a lift of $\hat{\theta} \in \mathbb{R}$. It is in general not known how to lift the angle $\hat{\theta}$ (when dimension is higher than 4), which plays an important role in studying (1.2). Hence, many results are based on the large angle assumption

$$\sum_k \arctan(\lambda_k) \in \Big((n-2)\frac{\pi}{2}, n\frac{\pi}{2}\Big)$$

which lifts $\hat{\theta}$ to the same interval. This assumption also has the advantage of helping with the PDE theory.

We now introduce the conjectured relationship between solutions to the dHYM equation and stability. Following the work of Lejmi-Székelyhidi on the J-equation [19], Collins-Jacob-Yau integrated a certain positivity condition along subvarieties to develop a necessary class condition for existence, and conjectured it was a sufficient condition as well [3]. Specifically, for any irreducible analytic subvariety $V \subseteq X$, define the complex number:

$$Z_{[\alpha][\omega]}(V) := -\int_V e^{-\sqrt{-1}\omega+\alpha},$$

where by convention we only integrate the term in the expansion of order $\dim(V)$. Under the supercritical phase assumption, $Z_{[\alpha][\omega]}(X)$ lies in the upper half plane $\mathbb{H}$. The conjecture of Collins-J.-Yau posits that a solution to the dHYM equation exists if and only if

$$\pi > \arg Z_{[\alpha][\omega]}(V) > \arg Z_{[\alpha][\omega]}(X). \tag{1.5}$$

Later, when $n = 3$, Collins-Xie-Yau demonstrated a necessary Chern number inequality [6] (which has since been extended to $n = 4$ [12]), which is also useful for defining the lifted angle $\hat{\theta}$ algebraically. Collins-Yau further conjectured that such a Chern number inequality in higher dimension was needed [7]. Indeed, recently when $n = 3$, an example was found where the stability inequality (1.5) holds, but the Chern number inequality does not, and no solution to the dHYM equation exists [37]. We note that slightly weaker versions of the Collins-Jacob-Yau conjecture have been solved by Chen [2] (assuming uniform stability), and Chu-Lee-Takahashi [8] (in the projective case). Again all of the previous results rest on the large angle assumption.

A few results without the large angle assumption are presented in [15] and later in [14]. Both authors worked on projectivised vector bundles over projective spaces (e.g. the blowup of P^n is a special case of such). These spaces can be equipped with a symmetry called Calabi symmetry, allowing us to reduce (1.2) to an ODE. In this setting, the authors was able to prove a stability condition which tells if a solution to dHYM equation exists, and if the lifted angle $\hat{\theta}$ is defined. More precisely, they show that a solution exists in the blowup of P^n if and only if

$$\operatorname{Im}\left(\frac{\int_V e^{-i\omega+\alpha}}{\int_X e^{-i\omega+\alpha}}\right) \neq 0 \tag{1.6}$$

for all analytic subvarieties $V \subset X$, and gives the same sign for all subvarieties of the same dimension.

Singularities

Given an initial class α_0, we ask if the line bundle mean curvature flow (1.3) converges. By (1.4), it is evident that existence of a solution of the dHYM equation is a necessary condition for convergence. In this dissertation, we show that it is, however, not sufficient, and we give an example where a finite time singularity occurs. We also give another example with a long time singularity in the unstable case, i.e. when a solution of dHYM equation is known to be obstructed.

We construct our example on the blowup of P^n with the same Calabi Symmetry as above, since the stability condition is already well-understood. The symmetry allows us to reduce (1.3) to a parabolic PDE with one spacial variable, and explicitly describe the singularity that appears in our setting. We show the following theorem.

Theorem 1.0.1. *Let X be the blowup of $\mathbb{P}^n$ at a point. There exists a Kähler form ω, and cohomology class $[\alpha] \in H^{1,1}(X, \mathbb{R})$ admitting a representative α_0, for which the flow 1.3 achieves a finite-time singularity. Specifically, if $\lambda_{Max}(p,t)$ denotes the largest eigenvalue of $\omega^{-1}\alpha_t$ at a point $p \in X$, then there exists a sequence of points $\{x_k\} \subset X$ and times $t_k \to T < \infty$ such that*

$$\lim_{k\to\infty} \lambda_{Max}(x_k, t_k) = \infty.$$

The idea is to observe that the flow (1.3) can be reduced to a flow of graphs of functions in $\mathbb{R}^2$, due to the Calabi Symmetry. It also behaves similarly to the curve shortening flow. Hence, we can apply avoidance principle which promises the disjointness of subsolutions. Subsolutions are curves that evolve slower than the flow. We explicitly construct two subsolutions so that they force the graph to develop a first-order singularity, i.e. a vertical tangency, in finite time.

For infinite time, we need to impose the large angle assumption to help with the PDE theory, in which we can prove long time existence. From there, we are able to construct a long time singularity.

Theorem 1.0.2. *Let (X,ω) be the blowup of $\mathbb{P}^n$ at a point, $n \geq 3$, and consider a class $[\alpha] \in H^{1,1}(X,\mathbb{R})$. Assume ω, $\alpha_0 \in [\alpha]$ have Calabi-symmetry, and furthermore assume α_0 has supercritical phase, that is $\Theta(\alpha_0) > (n-2)\frac{\pi}{2}$. Then the flow (1.3) beginning at α_0 exists for all time.*

Theorem 1.0.3. *Let (X,ω) be the blowup of $\mathbb{P}^n$ at a point, $n \geq 3$. There exists classes $[\alpha]$ and $[\omega]$, which are semi-stable in the sense of (1.5), where the flow (1.3) exists for all time and becomes singular at time $t = \infty$ along the destabilizing subvariety.*

It is perhaps not surprising that a singularity can appear even when the solution to dHYM equation exists. The line bundle mean curvature flow is motivated by the Lagrangian mean curvature flow (LMCF), and Neves demonstrated that singularity formations are abundant along the flow [22]. The singularities he constructs, however, is of second-order, i.e. a curvature blow-up. It will be interesting if we could find graphical Lagrangians that develop first-order singularity similar to our examples.

The dissertation is organized as follow. Chapter 2 covers some background of complex geometry. In Chapter 3, we define the blowup of P^n and explain the Calabi symmetry. In Chapter 4, we introduce some early results of the line bundle mean curvature flow (1.3), and rewrite it under the setting of Calabi symmetry. We then prove the main theorems in Chapter 5, and we conclude with some future directions at the end of that chapter.

Chapter 2

Complex Geometry Background

In this chapter, we provide some background information for complex geometry which will be useful in understanding the deformed Hermitian Yang–Mills equation. The materials mainly follows from [11].

2.1 Complexified Tangent Bundle and Differential Forms

Let M be a compcat $2n$-dimensional real manifold. It is a complex manifold if there exists an atlas in which the transition maps are holomorphic. In local coordinates, the tangent space at a point $p \in M$ is given by

$$T_pM = \operatorname{span}\left\{ \frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial y^1} \cdots, \frac{\partial}{\partial y^n} \right\}.$$

On each coordinates patch, we define

$$J : \frac{\partial}{\partial x^i} \mapsto \frac{\partial}{\partial y^i}; \quad \frac{\partial}{\partial y^i} \mapsto -\frac{\partial}{\partial x^i}.$$

Since the transition maps satisfy Cauchy–Riemann equations, J is invariant under change of coordinates, and defines a complex structure $J : TM \to TM$ globally. Define the complexified tangent space $TM^{\mathbb{C}} := TM \otimes \mathbb{C}$ and J extends naturally to $TM^{\mathbb{C}}$. Notice that $J^2 = -I$, so $\sqrt{-1}$ and $-\sqrt{-1}$ are the eigenvalues of J. Denotes $T^{1,0}M$ and $T^{0,1}M$ to be their eigenspaces respectively, so that $TM^{\mathbb{C}} = T^{1,0}M \oplus T^{0,1}M$. To be precise, let

$$\frac{\partial}{\partial z^i} := \frac{1}{2}\left(\frac{\partial}{\partial x^i} - \sqrt{-1} J \frac{\partial}{\partial x^i} \right) = \frac{1}{2}\left(\frac{\partial}{\partial x^i} - \sqrt{-1} \frac{\partial}{\partial y^i} \right),$$

$$\frac{\partial}{\partial \bar{z}^i} := \frac{1}{2}\left(\frac{\partial}{\partial x^i} + \sqrt{-1}J\frac{\partial}{\partial x^i}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x^i} + \sqrt{-1}\frac{\partial}{\partial y^i}\right).$$

Then,

$$T^{1,0}M := \left\{X \in TM^{\mathbb{C}} : JX = \sqrt{-1}X\right\} = \operatorname{span}\left\{\frac{\partial}{\partial z^i}\right\}_{i=1}^{n},$$
$$T^{0,1}M := \left\{X \in TM^{\mathbb{C}} : JX = -\sqrt{-1}X\right\} = \operatorname{span}\left\{\frac{\partial}{\partial \bar{z}^i}\right\}_{i=1}^{n}.$$

To define differential forms, we need to dualise the above construction. Write $\Omega^1(M,\mathbb{C}) := T^*M \otimes \mathbb{C}$, as opposed to $\Omega^1(M,\mathbb{R}) := T^*M$. Let $\Omega^{1,0}M$ and $\Omega^{0,1}M$ be the duals of $T^{1,0}M$ and $T^{0,1}M$ respectively. Similarly, let,

$$dz^i := dx^i + \sqrt{-1}dy^i,$$
$$d\bar{z}^i := dx^i - \sqrt{-1}dy^i.$$

Then,

$$\Omega^{1,0}M := \left\{\alpha \in \Omega^1(M,\mathbb{C}) : J\alpha = \sqrt{-1}\alpha\right\} = \operatorname{span}\left\{dz^i\right\}_{i=1}^{n},$$
$$\Omega^{0,1}M := \left\{\alpha \in \Omega^1(M,\mathbb{C}) : J^*\alpha = -\sqrt{-1}\alpha\right\} = \operatorname{span}\left\{d\bar{z}^i\right\}_{i=1}^{n}.$$

The higher order complex differential forms are defined the same way as with real forms. Define

$$\Omega^{p,q}M := \underbrace{\Omega^{1,0}M \wedge \Omega^{1,0}M}_{p \text{ times}} \wedge \underbrace{\Omega^{0,1}M \wedge \Omega^{1,0}M}_{q \text{ times}}.$$

Write $\Omega^{p,q} = \Omega^{p,q}M$ when there is no confusion. This gives a decomposition

$$\Omega^r(M,\mathbb{C}) = \bigoplus_{p+q=r} \Omega^{p,q}.$$

This also decomposes the exterior derivative:

$$\partial = \mathrm{proj}_{\Omega^{p+1,q}} \circ d : \Omega^{p,q} \to \Omega^{p+1,q},$$

$$\bar{\partial} = \mathrm{proj}_{\Omega^{p,q+1}} \circ d : \Omega^{p,q} \to \Omega^{p,q+1}.$$

They satisfy $\partial^2 = 0$ and $\bar{\partial}^2 = 0$, which allows us to define the Dolbeault cohomology

$$H^{p,q}(M, \mathbb{C}) := \ker\left(\bar{\partial} : \Omega^{p,q} \to \Omega^{p,q+1}\right) / \mathrm{im}\left(\bar{\partial} : \Omega^{p,q-1} \to \Omega^{p,q}\right).$$

2.2 Riemmanian Metric and Fundamental Form

Let $g : TM \otimes TM \to \mathbb{R}$ be a (real) Riemannian metric on M such that it is compatible with J, i.e. $g(-,-) = g(J-, J-)$. Let

$$\mathcal{A} = \left\{ \frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial y^1} \cdots, \frac{\partial}{\partial y^n} \right\}$$

be a basis of TM. Then, as a matrix in a coordinates system,

$$[g]_{\mathcal{A}} = \begin{pmatrix} g_{jk} & 0 \\ 0 & g_{jk} \end{pmatrix}_{1 \le j,k \le n}.$$

Extend g to a bilinear form $g : TM^{\mathbb{C}} \otimes TM^{\mathbb{C}} \to \mathbb{C}$. If we use a different basis,

$$\mathcal{B} = \left\{ \frac{\partial}{\partial z^1}, \cdots, \frac{\partial}{\partial z^n}, \frac{\partial}{\partial \bar{z}^1} \cdots, \frac{\partial}{\partial \bar{z}^n} \right\}.$$

Then,

$$[g]_{\mathcal{B}} = \frac{1}{2} \begin{pmatrix} 0 & g_{jk} \\ g_{jk} & 0 \end{pmatrix}_{1 \le j,k \le n}.$$

Definition 2.2.1. The fundamental form $\omega \in \Omega^2(M, \mathbb{R})$ associated to the metric g is defined as

$$\omega(-,-) = g(J-,-).$$

This real form can be extended to $\Omega^2(M, \mathbb{C})$. In coordinates, we have

$$[\omega]_{\mathcal{A}} = \begin{pmatrix} 0 & g_{jk} \\ -g_{jk} & 0 \end{pmatrix}_{1 \leq j,k \leq n},$$

or

$$[\omega]_{\mathcal{B}} = \frac{\sqrt{-1}}{2} \begin{pmatrix} 0 & g_{jk} \\ -g_{jk} & 0 \end{pmatrix}_{1 \leq j,k \leq n},$$

For notation purpose in Einstein summation convention, write $g_{\bar{k}j} = g_{jk}$. Then,

$$\omega = \frac{\sqrt{-1}}{2} g_{\bar{k}j} dz^j \wedge d\bar{z}^k \in \Omega^{1,1} M \cap \Omega^2(M, \mathbb{R}).$$

Definition 2.2.2. M is a *Kähler manifold* if the fundamental form ω is closed. The metric g is called the *Kähler metric* and ω is called the *Kähler form*.

2.3 Analytic Hypersurface and Holomorphic Line Bundles

A holomorphic line bundle over a complex manifold M is a locally trivializing $\mathbb{C}^1$-bundle with holomorphic transition maps. To be more precise, let $\mathcal{U} = \{U_\alpha\}_{\alpha \in \Lambda}$ be a collection of charts that defines M. Let $t_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^*$ be holomorphic transition maps such that

- $t_{\alpha\beta} = 1/t_{\beta\alpha}$,

- $t_{\alpha\beta} t_{\beta\gamma} t_{\gamma\alpha} = 1$.

Then, the information $(\mathcal{U}, \{t_{\alpha\beta}\}_{\alpha,\beta \in \Lambda})$ forms a line bundle

$$L := \bigsqcup_{\alpha \in \Lambda} U_\alpha \times \mathbb{C} \Big/ (z_\alpha, v_\alpha) \sim (z_\beta, v_\beta) \text{ iff } z_\alpha = z_\beta \in U_\alpha \cap U_\beta \text{ and } v_\alpha = t_{\alpha\beta}(z_\alpha) v_\beta.$$

On a line bundle L, we can equip a Hermitian metric h, so that $(v, w) \mapsto \langle v, w \rangle_h := v\bar{w}h(z)$ is a Hermitian inner product for all $(z, v), (z, w) \in L$. In a local chart, $h = h_\alpha : U_\alpha \to (0, \infty)$ is a positive function, satisfying

$$h_\beta(z) = |t_{\alpha\beta}(z)|^2 h_\alpha(z). \tag{2.1}$$

A holomorphic section s is a map $M \to L$ given by $z \mapsto (z, s(z))$ such that when restricted to any chart U_α, the map $s_\alpha := s|_{U_\alpha} : U_\alpha \to \mathbb{C}$ is holomorphic. By the structure of the line bundle, s is well defined if and only if

$$s_\alpha(z) = t_{\alpha\beta}(z) s_\beta(z) \tag{2.2}$$

for all $z \in U_\alpha \cap U_\beta$. The zero locus of s defines an analytic hypersurface in M.

Definition 2.3.1. Suppose $V \subseteq M$. We say that V is an analytic hypersurface of M if there exists a covering $\mathcal{U} = \{U_\alpha\}_{\alpha \in \Lambda}$ of M such that $V \cap U_\alpha = f_\alpha^{-1}\{0\}$ for some holomorphic functions $f_\alpha : U_\alpha \to \mathbb{C}$.

To serve our purpose, we impose some additional assumptions on f_α. We require that f_α to be non-singular and for all $\alpha, \beta \in \Lambda$, f_α / f_β extends to a non vanishing function on $U_\alpha \cap U_\beta$. This allows us to define

$$t_{\alpha\beta} = \frac{f_\alpha}{f_\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^*. \tag{2.3}$$

Denote L_V as the line bundle formed by the transition maps $t_{\alpha\beta}$ above. Then, $f_\beta = t_{\alpha\beta} f_\alpha$ satisfies (2.2) and thus form a section of L_V. We prove the following theorem following [11].

Theorem 2.3.2. *Let h be any Hermitian metric on L_V. Then,*

$$\left[-\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log h \right] \in H^{1,1}(M, \mathbb{C}) \cap H^2(M)$$

is the Poincaré dual of $[V] \in H_{n-2}(M)$.

Proof. Suppose $[\alpha] \in H^{n-2}(M)$. We need to show that

$$\int_V \alpha = -\frac{\sqrt{-1}}{2\pi} \int_M \partial\bar{\partial} \log h \wedge \alpha.$$

First, we show that $\partial\bar{\partial}\log h$ defines a global closed form on M. Indeed, by (2.1),

$$\begin{aligned}\partial\bar{\partial}\log h_\beta &= \partial\bar{\partial}\left(t_{\alpha\beta}\bar{t}_{\alpha\beta}h_\alpha\right)\\ &= \partial\bar{\partial}\left(\log t_{\alpha\beta} + \log \bar{t}_{\alpha\beta} + \log h_\alpha\right)\\ &= \partial\bar{\partial}\log h_\alpha.\end{aligned}$$

The last equality follows because $t_{\alpha\beta}$ is holomorphic, and thus $\bar{\partial}t_{\alpha\beta} = \partial\bar{t}_{\alpha\beta} = 0$. Similarly, if s is a non-vanishing holomorphic section, then $\partial\bar{\partial}\log||s||_h^2 = \partial\bar{\partial}\log h$. Indeed,

$$\partial\bar{\partial}\log||s||_h^2 = \partial\bar{\partial}\log(s\bar{s}h) = \partial\bar{\partial}\log h.$$

Recall that f is a section of L_V, which vanishes exactly on V. Hence, away from a neighnorhood of V, we have $\partial\bar{\partial}\log h = \partial\bar{\partial}\log||f||_h^2$. Since we assume that f is non-singular, we can choose a coordinates such that $f(z) = z^1$, and let $N_\epsilon = \{|z^1| < \epsilon\}$ be a tubular neighborhood of $V = f^{-1}\{0\}$ for small $\epsilon > 0$. Then,

$$\begin{aligned}-\frac{\sqrt{-1}}{2\pi}\int_M \partial\bar{\partial}\log h \wedge \alpha &= \lim_{\epsilon\to 0} -\frac{\sqrt{-1}}{2\pi}\int_{M\setminus N_\epsilon} \partial\bar{\partial}\log h \wedge \alpha\\ &= \lim_{\epsilon\to 0} -\frac{\sqrt{-1}}{2\pi}\int_{M\setminus N_\epsilon} \partial\bar{\partial}\log ||f||_h^2 \wedge \alpha\\ &= \lim_{\epsilon\to 0} -\frac{\sqrt{-1}}{4\pi}\int_{M\setminus N_\epsilon} d(\bar{\partial}-\partial)\log ||f||_h^2 \wedge \alpha\\ &= \lim_{\epsilon\to 0} -\frac{\sqrt{-1}}{4\pi}\int_{-\partial N_\epsilon} (\bar{\partial}-\partial)\log ||f||_h^2 \wedge \alpha\\ &= \lim_{\epsilon\to 0} \frac{\sqrt{-1}}{4\pi}\int_{\partial N_\epsilon} (\bar{\partial}-\partial)\log (f\bar{f}h) \wedge \alpha\\ &= \lim_{\epsilon\to 0} \frac{\sqrt{-1}}{4\pi}\int_{\partial N_\epsilon} (\bar{\partial}-\partial)\left(\log f + \log\bar{f} + \log h\right) \wedge \alpha\end{aligned}$$

Since h is a metric on L_V over a compact manifold M, we have $|\log h| < M < \infty$. Hence,

$$\begin{aligned}\lim_{\epsilon\to 0}\int_{\partial N_\epsilon} \bar{\partial}\log h \wedge \alpha &= \lim_{\epsilon\to 0}\int_{N_\epsilon} d\bar{\partial}\log h \wedge \alpha\\ &= \lim_{\epsilon\to 0}\int_{N_\epsilon} \partial\bar{\partial}\log h \wedge \alpha\end{aligned}$$

$$= 0 \qquad \left(\text{as } \lim_{\epsilon \to 0} Vol(N_\epsilon) = 0\right).$$

Similarly, we have $\lim_{\epsilon\to 0} \int_{\partial N_\epsilon} \partial \log h \wedge \alpha = 0$. Hence,

$$\begin{aligned} -\frac{\sqrt{-1}}{2\pi} \int_M \partial\bar{\partial} \log h \wedge \alpha &= \lim_{\epsilon \to 0} \frac{\sqrt{-1}}{4\pi} \int_{\partial N_\epsilon} (\bar{\partial} - \partial) \left(\log f + \log \bar{f}\right) \wedge \alpha \\ &= \lim_{\epsilon \to 0} \frac{\sqrt{-1}}{4\pi} \int_{\partial N_\epsilon} \left(-\frac{\partial f}{f} + \frac{\bar{\partial} \bar{f}}{\bar{f}}\right) \wedge \alpha \\ &= \lim_{\epsilon \to 0} \frac{1}{2\pi} \mathrm{Im} \int_{\partial N_\epsilon} \frac{\partial f}{f} \wedge \alpha \end{aligned}$$

Since $f(z) = z^1$ and N_ϵ is a tubular neighborhood of V, we have $\partial N_\epsilon = \{|z^1| = \epsilon\} \times V$. So

$$\begin{aligned} \int_{\partial N_\epsilon} \frac{\partial f}{f} \wedge \alpha &= \int_{\{|z^1|=\epsilon\}} \frac{1}{z^1} dz^1 \cdot \int_V \alpha \\ &= 2\pi\sqrt{-1} \cdot \int_V \alpha \end{aligned}$$

The result follows immediately. □

Remark 2.3.3. When there is no confusion, we also write $[V] = \left[-\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log h\right]$ as a cohomology class. This class is in fact the Chern class of the line bundle L_V.

Chapter 3

Setting of the Calabi Symmetry

In studying extremal Kähler metric, Calabi explicitly constructs examples on a family of projective bundles over projective spaces [1]. The rich symmetry on those spaces allows us to reduce the complexity of the equation. The same symmetry is used to study the dHYM equations, first in [15] and then in [14]. Here, we provide details of Calabi symmetry on the blowup of P^n, which are described explicitly in coordinates.

3.1 Generating Class of Projective Space

Define the complex projective space as $P^n := \mathbb{C}^{n+1}\backslash\{0\}/(w \sim \lambda w)$ with w as the homogeneous coordinates, i.e.

$$\left[w^0 : w^1 : \cdots : w^n\right] = \left[\lambda w^0 : \lambda w^1 : \cdots : \lambda w^n\right].$$

Let U_i be the local charts covering P^n defined as

$$U_i := \left\{ \left[\frac{w^0}{w^i} : \cdots : \frac{w^{i-1}}{w^i} : 1 : \frac{w^{i+1}}{w^i} : \cdots : \frac{w^n}{w^i}\right] \in P^n : w^i \neq 0 \right\}.$$

Let H be a hyperplane in P^n defined as

$$H := \{[0 : w^1 : w^2 : \cdots : w^n]\} \subseteq P^n.$$

When restricted to U_i, $H \cap U_i$ is the zero locus of $f_i = w^0/w^i$. Hence, by (2.3), the transition functions

$$t_{ij} = \frac{f_i}{f_j} = \frac{w^j}{w_i}$$

defines a line bundle L_H over P^n. Define also $h_i : U_i \to \mathbb{R}$ by

$$h_i(w) = \frac{|w^i|^2}{\sum_{k=0}^n |w^k|^2}.$$

Since $w^i \neq 0$ on U_i, h_i is a positive function. It also satisfies (2.1) and hence it is a Hermitian metric on L_H. Identify $U_0 \cong \mathbb{C}^n$ and let $z^i = w^i/w^0$ be its coordinates. Notice that $U_0 = P^n \backslash H$. Then,

$$h_0(z) = \frac{1}{1+|z|^2}.$$

We use Theorem 2.3.2 to compute the Poincaré dual of H.

$$\begin{aligned} -\frac{\sqrt{-1}}{2\pi}\partial_j\bar{\partial}_k \log h_0 \, dz^j \wedge d\bar{z}^k &= \frac{\sqrt{-1}}{2\pi}\partial_j \left(\frac{z^k}{1+|z|^2}\right) dz^j \wedge d\bar{z}^k \\ &= \frac{\sqrt{-1}}{2\pi}\left(\frac{\delta_{jk}}{1+|z|^2} - \frac{z^k\bar{z}^j}{(1+|z|^2)^2}\right) dz^j \wedge d\bar{z}^k \end{aligned}$$

Let $g_{\bar{k}j} = \delta_{jk}/(1+|z|^2) - z^k\bar{z}^j/(1+|z|^2)^2$. The following lemma shows that $g_{\bar{k}j}$ is positive definite.

Lemma 3.1.1. *Suppose $(z^1, z^2, \cdots, z^n) \in \mathbb{C}$. Then the eigenvalues of*

$$a\delta_{jk} + bz^k\bar{z}^j$$

are $\lambda_1 = a + b|z|^2$ and $\lambda_2 = \lambda_3 = \cdots = \lambda_n = a$.

Proof. Let $A_{jk} = a\delta_{jk} + bz^k\bar{z}^j$. Then, as a matrix,

$$A = aI + bzz^*.$$

So, we have $Az = (a + b|z|^2)z$, which gives the first eigenvalue. Let w be a vector orthogonal to z, so $z^*w = 0$. Then, we have $Aw = aw$. Since there are $n-1$ linearly independent vectors orthogonal to z, this gives the rest of the $n-1$ eigenvalues. $\square$

The above lemma implies that $g_{\bar{k}j}$ is a metric. Hence, we have the following definition.

Definition 3.1.2. Let

$$\omega_{FS} := \frac{\sqrt{-1}}{2\pi}\left(\frac{\delta_{jk}}{1+|z|^2} - \frac{z^k\bar{z}^j}{(1+|z|^2)^2}\right) dz^j \wedge d\bar{z}^k.$$

This is a Kähler metric on P^n and is called the *Fubini–Study metric*. The class $[\omega_{FS}] = [H] \in H^2(P^n)$ generates the cohomology of P^n, as it is the Poincaré dual of H, which generates $H_{2n-2}(P^n)$.

Remark 3.1.3. Being a Poincaré dual implies that $[\omega_{FS}] \in H^2(P^n, \mathbb{Z})$ is an integral class.

Using the Fubini–Study metric, we can prove the first half of the Calabi symmetry. Suppose $\mathbb{C}^n$ is identified with $U_0 \subseteq P^n$. Let $u(\rho) : \mathbb{R} \to \mathbb{R}$ be a smooth function, where $\rho = \log|z|^2$. Write $\omega = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u$, which is defined on $\mathbb{C}^n\backslash\{0\}$.

Theorem 3.1.4.

1. *ω is then a Kähler form on $\mathbb{C}^n\backslash\{0\}$ if and only if $u'(\rho) > 0$ and $u''(\rho) > 0$,*

2. *Let $U_\infty(r) := u(-\log r) + a\log r$. If U_∞ extends smoothly to $r = 0$, then ω can be extended smoothly to $P^n\backslash\{0\}$, and $[\omega] = a[H]$ as a cohomology class on $P^n\backslash\{0\}$.*

Proof. 1. Notice that

$$\partial\bar{\partial}u = \left(\frac{u'}{e^\rho}\delta_{jk} + (u'' - u')\frac{z^k\bar{z}^j}{e^{2\rho}}\right) dz^j \wedge d\bar{z}^k, \tag{3.1}$$

where the derivatives u' and u'' are with respect to ρ. By Lemma 3.1.1, its eigenvalues are $\lambda_1 = u''/e^\rho$ and $\lambda_2 = \cdots = \lambda_n = u'/e^\rho$. Hence, ω is positive if $u' > 0$ and $u'' > 0$.

2. Recall that

$$\omega_{FS} = -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\frac{1}{1+|z|^2}.$$

So,

$$\omega - a\omega_{FS} = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\left(u\left(\log|z|^2\right) + a\log\frac{1}{1+|z|^2}\right).$$

We need to show that this is a global exact form on $P^n\backslash\{0\}$, which is to show that

$$f(z) := u\left(\log|z|^2\right) + a\log\frac{1}{1+|z|^2}$$

extends smoothly to $P^n \backslash \{0\}$. Indeed. suppose $z_\infty := [0, z^1, \cdots, z^n] \in P^n \backslash U_0$. Then

$$\begin{aligned} z_\infty &= \lim_{r \to 0^+} \left[1 : \frac{z^1}{r} : \frac{z^2}{r} : \cdots : \frac{z^n}{r} \right] \\ f\left(\frac{z^1}{r}, \frac{z^2}{r}, \cdots, \frac{z^n}{r}\right) &= u\left(-\log |rz|^2\right) + a \log \frac{r^2}{r^2 + |z|^2} \\ &= U_\infty\left(|rz|^2\right) - a \log |rz|^2 + a \log \frac{r^2}{r^2 + |z|^2}. \end{aligned}$$

This extends smoothly if U_∞ extends smoothly to $r = 0$.

$\square$

3.2 Generating Class of $Bl_0\mathbb{C}^n$

Let $Bl_0\mathbb{C}^n$ be the blowup of $\mathbb{C}^n$ at the origin. By definition, it is the total space of the tautological line bundle of P^{n-1}. In local coordinates, we can describe it as follow.

$$Bl_0\mathbb{C}^n := \left\{\left(w^0, w^1, \cdots, w^n\right) \in \mathbb{C} \times \mathbb{C}^n \backslash \{0\}\right\} / \left(w^0, w^1, \cdots, w^n\right) \sim \left(w^0/\lambda, \lambda w^1, \cdots, \lambda w^n\right),$$

where $\lambda \in \mathbb{C}^*$. Any equivalence class in $Bl_0\mathbb{C}^n$ can be represented as

$$\left[w^0; w^1 : w^2 : \cdots : w^n\right] \in Bl_0\mathbb{C}^n.$$

Similar to the previous section, let U_i' be the local charts covering $Bl_0\mathbb{C}^n$ defined as

$$U_0' := \left\{\left[1; w^0 w^1 : w^0 w^2 \cdots : w^0 w^n\right] \in Bl_0\mathbb{C}^n : w^0 \neq 0\right\}$$

$$U_i' := \left\{\left[w^0 w^i; \frac{w^1}{w_i} : \cdots : \frac{w^{i-1}}{w^i} : 1 : \frac{w^{i+1}}{w^i} : \cdots : \frac{w^n}{w^i}\right] \in Bl_0\mathbb{C}^n : w^i \neq 0\right\}.$$

Let E be the exceptional divisor in $Bl_0\mathbb{C}^n$ defined as

$$E := \{[0; w^1 : w^2 : \cdots : w^n]\} \subseteq Bl_0\mathbb{C}^n$$

E is topologically the same as P^{n-1}. When restricted to U_i', $E \cap U_i'$ is the zero locus of $f_i = w^0 w^i$ (we set $f_0 \equiv 1$ on U_0'). Then, by (2.3), the transition functions

$$t_{i0} = w^0 w^i, \;\; t_{ij} = \frac{w^i}{w^j}$$

defines a line bundle L_E on $Bl_0\mathbb{C}^n$. Similarly, we define $h_i : U_i' \to (0, \infty)$ by

$$h_0 = \sum_{k=1}^{n} \left| w^i w^0 \right|^2, \;\; h_i = \frac{\sum_{k=1}^{n} \left| w^k \right|^2}{\left| w^i \right|^2}.$$

It satisfies (2.1) and thus it is a Hermitian metric on L_E. Let $z^i = w^0 w^i$ as the coordinates of U_0'. Notice that $U_0' = Bl_0\mathbb{C}^n \backslash E \cong \mathbb{C}^n \backslash \{0\}$. We see that the cohomology class of

$$-\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log h_0 = -\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log |z|^2$$

generates $H^2(Bl_0\mathbb{C}^n)$, as it is the Poincaré dual of E, which generates $H_{2n-2}(Bl_0\mathbb{C}^n)$.

Remark 3.2.1. Strictly speaking, since $Bl_0\mathbb{C}^n$ is not compact, we need to modify the 2-form by a gluing function λ to make it compactly support, as the Poincaré dual of E should be a cohomology class in $H_c^2(Bl_0\mathbb{C}^n)$. This will be addressed in the next section.

3.3 The Blowup of P^n

Let $X = Bl_0 P^n$ be the blowup of P^n at one point. We can define it explicitly by replacing $U_0 \subseteq P^n$ in Section 3.1 with $Bl_0\mathbb{C}^n$. Then,

$$X \backslash (H \cup E) = U_0 \backslash \{0\} = U_0' \backslash \{0\} \cong \mathbb{C}^n \backslash \{0\}.$$

Let $z = \left(z^1, \cdots, z^n\right)$ be the coordinates on $X \backslash (H \cup E)$, and let $\rho = \log |z|^2$. As in Section 3.1, let $u(\rho) : \mathbb{R} \to \mathbb{R}$ be a smooth function and $\omega = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} u$. Calabi symmetry provides asymptotic conditions in which ω can be extended to X.

Theorem 3.3.1. *Let*

$$U_0(r) := u(\log r) - b \log r,$$

$$U_\infty(r) := u(-\log r) + a \log r.$$

If U_0 and U_∞ extends smoothly to $r = 0$, then ω extends smoothly to X, and $[\omega] = a[H] - b[E]$ as a cohomology class in $H^{1,1}(X, \mathbb{C}) \cap H^2(X, \mathbb{R})$.

Proof. Let $\lambda_1 : \mathbb{R} \to \mathbb{R}$ be a smooth function such that

$$\lambda_1(\rho) = \begin{cases} 1 & \text{if } \rho < -1 \\ 0 & \text{if } \rho > 1. \end{cases}$$

Write $\lambda_2 = 1 - \lambda_1$. As in the proof of Theorem 3.1.4, to extend $\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}(\lambda_2 u)$ to a closed form in the class $a[H]$, we need

$$\lambda_2 u \left(\log \left| \frac{z}{r} \right|^2 \right) + a \log \frac{1}{1 + |z/r|^2}$$

to extend smoothly as $r \to 0^+$. Similarly, to extend $\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}(\lambda_1 u)$ to a closed form in the class $-b[E]$, we need

$$\lambda_1 u \left(\log |zr|^2 \right) - b \log |rz|^2$$

to extend smoothly as $r \to 0^+$. Both are satisfied by the assumption on U_0 and U_∞. Hence,

$$\omega = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}(\lambda_1 u + \lambda_2 u)$$

extends smoothly to X and $\omega \in a[H] - b[E]$. □

Chapter 4

The Line Bundle Mean Curvature Flow

Let (X, ω) be a compact Kähler manifold with Kähler form ω, and A cohomology class $[\alpha] \in H^{1,1}(X, \mathbb{R})$ solves the *deformed Hermitian–Yang–Mills* (dHYM) equation if there exists a representative $\alpha \in [\alpha]$ satisfying

$$\mathrm{Im}\left(e^{-\sqrt{-1}\hat{\theta}}\,\frac{(\omega+\sqrt{-1}\alpha)^n}{\omega^n}\right)=0, \tag{4.1}$$

where $e^{-\sqrt{-1}\hat{\theta}} \in S^1$ is a fixed constant. First, we notice that this is a topological constant that depends only on the class $[\omega]$ and $[\alpha]$, which satisfies

$$\mathrm{Im}\left(e^{-\sqrt{-1}\hat{\theta}}\int_X \left([\omega]+\sqrt{-1}[\alpha]\right)^n\right)=0, \tag{4.2}$$

assuming that the integration is non-zero. Under this assumption, Jacob–Yau [16] defines the volume functional $V : [\alpha] \to \mathbb{R}$ by

$$V(\alpha) := \int_X \left|\frac{\left(\omega+\sqrt{-1}\alpha\right)^n}{\omega^n}\right| \frac{\omega^n}{n!} \geq \left|\int_X \left(\omega+\sqrt{-1}\alpha\right)^n\right|.$$

Equality holds iff

$$\arg\left(\frac{(\omega+\sqrt{-1}\alpha)^n}{\omega^n}\right) \equiv \text{constant} = \hat{\theta}, \tag{4.3}$$

which is a reformulation of (4.1). Hence, a solution to the dHYM equation is a global minimizer of the functional V.

At any given point $p \in X$, it is possible to choose a coordinates in which their differentials are orthonormal on T_pX with respect to the metric associated to ω. We can diagonalize α with this basis. So, locally at the point p, we can write

$$\omega = \frac{\sqrt{-1}}{2} \sum_{i=1}^{n} dz^i \wedge d\bar{z}^i, \quad \alpha = \frac{\sqrt{-1}}{2} \sum_{i=1}^{n} \lambda_i dz^i \wedge d\bar{z}^i.$$

Implicitly, λ_i's are the eigenvalues of $\omega^{-1}\alpha$. Define $\Theta : X \to \mathbb{R}$ as

$$\begin{aligned} \Theta(x) &:= \arg\left(\frac{(\omega + \sqrt{-1}\alpha)^n}{\omega^n}\right) \\ &= \arg\det\left(1 + \sqrt{-1}\,\omega^{-1}\alpha\right) \\ &= \arg\prod_{i=1}^{n}\left(1 + \sqrt{-1}\,\lambda_i\right) \\ &= \sum_{i=1}^{n} \arctan(\lambda_i), \end{aligned} \tag{4.4}$$

where we are choosing the principle branch of the arctan function. As a reformulation of (4.3), the dHYM equation seeks a representative $\alpha \in [\alpha]$ such that

$$\Theta(x) \equiv \hat{\theta}. \tag{4.5}$$

4.1 Early Results

We state some early results here as well as some general elliptic theory. Interested readers can refer to [16]. First, we have the uniqueness theorem.

Theorem 4.1.1. *[16]*

1. *Suppose $\phi(t) : X \to R$ is some time-dependent smooth function. Let $\alpha_\phi = \alpha + \sqrt{-1}\partial\bar{\partial}\phi$. Then,*

   $$\frac{d}{dt}\Theta(\phi(t)) = \Delta_{\omega + \alpha\omega^{-1}\alpha}\left(\frac{d}{dt}\phi(t)\right).$$

   Hence, the dHYM equation is elliptic.

2. *If a solution exists, i.e. α_ϕ solves the dHYM equation, then ϕ is unique up to a constant.*

Proof. 1. We will not prove this here, but we will show in the next section that the dHYM equation is elliptic when $X = Bl_0 P^m$, in the proof of Lemma 5.2.2.

2. Following [16]. First, we show that $\hat{\theta}$ is unique. suppose α_{ϕ_1} and α_{ϕ_2} both solves (4.5). Consider the point $p \in X$ that achieves the minimum of $\phi_1 - \phi_2$, which has semi-positive definite Hessian. If we write $\lambda_k(\alpha)$ as the k-th eigenvalue of $\omega^{-1}\alpha$, we have

$$\lambda_k\left(\alpha_{\phi_1-\phi_2}\right)(p) \geq 0$$
$$\lambda_k\left(\alpha_{\phi_1}\right)(p) \geq \lambda_k\left(\alpha_{\phi_2}\right)(p)$$
$$\Theta\left(\alpha_{\phi_1}\right)(p) \geq \Theta\left(\alpha_{\phi_2}\right)(p).$$

The last inequality follows from (4.4), by noticing that arctan is an increasing function. Since α_{ϕ_1} and α_{ϕ_2} solves (4.5), we have $\Theta\left(\alpha_{\phi_1}\right) \equiv \Theta\left(\alpha_{\phi_1}\right)(p) = \hat{\theta}_1$ and $\Theta\left(\alpha_{\phi_2}\right) \equiv \Theta\left(\alpha_{\phi_2}\right)(p) = \hat{\theta}_2$. Hence, $\hat{\theta}_1 \geq \hat{\theta}_2$. Interchanging ϕ_1 with ϕ_2, we see that $\hat{\theta}_1 = \hat{\theta}_2$.

So,

$$\begin{aligned} 0 = \hat{\theta}_1 - \hat{\theta}_2 &= \int_0^1 \frac{d}{dt}\Theta\left(\alpha_{t\phi_1+(1-t)\phi_2}\right) dt \\ &= \left(\int_0^1 \Delta_t \, dt\right)(\phi_1 - \phi_2). \end{aligned}$$

The last equality follows from part 1, where Δ_t is some time-dependent elliptic operator. By the strong maximum principle, $\phi_1 - \phi_2 \equiv$ constant.

□

Since we will be using the maximum principle quite often, it is convenient to state it here as a part of the general elliptic theory.

Definition 4.1.2. A differential operator $L : C^n(X) \to C^{n-2}(X)$ is an elliptic operator if in a local coordinate chart,

$$L(f) = \sum_{j,k} a_{jk} \frac{\partial^2 f}{\partial x^k \partial x^j} + \sum_i b_i \frac{\partial f}{\partial x^i} + cf,$$

where a_{jk}, b_i, c are smooth functions, and (a_{jk}) forms a positive definite matrix.

Theorem 4.1.3 (The maximum principles)**.** *Let X be a compact manifold and L be an elliptic operator, and c be a smooth function as in Definition 4.1.2. Let $f = f(t,x) : [0,T) \times X \to \mathbb{R}$ be a time-dependent smooth function.*

1. *Suppose $\left(\frac{\partial}{\partial t} - L\right) f \geq 0$, $f_0 > 0$ and $c \geq 0$. Then, $f_t \geq \min f_0$.*

2. *Suppose $\left(\frac{\partial}{\partial t} - L\right) f \leq 0$, $f_0 > 0$ and $c \leq 0$. Then, $f_t \geq \max f_0$.*

Proof. We will show the first statement. The second statement is similar. First, we can replace f by $f + \epsilon t^2$ and let $\epsilon \searrow 0$ at the end. This allows us to assume $(d/dt - L)f > 0$ instead. Now, suppose otherwise that $0 < f(t_0, x_0) = y_0 < \min f_0$ for the first time at $t = t_0$, i.e. $f(t,x) > y_0$ for all $t < t_0$. Then, $\nabla f = 0$. Also,

$$\frac{\partial}{\partial t} f(t_0, x_0) \leq 0, \quad \sum_{j,k} a_{jk} \frac{\partial^2 f}{\partial x^k \partial x^j} \geq 0.$$

Since $c \geq 0$, we see that $\left(\frac{\partial}{\partial t} - L\right) f \leq 0$, which is a contradiction. $\square$

There are numerous result regarding the relationship between the existence of solution to (4.1) and notions of geometric stability [2, 3, 4, 5, 8, 15, 23]. Different parabolic methods have been developed [10, 13, 16, 33, 34]. One of such is known as the line bundle mean curvature flow. Suppose $\alpha_\phi = \alpha + \sqrt{-1}\partial\bar{\partial}\phi$ as in Theorem 4.1.1. The time dependent smooth function ϕ is said to satisfy the line bundle mean curvature flow if

$$\frac{d}{dt}\phi(x) = \Theta(\alpha_\phi) - \hat{\theta}. \tag{4.6}$$

Suppose the flow exists and converges to ϕ_∞ smoothly as $t \to \infty$. Then

$$\Theta\left(\alpha_{\phi_\infty}\right) - \hat{\theta} = \lim_{t\to\infty} \frac{d}{dt}\phi_t = 0, \tag{4.7}$$

which solves the dHYM equation (4.5). This method is used in [16] to show existence results under some additional assumption.

Theorem 4.1.4. *[16] Suppose (X, ω) has non-negative bisectional curvature, and $\alpha_0 = \sqrt{-1}\partial\bar{\partial}\phi_0$*

satisfies the hypercritical phase assumption, that is,

$$\Theta(\alpha_0) > (n-1)\pi/2.$$

Then, the line bundle mean curvature flow (2.2) converges smoothly to the solution of the dHYM equation.

The hypercritical phase assumption is imposed for two reasons. First, the maximum principle guarantees that $\Theta(\alpha_t) \geq \min \Theta(\alpha_0) > (n-1)\pi/2$. So, the eigenvalues λ_i of $\omega^{-1}\alpha_{\phi_t}$ stay positive in view of equation (4.4). Together with the assumption of non-negative bisectional curvature, those eigenvalues are also bounded above. Second, the hypercritical phase assumption implies that $\Theta(\cdot)$ has convex level sets, in which Evans–Krylov theory applies [3]. In essence, Theorem 4.1.4 is the consequence of the following.

Theorem 4.1.5. *[16] Let λ_i's be the eigenvalues of $\omega^{-1}\alpha_{\phi_t}$. Suppose*

$$\epsilon_0 < \lambda_i < C(t).$$

Then, the flow (4.6) exists for all time.

This is the key theorem in proving Theorem 1.0.2 and to construct a singularity example in Theorem 1.0.3.

4.2 Flow Equations in the Blowup of P^n

Let $X = Bl_0P^n$, and let ω be a Kähler form and α be a closed $(1,1)$-form satisfying Calabi symmetry, as in Theorem 3.3.1. By rescaling, assume that $\omega \in a[H]-[E]$ and $\alpha \in p[H]-q[E]$. Then, in local coordinates of $X\backslash(H \cup E)$, there exists $u(\rho), v(\rho) : \mathbb{R} \to \mathbb{R}$ with $\rho = \log|z|^2$ such that

$$\omega = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u, \quad \alpha = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}v.$$

The asymptotic conditions in Theorem 3.3.1 shows that

$$\lim_{\rho\to-\infty} u'(\rho) = 1, \quad \lim_{\rho\to\infty} u'(\rho) = a.$$

Since ω is a Kähler form, Theorem 3.1.4 implies that $u'' > 0$ and so $a > 1$. Similarly, we have

$$\lim_{\rho\to-\infty} v'(\rho) = q, \quad \lim_{\rho\to\infty} u'(\rho) = p.$$

As in finding the eigenvalues of equation (3.1), the eigenvalues of $\omega^{-1}\alpha$ are v'/u' with multiplicity $(n-1)$, and v''/u'' with multiplicity one. Furthermore, since $u''(\rho) > 0$, we have a change of coordinates (Legendre transform)

$$x = u'(\rho) \in [1, a].$$

Here, x is defined at 1 and a due to the asymptotic conditions for u', and can be treated as a real variable. This allows us to define $f : [1, a] \to \mathbb{R}$ as a function representing $v'(\rho)$ in x, i.e.

$$f(x) = v'(\rho).$$

Notice that the asymptotic conditions for v' implies that

$$f(1) = q, \text{ and } f(a) = p.$$

Differentiate $f(x)$ with respect to ρ, we obtain

$$f'(x)u''(\rho) = v''(\rho).$$

Here, we allow a slight abuse of notations, where f' denotes derivative with respect to x, whereas u'' and v'' denotes the derivative with respect to ρ. In this notation, the eigenvalues of $\omega\alpha^{-1}$ are

$$\frac{f}{x} \text{ with multiplicity } (n-1), \text{ and } f' \text{ with multiplicity 1.}$$

This allows us to write down Θ explicitly:

$$\begin{aligned} \Theta(\alpha) &= \sum_{i=1}^{n} \arctan(\lambda_i) \\ &= \arctan\left(f'\right) + (n-1)\arctan\left(\frac{f}{x}\right). \end{aligned} \tag{4.8}$$

In our setting, since $\alpha = \sqrt{-1}\partial\bar{\partial}v$, we can write the line bundle mean curvature flow (4.6) as

$$\dot{v} = \arctan\left(f'\right) + (n-1)\arctan\left(\frac{f}{x}\right) - \hat{\theta}.$$

Differentiate with respect to ρ, we arrive at

$$\dot{f} = u'' \cdot \left(\frac{f''}{1+f'^2} + (n-1)\frac{xf'-f}{x^2+f^2}\right). \tag{4.9}$$

Hence, we can rephrase Theorem 1.0.1 as follows.

Theorem 1.0.1'. *There exists smooth function $f_t(x) = f(t,x) : [0,T) \times [1,a] \to \mathbb{R}$ satisfying (4.9), with $f(1) = q$ and $f(a) = p$, such that*

$$\lim_{(t,x)\to(T,x_0)} f_t'(x) = \infty.$$

4.3 Stability

As shown in (4.7), the stationary solution to equation (4.9) gives a solution to the dHYM equation (4.5), which is an ODE in our case:

$$\arctan\left(f'\right) + (n-1)\arctan\left(\frac{f}{x}\right) = \hat{\theta}.$$

The constant $\hat{\theta}$, up to mod 2π, is completely determined by the values a, p and q, satisfying equation (4.2). Solutions to this ODE are drawn in the Figure 4.1. The initial values $f(1) = q$ and $f(a) = p$ are determined by the classes $[\omega]$ and $[\alpha]$, which are said to be *stable* if the solution joining $(1, q)$ and (a, p) is graphical. They are *semi-stable* if the solution has a vertical tangency, and *unstable* if the solution is not graphical. The matter of stability is completely determined algebraically in this Calabi symmetry setting (see Lemma 1 in [15]). Our example in Theorem 1.0.1 can be created on classes in the stable case, where solution to the dHYM equation exists.

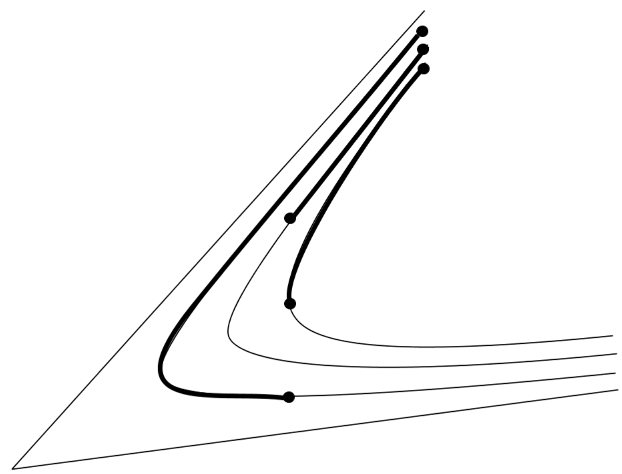

Figure 4.1: The unstable, stable, and semi-stable cases

Notions of stability is not known on a general Kähler manifold X. Most results requires an additional supercritical phase assumption $\Theta(\alpha) > (n-1)\pi/2$. Collins–Jacob–Yau prove that the complex number $\int_X -\exp(-\sqrt{-1}\,\omega + \alpha)$ lies in the upper half space in this case [3]. They further show that solution to the dHYM equation exists if

$$\pi > \arg \int_V -e^{-i\omega+\alpha} > \arg \int_X -e^{-i\omega+\alpha} \tag{4.10}$$

for any analytic irreducible subvariety $V \subseteq X$. Here, by convention, we only integrate the term in the expansion of order $\dim(V)$. Later, some necessary Chern number inequalities were discovered ([6] for $n = 3$ and [12] for $n = 4$). It is conjectured [7] that the stability inequality (1.6) with some Chern number inequalities are sufficient for the existence of the dHYM equations.

4.4 Comparison to the Curve Shortening Flow

The line bundle mean curvature flow in the Calabi setting (4.9) has some similarities with the curve shortening flow. In this section, we will discuss this similarity and demonstrate the avoidance principle, which will be the main technique in proving our main results in the next chapter.

Let $\gamma_t(s) : I \subseteq \mathbb{R} \to \mathbb{C}$ be a family of curves, where $I \subseteq \mathbb{R}$ is an interval and s is the arclength parameter. Let κ be the usual signed curvature

$$\kappa = \frac{d}{ds} \arctan \gamma',$$

and $\mathbf{N}$ be the unit normal vector

$$\mathbf{N} = e^{\sqrt{-1}\frac{\pi}{2}}\, \gamma'.$$

The family γ_t is said to satisfy the *curve shortening flow* if

$$\dot{\gamma} = \kappa \mathbf{N}. \tag{4.11}$$

Let $\langle -, - \rangle$ be the usual inner product. As the name suggested, the curve shortens as it evolves:

$$\begin{aligned} \frac{d}{dt}\int_I ds &= \int_I \left\langle \dot{\gamma}', \gamma' \right\rangle ds \\ &= \int_I \left\langle \frac{d}{ds}(\kappa \mathbf{N}), \gamma' \right\rangle dx \\ &= -\int_I \kappa^2 ds < 0. \end{aligned}$$

Example 4.4.1. Let γ_t be circles of radius $\sqrt{R^2 - 2t}$, with a fixed center. Then, γ_t satisfies the curve shortening flow (4.11). It becomes extinct when $t = R^2/2$.

In the case when $\gamma(x) = (x, f(x))$ is a graph of function, we compute

$$\dot{\gamma} = \big(0, \dot{f}\big), \qquad \mathbf{N} = \frac{1}{\sqrt{1 + f'^2}}(-f', 1) \quad \text{and} \quad \kappa = \frac{f''}{(1 + f'^2)^{3/2}}.$$

If we take inner product of equation (4.11) with $\mathbf{N}$, we have

$$\dot{f} = \frac{f''}{1 + f'^2}, \tag{4.12}$$

which is the same as the first term in equation (4.9).

Theorem 4.4.2 (The Avoidance Principle)**.** *Suppose γ_t and ζ_t are two families of curves satisfying the curve shortening flow (4.11). If γ_0 does not intersect ζ_0, then the distance between the two curve is non-decreasing along the flow, as long as it is defined.*

Proof. Locally, we can describe the curves as graph of functions f and g that satisfy (4.12). Since the two curves are disjoint at $t = 0$, we can assume $f_0 - g_0 > 0$. By the maximum principle, $f_t - g_t \geq \min(f_{t_0} - g_{t_0})$ for all $t_0 \geq 0$. This shows that the distance between two curves is non-decreasing. $\square$

Our flow (4.9) can be described as a flow of curve as well. Let

$$\xi = \frac{d}{ds} \arctan \gamma = \frac{1}{\sqrt{x'^2 + y'^2}} \cdot \frac{xy' - x'y}{x^2 + y^2}$$

be an extrinsic quantity. Consider the flow

$$\dot{\gamma} = u''(\gamma)(\kappa + (n-1)\xi)\mathbf{N}. \tag{4.13}$$

Then, when $\gamma(x) = (x, f(x))$ is a graph of function, (4.13) reduces to the line bundle mean curvature flow (4.9).

Chapter 5

The Proofs of the Main Theorems

Throughout the chapter, we assume that $\omega = \sqrt{-1}\partial\bar{\partial}u$ satisfies Calabi symmetry. The Kähler form extends from $\mathbb{C}\backslash\{0\}$ to $X = Bl_0P^n$ in the class $a[H] - [E]$. We further assume that $u'' < R$ and $u''(x) \geq k(x-1)(a-x)$ for $x \in [1, a]$. This is possible because the Calabi symmetry assumption requires u'' to be positive on $(1, a)$ and vanishes at the boundary. These assumptions are for the ease of presentation, and we believe that similar results should hold for a general ω satisfying Calabi symmetry.

5.1 A Finite Time Singularity

Consider a real number $R > 1$ (to be determined later), and set $a = 6R$. As above, Let $u : [1, a] \to \mathbb{R}$ be a smooth function representing a Kähler form satisfying Calabi symmetry. Consider a class $[\alpha] = p[H] - q[E]$ and assume $p \geq a$. Define a representative α_0 via the function $f_0(x)$, which has a graph such as in Figure 5.1. We construct a family of shrinking circles, and a traveling family of hyperbolas, which evolve slower than the flow (4.13). If f_t is the evolution of f_0 via the line bundle mean curvature flow (4.9), and f_0 avoids the initial circle and hyperbola at time $t = 0$, then we see by the avoidance principle f_t must avoid these families for all time. The specific path of hyperbolas then force f_t to achieve vertical slope at some finite time.

We first construct our family of hyperbolas. Observe that both κ and ξ are invariant under orthogonal transformation. Hence, by interchanging the x and y coordinates, we have the following lemma.

Lemma 5.1.1. *Suppose $y = f_t(x)$ satisfies the flow (4.9). If the inverse $x = f_t^{-1}(y) =: h_t(y)$ exists,*

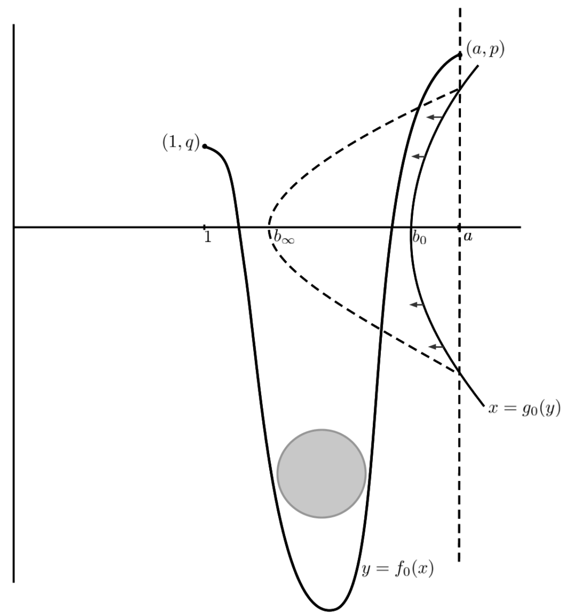

Figure 5.1: The graph of a function f_0 which forms a singularity.

then $h_t(y)$ satisfies

$$\dot{h} = u''(h(y)) \left(\frac{h''}{1+h'^2} + (n-1)\frac{yh'-h}{y^2+h^2} \right). \tag{5.1}$$

Denote the right hand side of (5.1.1) by $L(h)$. We now construct a family of hyperbolas g_t that are sub-solution to (5.1.1), i.e. $\dot{g} - L(g) \geq 0$.

Lemma 5.1.2. *Suppose $b(t) : [0,T) \to \mathbb{R}$ satisfies the initial value problem:*

$$\dot{b} = -\frac{kb_\infty(b_\infty - 1)(a^2 - b_0^2)(b - b_\infty)b^3}{a(a^2 - b_\infty^2)(2a^2 - b_\infty^2)^2} \tag{5.2}$$

where $1 < b_\infty < b_0 < a$ are constants. Then

$$g_t(y) := \sqrt{\frac{a^2 - b^2}{a^2 - b_\infty^2} y^2 + b^2}$$

is a sub-solution to the equation (5.1) for $y \in \left[-\sqrt{a^2 - b_\infty^2}, \sqrt{a^2 - b_\infty^2} \right]$.

Proof. For simplicity, write

$$m = \frac{a^2 - b^2}{a^2 - b_\infty^2}, \qquad 1 - m = \frac{b^2 - b_\infty^2}{a^2 - b_\infty^2}.$$

We also write $g = g_t$ for notational simplicity. Notice that $b_0 \geq b > b_\infty$ from the initial value problem, so $m < 1$. We compute

$$g' = \frac{my}{\sqrt{my^2 + b^2}} = \frac{my}{g},$$

which in turn gives

$$g'' = \frac{m}{g} - \frac{myg'}{g^2} = \frac{m}{g} - \frac{m^2y^2}{g^3} = \frac{mg^2 - m^2y^2}{g^3} = \frac{mb^2}{g^3}.$$

Furthermore the two expressions from (5.1) can be written as

$$\frac{g''}{1 + g'^2} = \frac{mb^2}{g(g^2 + m^2y^2)},$$

and

$$\frac{yg' - g}{y^2 + g^2} = \frac{my^2 - g^2}{g(y^2 + g^2)} = \frac{-b^2}{g(g^2 + y^2)}.$$

Thus

$$\begin{aligned}
L(g) &:= u''(g(y)) \left(\frac{g''}{1 + g'^2} + (n-1) \frac{yg' - g}{y^2 + g^2} \right) \\
&= u''(g(y)) \left(\frac{mb^2}{g(g^2 + m^2y^2)} - (n-1) \frac{b^2}{g(g^2 + y^2)} \right) \\
&\leq u''(g(y)) \left(\frac{mb^2}{g(g^2 + m^2y^2)} - \frac{b^2}{g(g^2 + y^2)} \right) \\
&= u''(g(y)) \frac{-(1-m)b^4}{g(g^2 + m^2y^2)(g^2 + y^2)} \\
&\leq k(g-1)(a-g) \frac{-(1-m)b^4}{g(g^2 + m^2y^2)(g^2 + y^2)} \leq 0,
\end{aligned}$$

where the last inequality follows from our assumption $u''(x) \geq k(x-a)(a-x)$.

Now, observe that

$$(a-g)(a+g) = a^2 - my^2 - b^2 = m\left(\frac{a^2-b^2}{m} - y^2\right) = m(a^2 - b_\infty^2 - y^2).$$

The right hand side is non-negative when $y \in \left[-\sqrt{a^2-b_\infty^2}, \sqrt{a^2-b_\infty^2}\right]$. As a result

$$\begin{aligned} L(g) &\leq \frac{-k(g-1)m(a^2-b_\infty^2-y^2)(1-m)b^4}{g(a+g)(g^2+m^2y^2)(g^2+y^2)} \\ &= \frac{-k(a^2-b_\infty^2-y^2)(g-1)(a^2-b^2)(b+b_\infty)(b-b_\infty)b^4}{g(a+g)(a^2-b_\infty^2)^2(g^2+m^2y^2)(g^2+y^2)}, \end{aligned}$$

where we plugged in for m and $(1-m)$. Because the above expression is negative, the inequalities $m < 1$, $b_\infty \leq g \leq a$, and $b_\infty < b \leq b_0$, allow us to conclude

$$L(g) \leq \frac{-k(a^2-b_\infty^2-y^2)(b_\infty-1)(a^2-b_0^2)2b_\infty(b-b_\infty)b^4}{g(2a)(a^2-b_\infty^2)^2(2a^2-b_\infty^2)^2}.$$

Next, we turn to the evolution of g:

$$\begin{aligned} \dot{g} = \frac{\dot{m}y^2 + 2b\dot{b}}{2g} = \frac{\frac{-2b\dot{b}}{a^2-b_\infty^2}y^2 + 2b\dot{b}}{2g} &= \frac{b\dot{b}}{g}\left(1 - \frac{y^2}{a^2-b_\infty^2}\right) \\ &= \frac{b\dot{b}(a^2-b_\infty^2-y^2)}{g(a^2-b_\infty^2)} \leq 0. \end{aligned}$$

Putting everything together we arrive at

$$\dot{g} - L(g) \geq \frac{b(a^2-b_\infty^2-y^2)}{g(a^2-b_\infty^2)}\left(\dot{b} + \frac{k(b_\infty-1)(a^2-b_0^2)b_\infty(b-b_\infty)b^3}{a(a^2-b_\infty^2)^2(2a^2-b_\infty^2)}\right).$$

The right hand side is zero by the initial value problem. Hence we have demonstrated $\dot{g} - L(g) \geq 0$.

$\square$

Set $b_\infty = R$, and $b_0 = 5R$. Recall $a = 6R$, so $1 < b_\infty < b_0 < a$. Note there exists a constant $M > 0$ such that

$$C_1 := k\frac{b_\infty(b_\infty-1)(a^2-b_0^2)}{a(a^2-b_\infty^2)(2a^2-b_\infty^2)^2} \geq \frac{1}{MR^3}.$$

Now, the differential equation (5.2) is separable, yielding

$$-C_1 dt = \frac{db}{(b - b_\infty)b^3},$$

which has the solution

$$-C_1 t + C_0 = \frac{b_\infty^2 + 2b^2 \log(b - b_\infty) + 2b_\infty b - 2b^2 \log(b)}{2b_\infty^3 b^2}$$

where C_0 is given by the initial value $b(0) = b_0 = 5R$. Plugging in $t = 0$ we see directly that

$$C_0 = \frac{11/50 + \log(4/5)}{R^3}.$$

Let T be the time such that $b(T) = 2R$. Then, we have

$$T = \frac{1}{C_1}\left(\frac{11/50 + \log(4/5)}{R^3} - \frac{5/8 - \log(2)}{R^3}\right) \leq A \tag{5.3}$$

for some constant A.

Proposition 5.1.3. *Let $\gamma(t)$ satisfy (4.13). If $\gamma(0)$ does not intersect the hyperbola $g_0(y)$, then $\gamma(t)$ does not intersect $g_t(y)$ for as long as the flow is defined.*

Proof. Suppose $p = (x_0, y_0)$ is the first point of intersection of the two curves, occurring at time t_0. Since the hyperbola g_t never achieves horizontal slope, we can assume near p that $\gamma(t)$ is a graph of a function $h_t(y)$ over the ball $B_\delta(y_0)$ in y-axis solving (5.1). Without loss of generality, for $0 \leq t < t_0$ assume that $g_t(y) > h_t(y)$ over $B_\delta(y_0)$. Then over the region $B_\delta(y_0) \times [0, t_0)$, we see $(\frac{d}{dt} - L)(g - h) \geq 0$, yet $g - h > 0$ on the parabolic boundary. The result follows from the maximum principle. $\square$

Next we turn to the family of shrinking circles which act as a barrier. Since ξ is relatively small for a curve far away from the origin, (4.13) behaves similarly to the curve shortening flow in this case. The idea is to consider a family of circles far away from the origin which evolve slightly faster than curve shortening flow, in order to absorb the small ξ term.

Proposition 5.1.4. *For $R = a/6 > 1$ as above, assume the graph of $f_0(x)$ does not intersect the ball $B_R(3R, y_0)$. Then, for y_0 sufficiently negative, the family of shrinking balls $B_{\sqrt{R^2-4Rt}}(3R, y_0)$ does not intersect the family of graphs of $f_t(x)$ evolving via via (4.9), as long as the flow is defined.*

Proof. Locally, we can write $\phi_t(x) = -\sqrt{r(t)^2 - (x-3R)^2} + y_0$ as equation representing the lower boundary of the shrinking balls, where $r(t) = \sqrt{R^2 - 4Rt}$. Direct computation gives

$$u'' \frac{\phi''}{1+\phi'^2} - \dot{\phi} = \frac{u'' - 2R}{\sqrt{r^2 - (x-3R)^2}} < \frac{-R}{\sqrt{r^2 - (x-3R)^2}},$$

since by assumption $u'' < R$. Suppose $t = t_0$ is the first time the graph of ϕ_t intersects f_t from above at a point x_0. At this point of intersection we have $f'_{t_0}(x_0) = \phi'_{t_0}(x_0)$, $\dot{f}_{t_0}(x_0) \geq \dot{\phi}_{t_0}(x_0)$, and we can assume $f_t(x) < \phi_t(x)$ for all $t < t_0$, and so $f''_{t_0}(x_0) \leq \phi''_{t_0}(x_0)$. Then at $t = t_0$, $x = x_0$, we have

$$\begin{aligned}
\dot{f} - \dot{\phi} &= -\dot{\phi} + u'' \left(\frac{f''}{1+f'^2} + (n-1)\frac{x_0 f' - f}{x_0^2 + f^2} \right) \\
&\leq -\dot{\phi} + u'' \left(\frac{\phi''}{1+\phi'^2} + (n-1)\frac{x_0 \phi' - \phi}{x_0^2 + \phi^2} \right) \\
&< -\frac{R}{\sqrt{r^2 - (x_0-3R)^2}} + u''(n-1)\frac{x_0\phi' - \phi}{x_0^2 + \phi^2}.
\end{aligned}$$

To achieve a contradiction we need to show that for y_0 sufficiently negative the right hand side above is negative. To control the ϕ' term we can compute directly

$$-\frac{R}{\sqrt{r^2 - (x_0-3R)^2}} + \frac{u''(n-1)x_0\phi'}{x_0^2 + \phi^2} = \frac{-R(x_0^2 + \phi^2) + (n-1)u''(x_0 - 3R)}{(x_0^2 + \phi^2)\sqrt{r^2 - (x_0-3R)^2}}.$$

Recall that by assumption $u'' < R$. Choose y_0 sufficiently negative to ensure $-(x_0^2 + \phi^2) + (n-1)(x_0 - 3R) \leq -\frac{1}{2}(x_0^2 + \phi^2)$. Then

$$-\frac{R}{\sqrt{r^2 - (x_0-3R)^2}} + \frac{u''(n-1)x_0\phi'}{x_0^2 + \phi^2} \leq \frac{-R}{2\sqrt{r^2 - (x_0-3R)^2}} < -\frac{1}{2}$$

since $r < R$. We have now demonstrated that

$$\dot{f} - \dot{\phi} < -\frac{1}{2} - \frac{u''(n-1)\phi}{x_0^2 + \phi^2}.$$

The function ϕ is negative, so the second term on the right hand side above is positive. However, we can choose y_0 sufficiently negative so that this term is less than $\frac{1}{2}$, and the result follows. $\square$

We now demonstrate the existence of a singularity using our two sub-solutions constructed above.

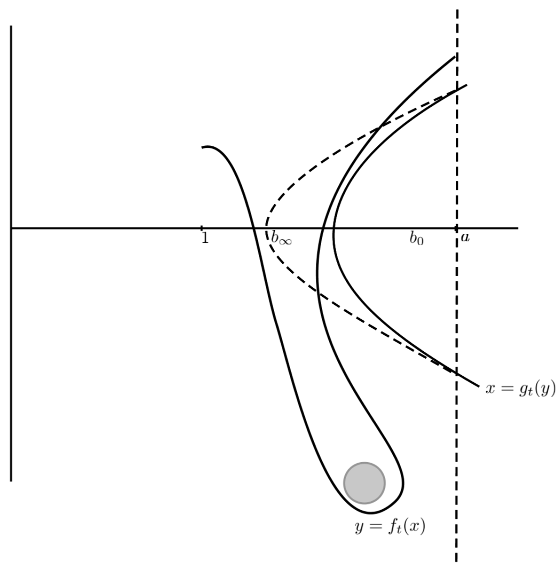

Figure 5.2: The maximum principle forces f_t to achieve vertical slope.

For $R > 1$, set $b_\infty = R$, $b_0 = 5R$, and $a = 6R$. Consider the circle of radius R centered around $(3R, y_0)$, with y_0 sufficiently negative so that the hypothesis of Proposition 5.1.4 is satisfied. The right side of the circle lies on the line $x = 4R$. Note that the vertex of the hyperbola $g_0(y)$ lies on the line $x = b_0 = 5R$. Furthermore, the hyperbola intersects $x = a$ at $y = \pm\sqrt{35R^2}$. Since $p > a = 6R$, we see (a, p) lies above the top of the hyperbola $g_0(y)$. Thus, it is possible to choose a function $f_0 : [1, a] \to \mathbb{R}$ with $f_0(1) = q$, $f_0(a) = p$, such that f_0 goes below $B_R(3R, y_0)$, then increases above the hyperbola $g_0(y)$ before arriving at (a, p).

Let $f_t(x)$ be the solution of (4.9) starting at $f_0(t)$. By Proposition 5.1.3 and Proposition 5.1.4, $f_t(x)$ can not intersect $g_t(y)$ nor $B_{\sqrt{R^2-4Rt}}(3R, y_0)$ as long as the flow is defined. Note it takes time $t = R/4$ for $B_{\sqrt{R^2-4Rt}}(3R, y_0)$ to shrink a point. Also, if T is the time the hyperbola $g_T(y)$ has pushed out to the line $x = 2R$, as we have seen by (5.3) there exists a constant A such that $T \leq A$. Hence, choose R large enough to ensure $A < R/4$ and thus $T < R/4$, which implies the hyperbola will push past the center of the shrinking circles before they completely disappear. This forces f_t

to first have a vertical tangency, as illustrated in Figure 5.2, demonstrating the existence of a finite time singularity and proving Theorem 1.0.1.

5.2 Long Time Existence

The example above shows that a finite-time singularity for the flow (4.9) can occur in the interior of the interval $(1, a)$. In particular, one can not always expect $\sup_{(1,a)} |f_t'(x)|$ to stay bounded for finite time. However, we can show that, for finite time, the first derivative stays bounded on the boundary.

Proposition 5.2.1. *Suppose $f_t(x)$ is defined on $(t, x) \in [0, T) \times [1, a]$. Then, there exists uniform constants A, B such that*

$$|f_t'(1)| + |f_t'(a)| \leq Ae^{Bt}.$$

Proof. We will only show that $|f'(1)| < C(T)$. The other end point is treated similarly. Consider $g_t(x) = q + Ae^{B(n-1)t}(x-1)$. Choose $A \gg 0$ sufficiently large to ensure both $Ae^{B(n-1)t} \geq 2\max\{|q|, |q|^{-1}\}$ and $f_0 < g_0$ for all $x \in (1, a]$. Choose $B \gg 0$ so that $u'' < B(x-1)$. We claim that $f_t < g_t$ for all time $t \in [0, T)$.

Suppose not, and assume the curves touch for the first time at $x = x_0 > 1$ and $t = t_0$. Then, $f_{t_0}(x_0) = g_{t_0}(x_0)$, $f_{t_0}'(x_0) = g_{t_0}'(x_0)$, $f_{t_0}''(x_0) \leq g_{t_0}''(x_0)$, and $\dot{f}_{t_0}(x_0) \geq \dot{g}_{t_0}(x_0)$. Thus, when $x = x_0$, $t = t_0$ we have

$$\begin{aligned}
\dot{f} &= u'' \left(\frac{f''}{1+f'^2} + (n-1)\frac{x_0 f' - f}{x_0^2 + f^2} \right) \\
&\leq B(x_0 - 1) \left(\frac{g''}{1+g'^2} + (n-1)\frac{x_0 g' - g}{x_0^2 + g^2} \right) \\
&= B(x_0-1)(n-1)\frac{Ax_0 e^{B(n-1)t_0} - q - Ae^{B(n-1)t_0}(x_0-1)}{x_0^2 + (q + Ae^{B(n-1)t_0}(x_0-1))^2} \\
&< ABe^{B(n-1)t}(x_0-1)(n-1)\frac{1 - A^{-1}e^{-B(n-1)t_0}q}{1+q^2},
\end{aligned}$$

since $x_0^2 + (q + Ae^{B(n-1)t_0}(x_0-1))^2 > 1 + q^2$. Furthermore by assumption on A we have $-A^{-1}e^{-B(n-1)t_0}q \leq \frac{1}{2}q^2$, and so

$$\frac{1 - A^{-1}e^{-B(n-1)t_0}q}{1+q^2} \leq 1.$$

Hence,

$$\dot{f} < ABe^{B(n-1)t_0}(x_0 - 1)(n-1) = \dot{g},$$

a contradiction. Thus g_t serves as a barrier giving an upper bound for the derivative $f'(1) \leq Ae^{B(n-1)t}$. The lower bound is treated similarly. $\square$

We now turn to the case where we do have long time existence, namely when $n \geq 3$ and α_0 has supercritical phase, $\Theta(\alpha_0) > (n-2)\frac{\pi}{2}$.

Lemma 5.2.2. *The supercritical phase condition is preserved along the flow.*

Proof. We compute the evolution of $\Theta = \Theta(\alpha_t)$ using (4.8) and (4.9).

$$\begin{aligned} \frac{d}{dt}\Theta(\alpha_t) &= \frac{d}{dt}\left(\arctan(f') + (n-1)\arctan\left(\frac{f}{x}\right)\right) \\ &= \frac{\dot{f}'}{1+f'^2} + (n-1)\frac{x\dot{f}}{x^2+f^2} \\ &= \frac{u''}{1+f'^2}\Theta'' + \left((n-1)\frac{xu''}{x^2+f^2} + \frac{1}{1+f'^2}\frac{du''}{dx}\right)\Theta'. \end{aligned} \tag{5.4}$$

This shows that the dHYM equation is elliptic, and the result follows from the maximum principle. $\square$

Lemma 5.2.3. *We have $f_t(x) > 0$ for all $t \geq 0$.*

Proof. Suppose there exists a time t_0 and a point x_0 where $f_{t_0}(x_0) \leq 0$, which implies $\arctan\left(\frac{f}{x_0}\right) \leq 0$. Let $\Theta(x_0) := (n-1)\arctan\left(\frac{f}{x_0}\right) + \arctan(f')$, and so super critical phase implies

$$\arctan(f') > (n-2)\frac{\pi}{2},$$

which is impossible for $n \geq 3$. $\square$

Lemma 5.2.4. *There exists a uniform constant C so that $f_t'(x) > -C$ for all $t \geq 0$.*

Proof. By the supercritical phase condition

$$\arctan(f_t') > (n-2)\frac{\pi}{2} - (n-1)\arctan\left(\frac{f_t}{x}\right).$$

Since $x \geq 1$ and $f_t \leq C$ by the maximum principle, there exists an $\epsilon > 0$ so that $\arctan\left(\frac{f_t}{x}\right) < \frac{\pi}{2} - \epsilon$. Thus

$$\arctan\left(f_t'\right) > -\frac{\pi}{2} + (n-1)\epsilon.$$

This gives a lower bound for f_t'. $\square$

Proposition 5.2.5. *A solution $f_t(x)$ to (4.9) has bounded first derivative for all times $T < \infty$. In particular, there exists uniform constants A, B so that*

$$\sup_{x \in [1,a]} |f_t'(x)| \leq A(1+t)e^{Bt}.$$

Proof. By the previous lemma we only need an upper bound for f_t'. By Proposition 5.2.1 we have

$$A^{-1}e^{-Bt}\left(|f_t'(1)| + |f_t'(a)|\right) \leq 1.$$

As a result if $\sup_{x\in[1,a]} A^{-1}e^{-Bt}|f_t'(x)|$ is large, this supremum must be achieved at an interior point. Let x_0 be the interior max. At this point we have $f_t'(x_0) > 0$, $f_t''(x_0) = 0$, and $f_t'''(x_0) \leq 0$. By direct computation at x_0 it holds

$$\begin{aligned}
\dot{f}' &= \frac{d}{dx}\left(u''\left(\frac{f''}{1+f'^2} + (n-1)\frac{xf'-f}{x+f^2}\right)\right)\\
&\leq \frac{du''}{dx}(n-1)\frac{x_0f'-f}{x_0^2+f^2} + u''\frac{d}{dx}\left(\frac{f''}{1+f'^2} + (n-1)\frac{xf'-f}{x^2+f^2}\right)\\
&\leq Cf' + u''\left(\frac{f'''}{1+f'^2} - (n-1)\frac{2(x_0f'-f)(x_0+ff')}{(x_0^2+f^2)^2}\right),
\end{aligned}$$

where we repeatedly plugged in that $f''(x_0) = 0$. Since f is positive the term $-2x_0f(f')^2$ is negative, and thus

$$\dot{f}' \leq Cf' + u''2(n-1)\frac{fx_0 + f^2f' - x_0^2f'}{(x_0^2+f^2)^2} \leq Cf' + C$$

for some constant C.

Now, consider the function $A^{-1}e^{-Bt}f_t'(x) - Ct$, and by making B larger, if necessary, we can assume $B \geq C$. At an interior maximum we see

$$\frac{d}{dt}\left(A^{-1}e^{-Bt}f' - Ct\right) \leq 0,$$

from which the result follows. $\square$

We remark that the above proof fails when the function f is not positive, since then the term $-2x_0 f(f')^2$ is positive. Thus the best inequality one can derive in this case is $\dot{f}' \leq C f'^2$, which is certainly not enough to prevent a finite time singularity, as we have demonstrated. We are now ready to prove our second main result.

Proof of Theorem 1.0.2. Let $\alpha_t := \alpha_0 + i\partial\bar{\partial}\phi_t$, be the solution to (4.6) starting at α_0, and assume the flow is defined for $t \in [0, T)$ for some time $T < \infty$. By proposition 5.2.5, all the eigenvalues of $\omega^{-1}\alpha_t$ are bounded uniformly by a constant C_T. From here the result follows from the argument outlined in Proposition 5.2 in [16].

The idea is that once the eigenvalues are bounded, the operator Δ_η is uniformly elliptic. Given $\Theta(\alpha_t)$ solves the heat equation (5.4), the parabolic estimates of Krylov-Safonov ([18] Theorem 11, Section 4.2) imply $\Theta(\alpha_t)$ is in C^α in time which gives ϕ_t is uniformly bounded in $C^{1,\alpha}$ in time. Now, the uniform eigenvalue bounds also imply ϕ_t has bounded C^2 norm. The supercritical phase assumption implies the operator $\Theta(\cdot)$ has convex level sets, which allows us to apply Evans-Krylov theory (see Section 6 of [3]). This gives uniform $C^{2,\alpha}$ bounds for ϕ_t which can be bootstrapped to higher order estimates. Thus we get smooth convergence $\phi_t \to \phi_T$ to some limit, which allows us to continue the flow past the time T. $\square$

5.3 Singular Behavior at $t = \infty$

In this session we construct an example which singularity at infinite times occurs at the exceptional divisor $[E]$. We first assume the existence of a stationary solution γ_∞ to (4.13). We write $\gamma_\infty(\theta) = (x_\infty(\theta), y_\infty(\theta)) = (r_\infty(\theta)\cos\theta, r_\infty(\theta)\sin\theta)$, with $\theta \in [\theta_{\min}, \theta_{\max}]$. We also require

$$1 \leq x_\infty(\theta) \leq a \quad \text{and} \quad x_\infty(\theta_{\min}) = x_\infty(\theta_{\max}) = a. \tag{5.5}$$

We prove the following.

Proposition 5.3.1. *Suppose*

$$r'_\infty \geq 0 \quad \textit{and} \quad \frac{r'_\infty}{r_\infty} \leq 2\tan\theta. \tag{5.6}$$

Then, There exists a subsolution $\gamma_t(\theta) = (r_t(\theta)\cos\theta, r_t(\theta)\sin\theta)$ such that $\gamma_0(\theta) = (a, a\tan\theta)$ and $\gamma_t \to \gamma_\infty$ uniformly as $t \to \infty$.

Proof. Our first step is to write down (4.13) in polar coordinates. Note that $\dot\gamma = (\dot r\cos\theta, \dot r\sin\theta)$, with the normal vector to γ given by

$$\mathbf{N} = \frac{1}{(r'^2 + r^2)^{1/2}}(-r'\sin\theta - r\cos\theta, r'\cos\theta - r\sin\theta).$$

Thus $\langle\dot\gamma, \mathbf{N}\rangle = -\frac{\dot r r}{(r'^2+r^2)^{1/2}}$. In this case the extrinsic quantity ξ is simply $\xi = \frac{d}{ds}\theta = \frac{1}{(r'^2+r^2)^{1/2}}$. The curvature of a plane curve in polar coordinates is given by $\kappa = \frac{2r'^2 - rr'' + r^2}{(r'^2+r^2)^{\frac{3}{2}}}$. Hence taking the dot product of (4.13) with $\mathbf{N}$ we arrive at

$$\dot r r = -u''\left(\frac{2r'^2 - rr'' + r^2}{r'^2 + r^2} + (n-1)\right).$$

Because γ_∞ is stationary, we have

$$\frac{2r_\infty'^2 - r_\infty r_\infty'' + r_\infty^2}{r_\infty'^2 + r_\infty^2} + (n-1) = 0. \tag{5.7}$$

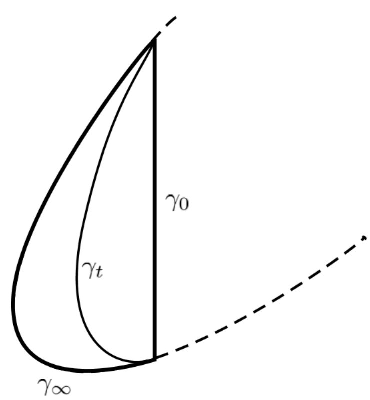

Figure 5.3: γ_t being the interpolation between γ_0 and γ_∞.

Now, let $b = b(t) : [0,\infty) \to \mathbb{R}$ be an increasing function to be determined later. We use $b(t)$ to define $r_t(\theta)$ by

$$\frac{1}{r_t^2(\theta)} = \frac{1}{1+b}\left(\frac{b}{r_\infty^2(\theta)} + \frac{\cos^2\theta}{a^2}\right). \tag{5.8}$$

For an appropriate choice of $b(t)$, we will show that the family of curves $\gamma_t(\theta) = (r_t(\theta)\cos\theta, r_t(\theta)\sin\theta)$

gives a subsolution to (4.13). The curve is shown in Figure 5.3 as an interpolation between γ_0 and γ_∞. Differentiating (5.8) with respect to θ, and suppressing dependence on t and θ from our notation for simplicity, we have

$$\frac{r'}{r^3} = \frac{1}{1+b}\left(\frac{br'_\infty}{r_\infty^3} + \frac{\sin(2\theta)}{2a^2}\right)$$

as well as

$$\frac{r''}{r^3} - \frac{3r'^2}{r^4} = \frac{1}{1+b}\left(\frac{br''_\infty}{r_\infty^3} - \frac{3br'^2_\infty}{r_\infty^4} + \frac{\cos(2\theta)}{a^2}\right).$$

So,

$$\begin{aligned}\frac{2r'^2 - rr'' + r^2}{r^4} &= -\left(\frac{r''}{r^3} - \frac{3r'^2}{r^4}\right) + \frac{1}{r^2} - \left(\frac{r'}{r^3}\right)^2 r^2 \\ &= \frac{1}{1+b}\Bigg(-\frac{br''_\infty}{r_\infty^3} + \frac{3br'^2_\infty}{r_\infty^4} - \frac{\cos(2\theta)}{a^2} + \frac{b}{r_\infty^2} + \frac{\cos^2\theta}{a^2} \\ &\qquad - \left(\frac{br'_\infty}{r_\infty^3} + \frac{\sin(2\theta)}{2a^2}\right)^2\left(\frac{b}{r_\infty^2} + \frac{\cos^2\theta}{a^2}\right)^{-1}\Bigg).\end{aligned}$$

By (5.7),

$$-\frac{br''_\infty}{r_\infty^3} + \frac{3br'^2_\infty}{r_\infty^4} + \frac{b}{r_\infty^2} = \frac{-b}{r_\infty^4}\left((n-1)(r'^2_\infty + r_\infty^2) - r'^2_\infty\right).$$

Now, for notational simplicity, set

$$A = \frac{br'_\infty}{r_\infty^3} + \frac{\sin(2\theta)}{2a^2}, \quad B = \frac{b}{r_\infty^2} + \frac{\cos^2\theta}{a^2}.$$

Then returning to the above we see

$$\begin{aligned}\frac{2r'^2 - rr'' + r^2}{r'^2 + r^2} &= \frac{1}{r^2}\frac{2r'^2 - rr'' + r^2}{r^4}\left(\left(\frac{r'}{r^3}\right)^2 + \left(\frac{1}{r^2}\right)^2\right)^{-1} \\ &= \frac{B}{A^2+B^2}\left(\frac{-b}{r_\infty^4}\left((n-1)(r'^2_\infty + r_\infty^2) - r'^2_\infty\right) + \frac{\sin^2\theta}{a^2} - \frac{A^2}{B}\right).\end{aligned}$$

Hence

$$\begin{aligned}\frac{2r'^2 - rr'' + r^2}{r'^2 + r^2} + (n-1) = \frac{1}{A^2+B^2}\Bigg(&-(n-1)\frac{Bb}{r_\infty^2} - (n-2)\frac{Bbr'^2_\infty}{r_\infty^4} \\ &+ \frac{B\sin^2\theta}{a^2} + (n-2)A^2 + (n-1)B^2\Bigg).\end{aligned}$$

We now compute

$$-(n-1)\frac{Bb}{r_\infty^2}+(n-1)B^2=(n-1)B\left(B-\frac{b}{r_\infty^2}\right)=(n-1)B\frac{\cos^2\theta}{a^2},$$

and

$$A^2-\frac{Bbr_\infty'^2}{r_\infty^4}=\frac{br_\infty'\sin(2\theta)}{a^2r_\infty^3}+\frac{\sin^2(2\theta)}{4a^4}-\frac{br_\infty'^2\cos^2\theta}{a^2r_\infty^4}.$$

Combining these, we have

$$\begin{aligned}\frac{2r'^2-rr''+r^2}{r'^2+r^2}+(n-1)=&\frac{(n-1)B\cos^2\theta+B\sin^2\theta}{a^2(A^2+B^2)}+\frac{(n-2)\sin^2(2\theta)}{4a^4(A^2+B^2)}\\&+\frac{n-2}{A^2+B^2}\left(\frac{br_\infty'\sin(2\theta)}{a^2r_\infty^3}-\frac{br_\infty'^2\cos^2\theta}{a^2r_\infty^4}\right).\end{aligned}$$

By assumption,

$$r_\infty'\geq 0\quad\text{and}\quad\frac{r_\infty'}{r_\infty}\leq 2\tan\theta,$$

which implies

$$\frac{br_\infty'\sin(2\theta)}{a^2r_\infty^3}-\frac{br_\infty'^2\cos^2\theta}{a^2r_\infty^4}\geq 0.$$

Additionally, $r_\infty, \sin\theta$ and $\cos\theta$ are all bounded above and below away from zero. This implies there exists a constant C_1 so that, for large b,

$$\frac{2r'^2-rr''+r^2}{r'^2+r^2}+(n-1)\geq\frac{C_1}{b}.$$

Returning to (5.8), we take the derivative of both sides in t

$$-\frac{2\dot r}{r^3}=-\frac{\dot b}{(1+b)^2}\left(\frac{b}{r_\infty^2}+\frac{\cos^2\theta}{a^2}-\frac{1+b}{r_\infty^2}\right)=-\frac{\dot b}{(1+b)^2}\left(\frac{\cos^2\theta}{a^2}-\frac{1}{r_\infty^2}\right).$$

Multiplying by $-r^4$ and plugging in the square of (5.8) for r^4 gives

$$\begin{aligned}2r\dot r&=\left(\frac{\cos^2\theta}{a^2}-\frac{1}{r_\infty^2}\right)\left(\frac{b}{r_\infty^2}+\frac{\cos^2\theta}{a^2}\right)^{-2}\dot b\\&=(r_\infty x-ra)\left(\frac{r_\infty x+ra}{a^2r^2r_\infty^2}\right)\left(\frac{b}{r_\infty^2}+\frac{\cos^2\theta}{a^2}\right)^{-2}\dot b\\&\geq(r_\infty x-ra)\frac{C_2}{b^2}\dot b\end{aligned}$$

for some $C_2 > 0$ whenever b is large. Note that the polar curves $r(\theta)$ intersect the line $x = a$ to the zeroth order, which implies there exists a constant $C_3 > 0$ for which

$$0 \geq \inf_{x\in[a-\epsilon,a]} \left(u''^{-1}(r_\infty x - ra)\right) + \inf_{x\in[1,a]} (r_\infty x - ra) \geq -C_3. \tag{5.9}$$

Next, we use the same assumption on the background Kähler form as before, namely, for $x \in [1, a-\epsilon]$ we assume $u''(x) \geq k(x-1)$. This implies

$$\begin{aligned} u'' &\geq k(r\cos\theta - 1) \\ &= k\left(\sqrt{(1+b)\left(\frac{b}{r_\infty^2} + \frac{\cos^2\theta}{a}\right)^{-1}}\cos\theta - 1\right) \\ &= k\left(\sqrt{(1+b)\left(\frac{b}{(r_\infty\cos\theta)^2} + \frac{1}{a}\right)^{-1}} - 1\right) \\ &\geq k\left(\sqrt{(1+b)\left(b + \frac{1}{a}\right)^{-1}} - 1\right). \end{aligned}$$

For simplicity, write the right hand side above as $C(b)$, which is a smooth positive function approaching 0 as $b \to \infty$. Combining with (5.9) we arrive at,

$$\frac{2}{u''} r\dot{r} \geq -\frac{C_2C_3}{b^2}\left(1 + \frac{1}{C(b)}\right)\dot{b}.$$

If b solves the initial value problem

$$\dot{b} = 2\left(1 + \frac{1}{C(b)}\right)^{-1}\frac{C_1}{C_2C_3}b; \qquad b_0 \gg 0,$$

then $r_t(\theta)$ defines a subsolution:

$$\frac{1}{u''}r\dot{r} + \left(\frac{2r'^2 - rr'' + r^2}{r'^2 + r^2} + (n-1)\right) \geq 0.$$

$\square$

We now show that the assumption on r_∞ in Proposition 5.3.1 can be satisfied with an explicit example.

Lemma 5.3.2. *There exists a semi-stable class $[\alpha]$ and an initial representative α_0 which satisfies the assumptions in Theorem 1.0.2 such that corresponding stationary solution r_∞ satisfies (5.6) with $\gamma_\infty(\theta_{\max}) = (a, p)$.*

Proof. Suppose r_∞ satisfies (5.7). Let

$$\tan\beta := \frac{y'(\theta)}{x'(\theta)} = \frac{r'_\infty \sin\theta + r_\infty \cos\theta}{r'_\infty \cos\theta - r_\infty \sin\theta} = \frac{r'_\infty r_\infty^{-1} \tan\theta + 1}{r'_\infty r_\infty^{-1} - \tan\theta}.$$

As a result

$$\frac{r'_\infty}{r_\infty} = \cot(\beta - \theta) = \tan(\pi/2 - \beta + \theta).$$

Now, choose $q \gg 0$. One can solve the initial value problem (5.7) in a semi-stable configuration, and set θ_0 so $\gamma_\infty(\theta_0) = (1, q)$ and $\gamma'_\infty(\theta_0)$ is vertical, which implies $\beta(\theta_0) = \pi/2$. In particular at this point

$$r'_\infty(\theta_0) > 0 \quad \text{and} \quad \frac{r'_\infty(\theta_0)}{r_\infty(\theta_0)} = \tan(\theta_0) < 2\tan(\theta_0).$$

Thus, there exists a neighborhood of θ_0 where (5.6) holds.

We now check (5.5). At $\theta = \theta_0$,

$$\begin{aligned} x'_\infty &= r_\infty \cos\theta \left(\frac{r'_\infty}{r_\infty} - \tan\theta \right) = 0 \\ x''_\infty &= \frac{\cos\theta}{r_\infty} \left(-2 r_\infty r'_\infty \tan\theta + r_\infty r''_\infty - r_\infty^2 \right) \\ &= \frac{\cos\theta}{r_\infty} \left(-2 r'^2_\infty + r_\infty r''_\infty - r_\infty^2 \right) > 0, \end{aligned}$$

where last inequality follows from (5.7). Hence, x_∞ achieves local minimum at $\theta = \theta_0$. We choose a slightly greater than 1 such that $x_\infty(\theta_{\min}) = x_\infty(\theta_{\max}) = a$. Finally, let γ_0 be an initial curve connecting $(1, q)$ to (a, p) that lies on the interior of the region R bounded by γ_∞ and $x = a$. Note that the angle of the $(1, 1)$ form α_0 associated to γ_0 is given by

$$\Theta(\alpha_0) = (n-1)\arctan(\gamma_0) + \arctan(\gamma'_0) = (n-1)\theta + \beta.$$

The supercritical phase assumption in Theorem 1.0.2 is satisfied if we choose q large enough so θ is sufficiently close to $\pi/2$ over the entire region R $\square$

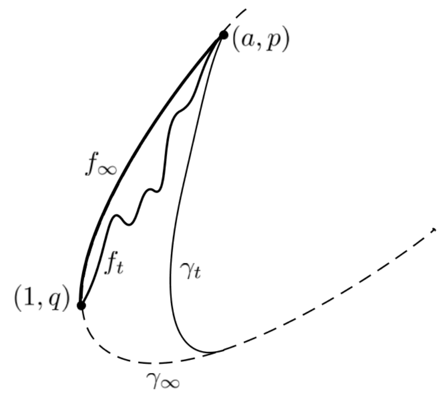

Figure 5.4: Singularity at $(1, q)$ at $t = \infty$.

Choose f_t to be above the curve γ_t at $t = t_0$ as in Figure 5.4. The avoidance principle guarantees that f_∞, f_t and γ_t are disjoint for all $t > t_0$. Using the subsolution in Theorem 5.3.1 together with the long time existence result in Theorem 1.0.2, a singularity is guaranteed to occur only at the point where γ_∞ achieves vertical tangency, which by construction, is at the point $(1, q)$, corresponding to the exceptional divisor of X.

5.4 Future Directions

There are many exciting and interesting questions which build off my previous results. One question would be to see whether similar singularity can be constructed on a more general Kähler manifold, perhaps by gluing our example in Theorem 1.0.1. The singularity we create occurs along an annulus region in the blowup of P^n, away from the essential divisors, and this is what we would need to glue in. The idea is as follows. Suppose we have a general Kähler manifold with a solution to the dHYM equation. We can fix a point and zoom-in the neighborhood, and replace an annulus in the neighborhood by the one from the blowup of P^n. It creates a new representative $\tilde{\alpha} \in [\alpha]$. We let $\tilde{\alpha}$ to flow along the line bundle mean curvature flow and hopefully, we can get a singularity in the annulus region before the information outside of the annulus comes in.

It would also be interesting to draw more parallels between the line bundle mean curvature flow and the graphical LMCF. For a graphical Lagrangian, let λ_i's be the eigenvalues of the Jacobian of the graph. We say that the Lagrangian is convex if $\lambda_i > 0$ for all i, and it is 2-convex if $(\lambda_i + \lambda_j)(1 - \lambda_i\lambda_j) > 0$ for all $i \neq j$. It is shown that both convexity and 2-convexity are preserved along the LMCF [27, 35], by computing the evolution of a certain tensor operator. The flow converges in this settings, and we hope to obtain similar results for the line bundle mean curvature flow, with curvature assumptions on the background Kähler manifold.

Finally, the location where long-time singularity happens is also of interest. Our example in Theorem 1.0.3 is constructed in the unstable case and the singularity occurs at the destabilizing subvariety, i.e. at the subvariety in which equation (1.6) fails to hold. One would expect this relationship between stability and singularity formation to hold in more general settings on the spaces where stability is defined.

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